Nambu–Jona-Lasinio model in a magnetic field with variable direction

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Homogeneous magnetic fields are known to act as strong catalysts of chiral symmetry breaking. The same trend is observed for a large class of magnetic fields with constant direction. This sort of stability with respect to the external field profile suggests that dynamical mass generation can really occur in some actual experimental conditions. Obviously, in order to ascertain the reliability of such a scenario, one should go beyond the simple background configurations that have been discussed up to now in the literature. Motivated by this consideration, we study the Nambu–Jona-Lasinio model minimally coupled to a background magnetic field with variable direction. For the chosen configuration we observe no trend to favor the massive phase. With respect to the zero-field case a larger coupling constant is required to break the chiral symmetry of the model. [S0556-2821(99)06204-9]

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I. INTRODUCTION

In the last decade, the influence of background fields on the mechanism of chiral symmetry breaking has received much attention. This interest was partly motivated by the anomalous e^+e^- peaks observed in heavy-ion collisions by the EPOS and ORANGE Collaborations at GSI [1]. In fact, a number of authors [2,3] attributed the origin of such structures to the formation of a new phase of QED, in which the electron mass acquires a dynamical contribution; the transition from the conventional phase to the new one should be triggered by the intense fields surrounding the colliding ions (see [4] for a critical review and a somewhat different perspective). The actual existence of the narrow e^+e^- peaks is still under debate [5] and, of course, the hypothetical new phase of QED does not represent the unique scenario which fits the experimental data. At any rate the physics of strong fields, be it accompanied or not by a new phase, is an interesting subject by itself, and it may also have implications for astrophysical phenomena [3,6], such as the generation of gamma ray bursts.

In the literature, the dynamical generation of mass in the presence of electromagnetic background fields has been investigated for QED and for some popular four-fermion interactions, namely the Nambu–Jona-Lasinio model [7] and the Gross-Neveu one [8]. Generally speaking, it turns out that magnetic fields act as strong catalysts for chiral symmetry breaking [9–11]. In the Nambu–Jona-Lasinio (NJL) model, for instance, one finds the interesting result that spontaneous mass generation takes place even for arbitrarily small four-fermion coupling constants (although in this case the generated mass is small as well). Up to now, the problem has been usually analyzed for unidirectional magnetic fields. In the homogeneous field case, the breaking of chiral symmetry can be understood [10] as a consequence of the dimensional reduction $D\rightarrow(D-2)$ in the dynamics of fermion pairing in a magnetic field. The occurrence of this dimensional reduction is intimately connected with the fact that the lowest Landau levels form a degenerate ground state with zero energy (in the massless limit). The generalization to inhomogeneous but unidirectional configurations [12,13] can be achieved using a notable result derived by Aharonov, Casher, and Jackiw [14,15] (ACJ), according to which the existence of zero-energy modes is ensured for any magnetic field with a fixed direction. This feature has been used in [12] to discuss the NJL model in the presence of the field $B_x=0, B_y=0, B_z=B(y)$. As in the uniform case, the associated "gap equation" admits nontrivial solutions corresponding to a massive phase. Moreover, it turns out [16] that the presence of gradients can be absorbed by simply introducing an effective magnetic field defined as the sum of the square moduli of the ground-state wave functions. As pointed out by Jackiw [15], the presence of zero-energy modes in the spectrum of a Dirac particle interacting with unidirectional magnetic fields is mainly due to a quantum-mechanical supersymmetry, that of the second-order Dirac Hamiltonian describing the electronic motion in the transverse plane perpendicular to $\boldsymbol{B}$. If we consider an arbitrary external field, for which the separation of variables is no longer possible, then we lose not only translational invariance but also supersymmetry. From this point of view, the background configurations considered in the literature appear as quite exceptional situations. Therefore, it is extremely important to reconsider the question of spontaneous mass generation under more general assumptions, where the "fixed direction" hypothesis is relaxed and where new phenomena and new trends are very likely to appear. In order to shed light on this point, we shall discuss the NJL model in the following axial symmetric background configuration:

$$B_z=B_r=0,\quad B_\phi=\frac{\alpha}{r^2}, \tag{1}$$

where cylindrical coordinates are used and α is a constant (in which we absorb the electric charge). It is well known that the Dirac equation can be exactly solved only for an extremely limited class [17] of external magnetic fields. Among those with variable direction, our choice (1) is nothing but the simplest one.

II. GENERAL FORMALISM

Before entering the details of the computation, we find it appropriate to recall some general considerations about the NJL model in the presence of external electromagnetic fields. Our starting point is the gap equation which has the form [7,9]

$$m = 2ig\,\mathrm{tr}\,S_A(x,x;m), \tag{2}$$

where $S_A(x,x;m)$ is the fermion propagator in the prescribed four vector potential $A_\mu(x)$. Strictly speaking, the gap equation should be considered as a functional equation in the unknown function $m(x)$. Accordingly, the propagator $S_A(x,x;m)$ should be computed in terms of the arbitrary space-dependent mass $m(x)$. To avoid such a difficult task, we follow [12] by keeping m fixed while computing the closed fermion loop involved in the gap equation. Using the ''proper time'' formalism introduced by Schwinger, the trace of the Green's function $S_A(x,x)$ can be written as

$$\mathrm{tr}\,S_A(x,x;m) = -i\int_{1/\Lambda^2}^{\infty} ds\, e^{-m^2 s}\frac{m}{4\pi^2}F(s;x). \tag{3}$$

Here $1/\Lambda^2$ is a small-s cutoff which regularizes the ultraviolet divergence of the closed fermion loop and $F(s;x)$ is a weighted sum over the eigenfunctions of the system:

$$F(s;x) = \frac{2\pi^{3/2}}{\sqrt{s}}\sum_\nu \Psi_\nu^\dagger(\mathbf{x})\Psi_\nu(\mathbf{x})e^{-\epsilon_\nu^2 s}, \tag{4}$$

ϵ_ν^2 being the eigenvalue of the second-order Dirac Hamiltonian. For a free fermion we have

$$F_0(s;x) = \frac{1}{s^2}, \tag{5}$$

whereas a fermion interacting with a uniform magnetic field B corresponds to

$$F_{\mathrm{const}}(s;x) = \frac{1}{s^2}eBs\,\coth(eBs). \tag{6}$$

For s small $F(s;x)$ behaves as $1/s^2$ in both cases. This is quite natural since the small-s regime corresponds to ultraviolet contributions to the trace of the fermion propagator. In order to separate this ultraviolet divergence, the gap equation will be written in the form

$$m = 2g\int_{1/\Lambda^2}^{\infty} ds\, e^{-m^2 s}\frac{m}{4\pi^2}\left[F(s;x) - \frac{1}{s^2}\right] + 2g\int_{1/\Lambda^2}^{\infty}\frac{ds}{s^2}e^{-m^2 s}\frac{m}{4\pi^2}, \tag{7}$$

which gives

$$\frac{2\pi^2}{g} = \int_{1/\Lambda^2}^{\infty} ds\, e^{-m^2 s}\left[F(s;x) - \frac{1}{s^2}\right] + \Lambda^2 e^{-m^2/\Lambda^2} - m^2 E_1(m^2/\Lambda^2), \tag{8}$$

that is

$$\frac{2\pi^2}{g\Lambda^2} = 1 - \frac{m^2}{\Lambda^2}\log\frac{\Lambda^2}{m^2} + \frac{1}{\Lambda^2}\int_0^{\infty} ds\, e^{-m^2 s} \times\left[F(s;x) - \frac{1}{s^2}\right] + O(m^2/\Lambda^2). \tag{9}$$

In the zero-field case the third term in the right-hand side (RHS) of this equation disappears, thereby giving the well-known result of the original NJL model. In the constant field case we are faced with a completely different behavior of $F(s;r)$ in the large-s domain. More precisely, for $s \gg 1/eB$, we see that

$$F_{\mathrm{const}}(s;x) \sim \frac{eB}{s}. \tag{10}$$

This feature has a dramatic consequence in the competition between massless and massive phases. Indeed, the asymptotic behavior of Eq. (10) corresponds to the following contribution to the trace of the fermion propagator:

$$\mathrm{tr}\,S_A(x,x;m) \sim -i\frac{meB}{4\pi^2}\ln\left(\frac{eB}{m^2}\right). \tag{11}$$

Inserting this in the gap equation, one obtains

$$\frac{2\pi^2}{g\Lambda^2} \simeq 1 + \frac{eB}{\Lambda^2}\ln\frac{eB}{m^2}. \tag{12}$$

Now the RHS is logarithmically divergent as $m \to 0$, so that we can find nontrivial solutions to the gap equation even for arbitrarily small four-fermion coupling constant. A similar result is obtained for any unidirectional background magnetic field, where the separation of motion along the field direction enables us to write

$$F(s;x) = \frac{\pi}{s}f(s;x). \tag{13}$$

Here $f(s;x)$ is a weighted sum over the wave functions describing the electronic motion in the transverse plane (x,y):

$$f(s;x) = \sum_\eta \psi_\eta^\dagger(\mathbf{x})\psi_\eta(\mathbf{x})e^{-\epsilon_\eta^2 s}, \tag{14}$$

ψ_η being the solution of the following eigenvalue equation:

$$[(p_x - eA_x)^2 + (p_y - eA_y)^2 - eB(x,y)\Sigma_3]\psi_\eta = \epsilon_\eta^2\psi_\eta. \tag{15}$$

The factor $1/s$ in Eq. (13) appears after integration over p_z. From the works of Aharonov, Casher, and Jackiw [14,15] we know that Eq. (15) admits a set of zero-energy modes whose number is proportional to the total magnetic flux. Moreover, when this flux is infinite, there exists an energy gap separating the excited levels from the degenerate ground state. From these lowest energy levels the function $F(s;x)$ receives a contribution proportional to $1/s$. On the opposite side, the excited states are exponentially suppressed by the Gaussian weight appearing in Eq. (14). As a consequence, for small values of m the RHS of the gap equation (9) still exhibits a logarithmic divergence which results in the breaking of chiral symmetry in the small-g regime.

III. RESULTS FOR THE GAP EQUATION

We are now in a position to understand the discussion of the NJL model in the magnetic-field profile given by Eq. (1), for which the ACJ result is not applicable any more. In cylindrical coordinates, the properly normalized eigenfunctions with energy k_0 can be written as follows (see [17] for details):

$$\Psi = \frac{1}{2\pi} e^{-i(k_2\theta + k_3 z)} \frac{1+i\Sigma_3}{\sqrt{2}} e^{i\Sigma_3\theta/2} \frac{N_{l,\zeta}(p,k_3)}{2\sqrt{\mu k_0}} \psi(r), \tag{16}$$

where

$$k_2 = l - \frac{1}{2}, \quad \mu = \sqrt{k_2^2+\alpha^2}, \quad p = \sqrt{k_0^2 - m^2 - k_3^2}. \tag{17}$$

Here l is an integer and

$$N_{l,\zeta}(p,k_3) = \sqrt{\frac{p}{\pi}} \frac{|\Gamma(\mu - i\nu_0)|}{2(2\mu+1)\Gamma(2\mu)} \times \left[\frac{p^2(\mu^2+\nu_0^2)}{c^2+p^2(\mu^2+\nu_0^2)}\right]^{1/2} e^{-\pi\nu_0/2} \tag{18}$$

with

$$\nu_0 = \frac{\alpha k_3}{p}, \quad c = k_2 k_3 - \zeta\mu\sqrt{k_3^2+p^2}. \tag{19}$$

The four components spinor function $\psi(r)$ can be written as

$$\psi(r) = \begin{pmatrix} \sqrt{k_0+m} \\ \zeta\sqrt{k_0-m} \end{pmatrix} \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix}, \tag{20}$$

where $\zeta = \pm 1, \sqrt{k_0 \pm m}$ are 2×2 matrices and

$$f_1(r) = \sqrt{\mu+k_2}\,\psi_1(r) + \sqrt{\mu-k_2}\,\psi_{-1}(r),$$
$$f_2(r) = \sqrt{\mu-k_2}\,\psi_1(r) - \sqrt{\mu+k_2}\,\psi_{-1}(r). \tag{21}$$

Finally, the functions ψ_1 and ψ_{-1} can be expressed in terms of the confluent hypergeometric functions:

$$\psi_1 = \frac{c}{\mu p}(2pr)^{\mu+1/2}\exp(-ipr)\Phi(\mu - i\nu_0 + 1, 2\mu+2, 2ipr),$$

$$\psi_{-1} = 2(2\mu+1)(2pr)^{\mu-1/2} \times \exp(-ipr)\Phi(\mu - i\nu_0, 2\mu, 2ipr). \tag{22}$$

Using Eqs. (16)–(22), the weighted sum $F(s;r)$ over the eigenfunctions of the system can be cast in the form

$$F(s;r) = \frac{2\pi^{3/2}}{\sqrt{s}} \int_0^\infty dp \int_{-\infty}^{\infty} dk_3 \sum_{l,\zeta} \frac{N_{l,\zeta}^2(p,k_3)}{4\pi^2} \times (|\psi_{-1}(r)|^2 + |\psi_1(r)|^2) e^{-s(p^2+k_3^2)}. \tag{23}$$

For small s we expect that $F(s;r) \sim 1/s^2$, therefore the separation of the ultraviolet divergence made in Eq. (7) is still appropriate (see below). For moderate values of s, one should resort to numerical computations. However, from the physical point of view, it is much more interesting to study the large-s behavior of $F(s;r)$, which is directly connected to the competition between the massless and the massive phase in the weak- coupling regime, that is, for small values of g. In particular, the magnetic field will do act as a catalyst only if the decrease of $F(s,r)$ is not faster than $1/s$ [see Eqs. (10) and (11)]. Fortunately, some analytic estimates are possible in the large-s regime, where the leading contribution to $F(s;r)$ comes from the lowest-lying energy levels. In particular we can neglect ψ_1 with respect to ψ_{-1} and set

$$\psi_{-1}(r) \sim 2(2\mu+1)(2pr)^{\mu-1/2}. \tag{24}$$

This leads to

$$F(s;r) = \frac{1}{2\pi^{3/2}\sqrt{s}} \int_0^\infty p\,dp \int_{-\infty}^{\infty} dk_3 \sum_{l,\zeta} \left[\frac{p^2(\mu^2+\nu_0^2)}{c^2+p^2(\mu^2+\nu_0^2)}\right] \times \left|\frac{\Gamma(\mu - i\nu_0)}{\Gamma(2\mu)}\right|^2 e^{-\pi\nu_0}(2pr)^{2\mu-1} e^{-s(p^2+k_3^2)}. \tag{25}$$

Now, by means of the trivial change of variables

$$p = \frac{z}{\sqrt{s}}\cos\phi, \quad k_3 = \frac{z}{\sqrt{s}}\sin\phi, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}, \tag{26}$$

we can easily extract the s dependence in Eq. (25); moreover, the integration over ϕ is decoupled from that on z which simply gives a Gamma function. The final result is

$$F(s;r) = \sum_l \frac{a(\mu;r)\Gamma(\mu+1)}{\Gamma^2(2\mu)} \frac{1}{s^{\mu+3/2}} \tag{27}$$

with

$$a(\mu;r) = \frac{(2r)^{2\mu-1}}{8\mu\,\pi^{3/2}} \int_{-\pi/2}^{\pi/2} d\phi (\cos\phi)^{2\mu} \times \left[\frac{\mu^2\cos^2\phi + \alpha^2\sin^2\phi}{\mu - k_2\sin\phi}\right] |\Gamma(\mu - i\,\alpha\tan\phi)|^2 e^{-\pi\,\alpha\tan\phi}. \tag{28}$$

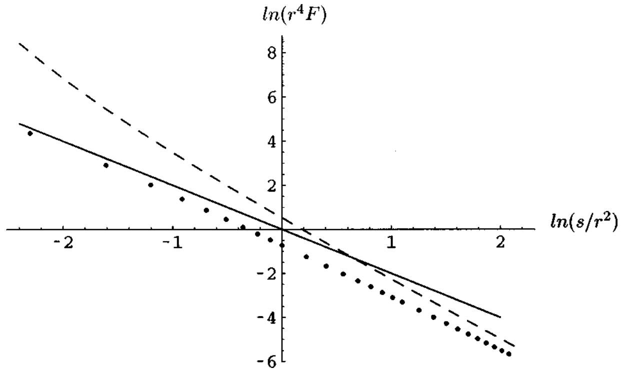

FIG. 1. Numerical computation of $F(s;r)$. The dotted line represents the logarithm of the dimensionless function $r^4F(s;r)$ versus the logarithm of s/r^2. The solid line and the dashed line are drawn as a cross check for the numerical computation and the analytical approximations used in the paper. Solid line: zero-field case $F_0(s;r)=1/s^2$. Dashed line: large-s approximation of $F(s,r)$ as given by Eqs. (27) and (28).

Recalling that $\mu=\sqrt{(l-1/2)^2+\alpha^2}$, we see that the decrease of $F(s;r)$ is faster than $1/s^2$. This means that no divergence emerges from the $m\rightarrow 0$ limit in RHS of the gap equation (9). To say it in a different way, we recover the conventional scenario in which chiral symmetry breaking is a strong-coupling phenomenon requiring a four-fermion coupling constant above a certain threshold.

In order to check our estimates and approximations we have performed a numerical computation of $F(s;r)$ as a function of s for $\alpha=1$. As far as the r dependence is concerned, it suffices to exploit the scaling property

$$F(s;\eta r)=\frac{1}{\eta^4}F\left(\frac{s}{\eta^2},r\right), \tag{29}$$

which follows from simple dimensional arguments or from a direct inspection of the wave functions and their normalization constants. Our results are shown in Fig. 1. The dotted line represents the logarithm of the dimensionless function $r^4F(s;r)$ versus the logarithm of s/r^2, while the solid line corresponds to the zero-field case $F_0(s;r)=1/s^2$. As expected, we see that $F(s,r)$ behaves as $F_0(s,r)$ for small values of s. The dashed line is the large$-s$ approximation of $F(s,r)$ as given by Eqs. (27) and (28). Once again the agreement is satisfactory. From the numerical computation it follows the important result that

$$F(s;r)<1/s^2. \tag{30}$$

As a consequence, the third term in the RHS of the gap equation (9) is negative. Obviously, this confirms that we need a larger coupling constant to form the massive phase in the presence of the magnetic field (1). More precisely, inequality (30) means that the required value of g is greater than $2\pi^2/\Lambda^2$, the threshold corresponding to the zero-field case. From the point of view of our approximate treatment this result is welcome. In fact, an intermediate value of g (larger with respect to the homogeneous case, but smaller than $2\pi^2/\Lambda^2$) could be interpreted as an artifact of our approximations. In particular, we have to keep in mind that a constant mass has been used in the computation of the closed fermion loop involved in the gap equation. Since this loop probes a region of space in which the magnetic field rapidly decays as $1/r^2$, one can suspect that the ''generated mass'' is an average quantity which underestimates the effect of the local field. Fortunately, such a possible drawback does not affect our main conclusion, for the effect of field (1) turns out to be opposite to that induced by unidirectional configurations. Finally, we find it appropriate to make a brief comment on the r dependence of the gap equation. Using the scaling property (29), one can easily show that the third term in the RHS of the gap equation (9) is a monotonic function of r going to zero as r tends to infinity. As a consequence, the generated mass is an increasing function of r whose asymptotic value is nothing but the mass scale of the NJL model in the zero-field case. Obviously, this behavior is exactly what we expect from a background field which tends to inhibit the formation of a massive phase.

IV. SUMMARY

Homogeneous magnetic fields are known to act as catalysts for chiral symmetry breaking. Several works show that a similar trend is present for inhomogeneous magnetic fields, provided that their direction is kept fixed. Naively, this fact suggests that dynamical mass generation exhibits a sort of stability with respect to the external field profile. In turn, such a stability would make reliable the scenario in which chiral symmetry breaking is induced by sufficiently strong fields which may be produced in laboratory conditions. Actually, one easily realizes that magnetic fields with fixed direction are extremely special configurations, since they all share a supersymmetry: that of the second-order Dirac Hamiltonian describing the electronic motion in the transverse plane [15]. Moreover, such a supersymmetry seems to be a crucial ingredient to obtain a dynamically generated mass. Thus, it is of fundamental importance to reconsider the question of spontaneous mass generation under more general assumptions, where the ''fixed direction'' hypothesis is relaxed. Motivated by these considerations, we have discussed the NJL model in the background configuration (1) representing one of the few examples of divergenceless magnetic field with variable direction for which the Dirac equation can be analytically solved. By taking advantage of the exact wave functions and energy levels we have estimated the leading contribution to the trace of the electron propagator which, in turn, determines the main features of the gap equation. As a first result, we recover the conventional scenario where chiral symmetry breaking is a strong-coupling phenomenon requiring a four-fermion coupling constant above a certain threshold. This is in striking contrast with the constant field case in which the massive phase is formed at the weakest attractive interaction between fermions. Furthermore, via a numerical computation, we observe that the external field (1) tends to restore the chiral symmetry of the model: that is, we need a larger coupling constant to form the massive phase. These conclusions corroborate our conjecture according to which the dynamical generation of mass is not so universal as one would expect by extrapolating the results obtained for homogeneous or unidirectional magnetic fields. It goes without saying that we need further investigations to clearly understand the role of the field gradients, with particular attention to those gradients which destroy the supersymmetry shared by all unidirectional field profiles.

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