Moving observers, nonorthogonal boundaries, and quasilocal energy

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The popular Hamilton-Jacobi method first proposed by Brown and York for defining quasilocal quantities such as energy for spatially bound regions assumes that the timelike boundary is orthogonal to the foliation of the spacetime. Such a restriction is undesirable for both theoretical and computational reasons. We remove the orthogonality assumption and show that it is more natural to focus on the foliation of the timelike boundary rather than the foliation of the entire four dimensional bound region. Reference spacetimes which define additional terms in the action are discussed in detail. To demonstrate this new formulation, we calculate the quasilocal energies seen by observers who are moving with respect to a Schwarzschild black hole.
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I. INTRODUCTION

Gravitational thermodynamics and its relationship to the Euclidean-action formulation of quantum gravity have been of increasing interest in recent years. This relationship was first explored by Gibbons and Hawking [1], who argued that the Euclidean gravitational action is equal to the grand canonical free energy times the reciprocal of the temperature associated with a black hole (or cosmological) event horizon [2]. A more recent extension of this work by Brown and York involved consideration of the formulation of the partition function for gravitating systems of finite spatial extent [3,4]. Starting from a spacelike foliation of a finite region of spacetime and a timelike vector field defining a flow of time, they studied the Einstein-Hilbert action using a Hamilton-Jacobi-type analysis. Decomposing the action according to the foliation and flow of time, they showed that natural candidates arose for quantities such as energy and momentum. These quantities were defined quasilocally, i.e. for a region of finite spatial extent containing a gravitating system.

For a number of reasons this analysis and its associated quasilocal quantities have generated much interest and found a multitude of uses. First, all physical systems with which we have any experience have a finite spatial boundary. Indeed one of the central concepts in thermodynamics is that of a system and a reservoir that are separated by a partition. Quasilocal quantities admit a physical realization of these concepts so that thermodynamics may be applied in a sensible way. As such, in the literature this analysis has been used extensively in the study of black hole thermodynamics (for example in [4–6]). Among other places, this work has found application in studies of the distribution of gravitational energy in a variety of spacetimes (for example [7]) and also in examining the quantum mechanical creation of pairs of black holes (for example [8]). A very similar Hamiltonian decomposition of the action has also been executed by Hawking and Horowitz [9].

However, an acknowledged incompleteness exists in the quasilocal formulation in that (apart from two exceptions mentioned below) the spacetime foliation is always assumed to be orthogonal to the timelike boundary. While this is the case for many standard examples (such as black holes surrounded by a set of stationary observers) it is a fairly strong restriction. For example, within the confines of this orthogonality assumption it is extremely difficult to calculate the quasilocal quantities seen by observers who are falling into a black hole. Furthermore, when one considers variations of the metric (as one actually does during the quasilocal Hamilton-Jacobi analysis) the orthogonality assumption implies that the variations are not general, but instead restricted to those that preserve the orthogonality.

The requirement that the timelike boundary of the finite region be orthogonal to its spacelike boundary was dropped by Hayward [10], who considered how the basic Hilbert action I should be modified so that solving $\delta I=0$ for general variations of the metric (subject to the boundary condition that boundary metrics should be held constant) will produce the Einstein equations in the usual way. However, this was from a purely Lagrangian viewpoint—no consideration was given as to how these variations would decompose in accordance with the spacetime foliation. That approach was recently considered by Hawking and Hunter [11] and Lau [12], who addressed the nonorthogonal situation from a Hamiltonian perspective.

In Ref. [11] Hayward's action was broken down according to the foliation of the spacetime, a Hamiltonian proposed, and two sample calculations performed where the boundaries were nonorthogonal. However, there was no attempt made in this treatment to consider the variation of the action and show an agreement between the quasilocal quantities suggested by that approach and the direct Hamiltonian deconstruction of the action. Furthermore, in order to deal with the nonorthogonal intersections, the authors found it necessary to impose somewhat complicated restrictions on background comparison spacetimes.

Lau's main interest in [12] was to reformulate the quasilocal quantities of [3] in terms of Ashtekar variables. Treating nonorthogonal boundaries was a matter of secondary concern. Thus, although certain elements of his discussion are similar to some of the developments of this paper, the focus

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is quite different. In particular he did not decompose the action I, and the decomposition of δI was with respect to variations of the Ashtekar variables rather than metric variables. He did not discuss background terms in detail and did not calculate any examples.

In this paper we shall consider both a decomposition of Hayward's action and a decomposition of the variation of that action and show that they agree in their natural candidates for the quasilocal quantities. In doing this, we shall focus on the boundary lapse and shift functions rather than the full spacetime lapse and shift as was the case in [11]. This will result in less complicated decompositions that also require less stringent restrictions on the comparison spacetime, in contrast with the approach in Ref. [11]. We shall also argue that the boundary lapse and shift functions are the natural lapse and shift to consider.

Before turning to those decompositions it is necessary to set out quite a few definitions. Those definitions will be the subject of Sec. II. In Sec. III we will perform the decompositions, examine the quasilocal quantities that naturally arise from those decompositions, see how those quantities relate to conserved charges, and finally examine the background terms in some detail. Section IV is made up of applications of the theory of Sec. III. In that section we shall calculate the quasilocal energy (and other quantities) seen by a spherically symmetric set of observers undergoing a variety of motions in Schwarzschild spacetime.

II. DEFINITIONS

Consider a region $\mathcal{M}$ of an n-dimensional spacetime with metric tensor field $g_{\alpha\beta}$ and on that region define a timelike vector field T^α and a spacelike $(n-1)$-dimensional hypersurface Σ_0. This field and surface are sufficient to define a notion of time over $\mathcal{M}$. As a start, we let Σ_0 be an "instant" in time. That is we choose to define all events happening on that surface as happening simultaneously. Next, consider a set of observers at locations $x^\alpha_{A0} \in \Sigma_0$ (where the A index labels the individual observers). The past and future locations of these observers are uniquely determined if we specify that they must follow the paths $x^\alpha_A(t)$ through spacetime where these paths satisfy the differential equation $dx^\alpha_A/dt = T^\alpha$ subject to the initial condition $x^\alpha_A(t_0) = x^\alpha_{A0}$. We then define "instants" of time to be $t=$const surfaces. We label them Σ_t and define the notion of past and future by saying that if $t_1 < t_2$, then Σ_1 "happens" before Σ_2. By construction the t_0 hypersurface is Σ_0. Thus, from the vector field and original hypersurface we have imposed an observer dependent notion of time on our manifold according to the constructed time coordinate t. Next, we may break up T^α into its components perpendicular and parallel to the Σ_t by defining a lapse function N and a shift vector field V^α so that

$$T^\alpha = N u^\alpha + V^\alpha, \tag{1}$$

where u^α is defined so that at each point in $\mathcal{M}$ it is the future pointing unit normal vector to the appropriate spacelike hypersurface Σ_t, and $V^\alpha u_\alpha = 0$. The lapse and shift then tell us how observers being swept along with the time flow T^α move through space and time relative to the foliation.

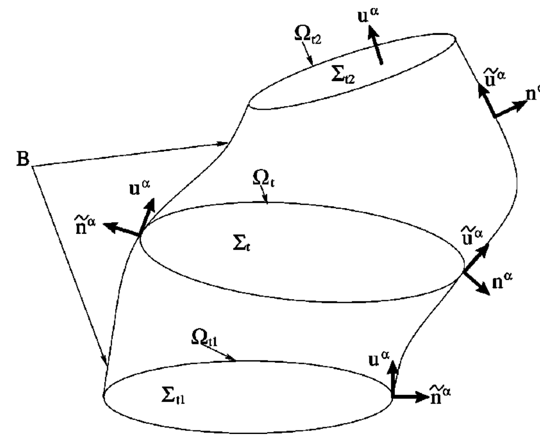

FIG. 1. The region M of Lorentzian spacetime, its assorted normal vector fields, and a typical element of the foliation.

We specialize to the situation of interest for this paper. On the surface Σ_0 define a surface Ω_0 that is topologically an $(n-2)$-sphere. Ω_0 locally bifurcates Σ_0—we will pick one of the regions as "inside" and the other as "outside."[1] Now, propagate this surface through time according to the time flow T^α. Then, by the local continuity of the time flow, Ω_t will (at least locally) topologically remain an $(n-2)$-sphere in the hypersurface Σ_t and still divide the "inside" from the "outside." Choosing time coordinates t_1 and t_2 we define an $(n-1)$ dimensional timelike hypersurface $B = \{\cup_t \Omega_t : t_1 \leq t \leq t_2\}$. We then define $M \subset \mathcal{M}$ as the region "inside" B bounded by the surfaces Σ_{t1} and Σ_{t2}. Here Σ_t and Ω_t represent foliations of M and B respectively. Figure 1 illustrates these concepts for a three dimensional $\mathcal{M}$.

We define unit normal vector fields for the various hypersurfaces. Already we have defined u^α as the timelike unit normal vector field to the Σ_t surfaces. Similarly, we may define $\tilde{u}^\alpha$ as the forward pointing timelike unit normal vector field to the surfaces Ω_t in the hypersurface B. The spacelike outward-pointing unit normal vector field to B is defined as n^α. Then, by construction $\tilde{u}^\alpha n_\alpha = 0$ and $T^\alpha n_\alpha = 0$. We further define $\tilde{n}^\alpha$ as the vector field defined on B such that $\tilde{n}^\alpha$ is the unit normal vector to Ω_t in Σ_t (Ω_t viewed as a surface in Σ_t). By construction $u^\alpha \tilde{n}_\alpha = 0$.

We define the scalar field $\eta = u^\alpha n_\alpha$ over B. If $\eta = 0$ everywhere, then the foliation surfaces are orthogonal to the boundary B (the case dealt with in Refs. [3,9]),[2] and the vector fields with the tildes are equal to their counterparts without tildes. We express $\tilde{u}^\alpha$ and n^α in terms of u^α and $\tilde{n}^\alpha$ (or vice versa) as

[1]Globally of course the "inside" and "outside" could be connected. Consider for example the case where Σ_0 is a two-torus, and Ω_0 is a homotopically nontrivial circle in that surface.

[2]The definitions of u^α and n^α are consistent with Ref. [3], but interchanged with respect to those in Ref. [9].

$$n^\alpha = \frac{1}{\lambda}\tilde{n}^\alpha - \eta u^\alpha \quad \text{and} \quad \tilde{u}^\alpha = \frac{1}{\lambda}u^\alpha - \eta\tilde{n}^\alpha \tag{2}$$

or

$$\tilde{n}^\alpha = \frac{1}{\lambda}n^\alpha + \eta\tilde{u}^\alpha \quad \text{and} \quad u^\alpha = \frac{1}{\lambda}\tilde{u}^\alpha + \eta n^\alpha, \tag{3}$$

where $\lambda^2 \equiv 1/(1+\eta^2)$.

Note, too, that with $T^\alpha n_\alpha = 0$ we may write

$$T^\alpha = \tilde{N}\tilde{u}^\alpha + \tilde{V}^\alpha, \tag{4}$$

where we call $\tilde{N} \equiv \lambda N$ the boundary lapse and $\tilde{V}^\alpha \equiv \sigma^\alpha_\beta V^\beta$ the boundary shift.

Next consider the metrics induced on the hypersurfaces by the spacetime metric $g_{\alpha\beta}$. These may be written in terms of $g_{\alpha\beta}$ and the normal vector fields. $h_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta$ is the metric induced on the Σ_t surfaces, $\gamma_{\alpha\beta} \equiv g_{\alpha\beta} - n_\alpha n_\beta$ is the metric induced on B, and $\sigma_{\alpha\beta} \equiv h_{\alpha\beta} - \tilde{n}_\alpha\tilde{n}_\beta = \gamma_{\alpha\beta} + \tilde{u}_\alpha\tilde{u}_\beta$ is the metric induced on Ω_t. By raising one index of these metrics we obtain projection operators into the corresponding surfaces. These have the expected properties $h^\alpha_\beta u^\beta = \gamma^\alpha_\beta n^\beta = \sigma^\alpha_\beta n^\beta = \sigma^\alpha_\beta u^\beta = 0$, and $h^\alpha_\beta h^\beta_\gamma = h^\alpha_\gamma$, $\gamma^\alpha_\beta\gamma^\beta_\gamma = \gamma^\alpha_\gamma$, and $\sigma^\alpha_\beta\sigma^\beta_\gamma = \sigma^\alpha_\gamma$.

On choosing a coordinate system $\{x^1, x^2, x^3\}$ on the surface Σ_0 we define $h = \det(h_{\alpha\beta})$ (where in this case we take $h_{\alpha\beta}$ as the coordinate representation of that metric tensor). We then map this coordinate system to each of the other Σ_t surfaces using the time flow; combining this set of coordinates on each surface with the time coordinate $x^0 \equiv t$ we have a coordinate system over all of M. We define $g = \det(g_{\alpha\beta})$. Similarly, choosing a coordinate system on Ω_t we define $\sigma = \det(\sigma_{\alpha\beta})$. Again, using the time flow to extend the coordinate system over all of B, we define $\gamma = \det(\gamma_{\alpha\beta})$. It is then not hard to show [11] that

$$\sqrt{-g} = N\sqrt{h} \quad \text{and} \quad \sqrt{-\gamma} = \tilde{N}\sqrt{\sigma}. \tag{5}$$

We also define the following extrinsic curvatures. Taking ∇_α as the covariant derivative on $\mathcal{M}$ compatible with $g_{\alpha\beta}$, the extrinsic curvature of Σ_t in $\mathcal{M}$ is $K_{\alpha\beta} \equiv -h^\gamma_\alpha h^\delta_\beta \nabla_\gamma u_\delta = -\frac{1}{2}\pounds_u h_{\alpha\beta}$, where $\pounds_u$ is the Lie derivative in the direction u^α. The extrinsic curvature of B in $\mathcal{M}$ is $\Theta_{\alpha\beta} = -\gamma^\gamma_\alpha\gamma^\delta_\beta\nabla_\gamma n_\delta$, while the extrinsic curvature of Ω_t in Σ_t is $k_{\alpha\beta} \equiv -\sigma^\gamma_\alpha\sigma^\delta_\beta\nabla_\gamma\tilde{n}_\delta$. Contracting each of these with the appropriate metric we define $K \equiv h^{\alpha\beta}K_{\alpha\beta}$, $\Theta \equiv \gamma^{\alpha\beta}\Theta_{\alpha\beta}$, and $k \equiv \sigma^{\alpha\beta}k_{\alpha\beta}$.

Finally, we define the following intrinsic quantities over $\mathcal{M}$ and Σ_t. On $\mathcal{M}$, the Ricci tensor, Ricci scalar, and Einstein tensor are $\mathcal{R}_{\alpha\beta}$, $\mathcal{R}$, and $G_{\alpha\beta}$ respectively. On Σ_t, D_α is the covariant derivative compatible with $h_{\alpha\beta}$, while $R_{\alpha\beta}$ and R are respectively the intrinsic Ricci tensor and scalar.

III. ANALYZING THE ACTION

For definiteness, we now take $\mathcal{M}$ to be four dimensional. The generalization to other dimensions is trivial. Given $M \subset \mathcal{M}$ as described above and allowing for the inclusion of a cosmological constant, Hayward's action [10] is

$$I = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}(\mathcal{R} - 2\Lambda) + \frac{1}{\kappa}\int_\Sigma d^3x\sqrt{h}K - \frac{1}{\kappa}\int_B d^3x\sqrt{-\gamma}\Theta + \frac{1}{\kappa}\int_\Omega d^2x\sqrt{\sigma}\sinh^{-1}(\eta) - \underline{I}, \tag{6}$$

where $\int_\Sigma = \int_{\Sigma_2 - \Sigma_1}$ and $\int_\Omega = \int_{\Omega_2} - \int_{\Omega_1}$, and if we choose a system of units where $c = G = 1, \kappa = 8\pi$. Here $\underline{I}$ is a functional of the boundary metrics on ∂M. For simplicity, in the next two sections we will take $\underline{I} = 0$, but in the Sec. III D we will allow it to be nonzero again.

A. Variation of the action

The variation of I with respect to the metric $g_{\alpha\beta}$ is [10]

$$\delta I = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}(G_{\alpha\beta} + \Lambda g_{\alpha\beta})\delta g^{\alpha\beta} - \int_\Sigma d^3x(P^{\alpha\beta}\delta h_{\alpha\beta}) + \int_B d^3x(\pi^{\alpha\beta}\delta\gamma_{\alpha\beta}) + \int_\Omega d^2x\left(\frac{1}{\kappa}\sinh^{-1}(\eta)\delta\sqrt{\sigma}\right), \tag{7}$$

where $P^{\alpha\beta} \equiv (\sqrt{h}/2\kappa)(K^{\alpha\beta} - Kh^{\alpha\beta})$ and $\pi^{\alpha\beta} \equiv (\sqrt{-\gamma}/2\kappa)(\Theta^{\alpha\beta} - \Theta\gamma^{\alpha\beta})$. If we consider variations of the metric that leave the boundary metrics $h_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ fixed, then all of the boundary terms are 0, and $\delta I = 0$ if and only if Einstein's equations hold over M. Thus I is the action that generates general relativity if we are considering variations of the metric over a bounded region of space such as M.

Now, $\gamma_{\alpha\beta}$ is fully defined if we specify $\tilde{N}$, $\tilde{V}^\alpha$, and $\sigma_{\alpha\beta}$. Thus variation of $\gamma_{\alpha\beta}$ is equivalent to a variation of these quantities, and we may rewrite the B term in the above with respect to such variations. During this calculation we repeatedly make use of the fact that $\delta u_\alpha \| u_\alpha, \delta n_\alpha \| n_\alpha$. This is true because these one-forms are defined by the requirement that $u_\alpha v^\alpha = n_\alpha w^\alpha = 0$ for all vector fields $v^\alpha \in T\Sigma_t, w^\alpha \in TB$. The metric does not figure in the definition of $T\Sigma_t$ or TB; nor is a metric required to calculate the action of a one-form on a vector, and so u_α and n_α are defined up to a normalizing factor independently of the metric. Thus, $\sigma^{\alpha\beta}\delta u_\beta = \sigma^{\alpha\beta}\delta\tilde{u}_\beta = \sigma^{\alpha\beta}\delta n_\beta = \sigma^{\alpha\beta}\delta\tilde{n}_\beta = 0$.

Expressing u^α and $\tilde{n}^\alpha$ in terms of $\tilde{u}^\alpha$ and n^α and writing $\gamma_{\alpha\beta} = \sigma_{\alpha\beta} - \tilde{u}_\alpha\tilde{u}_\beta$, we have

$$(\Theta^{\alpha\beta} - \Theta\gamma^{\alpha\beta})\delta\gamma_{\alpha\beta} = (\Theta^{\alpha\beta} - \Theta\sigma^{\alpha\beta})\delta\sigma_{\alpha\beta} + 2\Theta_{\alpha\beta}\tilde{u}^\alpha\delta\tilde{u}^\beta - 2\Theta\tilde{u}_\alpha\delta\tilde{u}^\alpha. \tag{8}$$

In the meantime, we may decompose $\Theta_{\alpha\beta}$ into its parts that are perpendicular and parallel to the Ω_t to obtain

$$\Theta_{\alpha\beta} = \tilde{k}_{\alpha\beta} + 2\tilde{u}_{(\alpha}\sigma^\gamma_{\beta)}\tilde{u}^\delta\nabla_\gamma n_\delta + \tilde{u}_\alpha\tilde{u}_\beta(n^\gamma\tilde{a}_\gamma), \tag{9}$$

where $\tilde{k}_{\alpha\beta} \equiv -\sigma_\alpha^\gamma \sigma_\beta^\delta \nabla_\gamma n_\delta$ (the extrinsic curvature of Ω_t in a hypersurface perpendicular to B) and $\tilde{a}_\alpha \equiv \tilde{u}^\beta \nabla_\beta \tilde{u}_\alpha$ is the acceleration of normal vector $\tilde{u}^\alpha$ along its length. Also, on contracting $\Theta^{\alpha\beta}$ with $\gamma_{\alpha\beta}$,

$$\Theta = \tilde{k} - n^\alpha \tilde{a}_\alpha, \tag{10}$$

where $\tilde{k} \equiv \sigma^{\alpha\beta} k_{\alpha\beta}$. Putting these three results (8),(9),(10) together, a few lines of algebra produces

$$(\Theta^{\alpha\beta} - \Theta \gamma^{\alpha\beta}) \delta\gamma_{\alpha\beta} = (\tilde{k}^{\alpha\beta} - [\tilde{k} - n^\gamma \tilde{a}_\gamma] \sigma^{\alpha\beta}) \delta\sigma_{\alpha\beta} + 2\sigma_\beta^\gamma n^\delta \nabla_\gamma \tilde{u}_\delta \delta\tilde{u}^\beta - 2\tilde{k}\tilde{u}_\beta \delta\tilde{u}^\beta. \tag{11}$$

To complete this deconstruction, recall that the time-flow vector field is defined independent of the metric. Therefore $\delta T^\alpha = 0$ which in turn implies that $\delta\tilde{u}^\alpha = (-1/\tilde{N})(\tilde{u}^\alpha \delta\tilde{N} + \delta\tilde{V}^\alpha)$. Applying this to Eq. (11), substituting the result back into the variation of the action (7), and recalling Eq. (5) we obtain the final result

$$\begin{aligned} \delta I = {} & \frac{1}{2\kappa} \int_M d^4x \sqrt{-g} (G_{\alpha\beta} + \Lambda g_{\alpha\beta}) \delta g^{\alpha\beta} \\ & - \int_\Sigma d^3x (P^{\alpha\beta} \delta h_{\alpha\beta}) + \int_\Omega d^2x \left(\frac{1}{\kappa} \sinh^{-1}(\eta) \delta\sqrt{\sigma} \right) \\ & - \int dt \int_{\Omega_t} d^2x \sqrt{\sigma} \left[\tilde{\varepsilon}\, \delta\tilde{N} - \tilde{\jmath}_\beta \delta\tilde{V}^\beta - \frac{\tilde{N}}{2} \tilde{s}^{\alpha\beta} \delta\sigma_{\alpha\beta} \right], \end{aligned} \tag{12}$$

where $\tilde{\varepsilon} \equiv (1/\kappa)\tilde{k}$, $\tilde{\jmath}_\alpha \equiv -(1/\kappa)\sigma_\alpha^\beta n^\delta \nabla_\beta \tilde{u}_\delta$, and $\tilde{s}^{\alpha\beta} \equiv (1/\kappa)(\tilde{k}^{\alpha\beta} - [\tilde{k} - n^\gamma \tilde{a}_\gamma]\sigma^{\alpha\beta})$.

From this result we can make a couple of useful observations. First, examining the initial and final hypersurfaces Σ_1 and Σ_2 and their boundaries Ω_1 and Ω_2 we see that $P^{\alpha\beta}$ is the Σ_t hypersurface momentum conjugate to $h_{\alpha\beta}$ while $(1/2\kappa)\sinh^{-1}(\eta)$ is the Ω_t hypersurface momentum conjugate to $\sqrt{\sigma}$. Second, we see that $-\sqrt{\sigma}\tilde{\varepsilon}$ is conjugate to the boundary lapse $\tilde{N}$, $\sqrt{\sigma}\tilde{\jmath}^\alpha$ is conjugate to the boundary shift $\tilde{V}^\alpha$, and $\frac{1}{2}\tilde{N}\sqrt{\sigma}\tilde{s}^{\alpha\beta}$ is conjugate to the boundary metric $\sigma_{\alpha\beta}$. Following the Hamilton-Jacobi analysis of [3], we identify $\tilde{\varepsilon}$, $\tilde{\jmath}^\alpha$, and $\tilde{s}^{\alpha\beta}$ as surface energy, momentum, and stress densities. If $\eta = 0$, these quantities coincide with those defined in [3]. Also note that each of these terms is explicitly independent of η. They are defined with respect to the foliation of B only.

B. Decomposing the action

We now decompose I with respect to the foliation. To start, the $\mathcal{R} - 2\Lambda$ term of Eq. (6) may be rewritten as

$$\begin{aligned} \int_M d^4x \sqrt{-g} (\mathcal{R} - 2\Lambda) & = \int_M d^4x \sqrt{-g} (R - 2\Lambda - K^2 + K_{\alpha\beta}K^{\alpha\beta}) \\ & \quad - 2\int_\Sigma d^3x \sqrt{h} K - 2\int_B d^3x \sqrt{-\gamma} (K\eta + n_\alpha a^\alpha), \end{aligned} \tag{13}$$

where $a^\alpha \equiv u^\beta \nabla_\beta u^\alpha$ is the acceleration of the foliation's unit normal vector field along its length. Next, if we substitute expressions for n^α and $\tilde{u}^\alpha$ from Eq. (2) into Eq. (10), then it is a simple matter to show that

$$\Theta + \eta K + n^\alpha a_\alpha = \frac{1}{\lambda} k + \lambda \tilde{u}^\alpha \nabla_\alpha \eta. \tag{14}$$

Combining these two results the following expression for I is obtained:

$$\begin{aligned} I = {} & \frac{1}{2\kappa} \int_M d^4x \sqrt{-g} (R - 2\Lambda - K^2 + K_{\alpha\beta}K^{\alpha\beta}) \\ & - \frac{1}{\kappa} \int_B d^3x \sqrt{-\gamma} \left(\frac{k}{\lambda} + \lambda \tilde{u}^\alpha \nabla_\alpha \eta \right) \\ & + \frac{1}{\kappa} \int_\Omega d^2x \sqrt{\sigma} \sinh^{-1}(\eta). \end{aligned} \tag{15}$$

Next, we apply the Einstein constraint equations. These are

$$\mathcal{H} \equiv -\frac{\sqrt{h}}{\kappa} G_{\alpha\beta} u^\alpha u^\beta = -\frac{\sqrt{h}}{\kappa} (R - 2\Lambda + K^2 - K_{\alpha\beta}K^{\alpha\beta}) = 0, \tag{16}$$

and

$$\mathcal{H}_\alpha \equiv \frac{\sqrt{h}}{\kappa} h_\alpha^\beta G_{\beta\gamma} u^\gamma = \frac{\sqrt{h}}{\kappa} (D_\beta K_\alpha^\beta - D_\alpha K) = 0. \tag{17}$$

Combining these constraints with the Lie derivative definition of the extrinsic curvature $K_{\alpha\beta} = -\frac{1}{2}\pounds_u h_{\alpha\beta} = (-1/2N)(\pounds_T h_{\alpha\beta} - 2D_{(\alpha}V_{\beta)})$ of Σ_t in $\mathcal{M}$, we may rewrite the integrand of the remaining bulk term with respect to these constraints, a time derivative of the hypersurface metric, and a total divergence term:

$$\begin{aligned} R - 2\Lambda - K^2 + K_{\alpha\beta}K^{\alpha\beta} & = \frac{2\kappa}{\sqrt{h}} P^{\alpha\beta} \pounds_T h_{\alpha\beta} - \frac{2\kappa}{\sqrt{h}} \mathcal{H} \\ & \quad - \frac{2\kappa}{\sqrt{-g}} V^\alpha \mathcal{H}_\alpha - \frac{4\kappa}{N} D_\alpha \left[\frac{1}{\sqrt{h}} P^{\alpha\beta} V_\beta \right], \end{aligned} \tag{18}$$

where $P^{\alpha\beta}$ is the hypersurface momentum for Σ_t, which we discussed above. Then, using Stokes theorem on the hypersurfaces Σ_t to move the total divergence term out to the boundaries Ω_t and applying Eq. (5) we may write the action as

$$I=\int_M d^4x(P^{\alpha\beta}\pounds_T h_{\alpha\beta}-N\mathcal{H}-V^\alpha H_\alpha)$$
$$-\frac{1}{\kappa}\int dt\int_{\Omega_t}d^2x\sqrt{\sigma}(Nk-V^\alpha[K_{\alpha\beta}-Kh_{\alpha\beta}]\tilde{n}^\beta$$
$$-\tilde{N}\lambda\tilde{u}^\alpha\nabla_\alpha\eta)+\frac{1}{\kappa}\int_\Omega d^2x\sqrt{\sigma}\sinh^{-1}(\eta). \quad (19)$$

Up to this point we have been working with the foliation of M and therefore with the lapse N, shift V^α, and normal vectors u^α and $\tilde{n}^\alpha$. On the term evaluated on B we now switch to work with the foliation of B and therefore the boundary lapse $\tilde{N}$, the boundary shift $\tilde{V}^\alpha$, and normal vectors $\tilde{u}^\alpha$ and n^α. Then

$$Nk=\frac{1}{\lambda^2}\tilde{N}\tilde{k}-\eta N\sigma^{\alpha\beta}\nabla_\alpha\tilde{u}_\beta, \quad (20)$$

$$-\tilde{n}^\alpha(K_{\alpha\beta}-Kh_{\alpha\beta})V^\beta=N\eta\sigma^{\alpha\beta}\nabla_\alpha\tilde{u}_\beta$$
$$-\tilde{N}\eta^2\tilde{k}+n^\alpha\tilde{V}^\beta\nabla_\beta\tilde{u}_\alpha+\lambda\tilde{V}^\beta\nabla_\beta\eta, \quad (21)$$

where $\tilde{k}\equiv\sigma^{\alpha\beta}\tilde{k}_{\alpha\beta}$. Combining these two results with Eq. (4), which can be used to show that

$$\int_\Omega d^2x\sqrt{\sigma}\sinh^{-1}(\eta)-\int dt\int_{\Omega_t}d^2x\sqrt{\sigma}\lambda\tilde{N}\tilde{u}^\alpha\nabla_\alpha\eta$$
$$=\int dt\int_{\Omega_t}d^2x[(\pounds_T\sqrt{\sigma})\sinh^{-1}(\eta)+\sqrt{\sigma}\lambda\tilde{V}^\alpha\nabla_\alpha\eta], \quad (22)$$

we obtain the following decomposition of the action:

$$I=\int_M d^4x(P^{\alpha\beta}\pounds_T h_{\alpha\beta}-N\mathcal{H}-V^\alpha H_\alpha)$$
$$+\frac{1}{\kappa}\int dt\int_{\Omega_t}d^2x(\pounds_T\sqrt{\sigma})\sinh^{-1}(\eta)$$
$$-\int dt\int_{\Omega_t}d^2x\sqrt{\sigma}(\tilde{N}\tilde{\varepsilon}-\tilde{V}^\alpha\tilde{\jmath}_\alpha), \quad (23)$$

where $\tilde{\varepsilon}$ and $\tilde{\jmath}^\alpha$ are the energy and surface momentum densities that we obtained from the variational calculation.

The terms of this expression will be familiar to anyone who is familiar with Refs. [3,9,11]. Specifically, $\tilde{\varepsilon}$ and $\tilde{\jmath}^\alpha$ are exactly the energy surface density and momentum surface density that the observers on the boundary would measure if the foliation of M were perpendicular to B. A little thought shows that these quantities are the ones that would be reasonable for observers restricted to surface B to measure. Such observers are cognizant of the foliation of the boundary (for the foliation has been defined to correspond to their notion of simultaneity), but being restricted to the surface they have no way of associating that foliation with a foliation of M as a whole. Viewed another way, there are no observers in the interior of M and therefore no unique way to extend the "instants" of time into that region. As such, it does not seem to physically make sense for the observers to measure the energy and momentum surface densities with respect to the foliation Σ_t, the lapse N, and the shift V^α that we have defined but they cannot observe. Rather, as observers travelling along B they would naturally (locally) extend the foliation of B into a foliation that is perpendicular to T^α and therefore in effect be considering a foliation $\tilde{\Sigma}_t$ (locally) defined around B with lapse $\tilde{N}$ and shift $\tilde{V}^\alpha$. Then they would measure the quantities that we have found naturally arise from the action.

We may also define a Hamiltonian. In elementary classical mechanics with one degree of freedom, the action I and Hamiltonian H are related by the equation $I=p\dot{q}-H$, where q is the variable giving the configuration of the system and $p=\partial I/\partial q$ is the conjugate momentum. Extending this to the system under consideration [3], $h_{\alpha\beta}$ and $\sqrt{\sigma}$ are configuration variables while $P^{\alpha\beta}$ and $(1/\kappa)\sinh^{-1}(\eta)$ are their conjugate momenta, and so the Hamiltonian is

$$H=\int_{\Sigma_t}d^3x[N\mathcal{H}+V^\alpha H_\alpha]+\int_{\Omega_t}d^2x\sqrt{\sigma}(\tilde{N}\tilde{\varepsilon}-\tilde{V}^\alpha\tilde{\jmath}_\alpha). \quad (24)$$

Again this quantity is indifferent to the intersection angle between the foliation of M and the boundary. For solutions to the Einstein equations, it is defined entirely with respect to the foliation of B. Note that this Hamiltonian does not agree with that proposed in [11] where the problem was approached from the point of view of the foliation of M rather than that of B.

C. Conserved charges

The discussion of conserved charges presented in [3] carries over exactly into this work. Thus if $\xi^\alpha\in TB$ is a vector field in the boundary B and $\pounds_\xi\gamma_{\alpha\beta}=0$ (i.e. it is a Killing vector field), then we may define an associated conserved charge

$$Q_\xi\equiv\int_\Omega d^2x\sqrt{\sigma}\xi^\alpha(\tilde{\varepsilon}\tilde{u}_\alpha+\tilde{\jmath}_\alpha). \quad (25)$$

If T^α is a Killing vector field, then the Hamiltonian H as defined above is a conserved charge. If there is an angular Killing vector field $\phi^\alpha\in T\Omega$, then the angular momentum

$$J_\phi\equiv\int_\Omega d^2x\sqrt{\sigma}\phi^\alpha\tilde{\jmath}_\alpha \quad (26)$$

is also a conserved charge.

D. Background terms

We now return to the reference term $\underline{I}$. Defined as it is as a functional of the boundary metrics, it is clear that for a metric variation that leaves the boundary metrics unchanged, $\delta \underline{I}=0$—therefore its exact form does not affect the equations of motion. This degree of freedom in the definition of I may equivalently be viewed as the freedom to define zero points of the energy, momentum, and Hamiltonian. Specifically, it allows us to choose a reference spacetime for which we wish these quantities to be zero. For asymptotically flat spacetimes we would normally choose Minkowski space as the reference spacetime, but other choices may be made if we are studying spacetimes with other asymptotic behaviors—for example asymptotically anti–de Sitter space [6].

Given a reference spacetime $(\underline{M},\underline{g}_{\alpha\beta})$, we embed $(\Omega,\sigma_{\alpha\beta})$ in that spacetime and define a vector field $\underline{T}^\alpha$ over the embedded $(\underline{\Omega},\underline{\sigma}_{\alpha\beta})$ such that $\underline{T}^\alpha \underline{T}_\alpha = T^\alpha T_\alpha$ and the components of $\pounds_{\underline{T}} \underline{\sigma}_{\alpha\beta} = \pounds_T \sigma_{\alpha\beta}$.[3] These conditions ensure that the boundary lapse and the components of the boundary shift vector as calculated from $\underline{T}^\alpha$ are equal to those calculated for T^α in the original spacetime. We then define

$$\underline{I} = \int dt \int_\Omega d^2x \sqrt{\sigma}[\tilde{N}\underline{\tilde{\varepsilon}} - \tilde{V}^\alpha \underline{\tilde{\jmath}}_\alpha], \tag{27}$$

where $\underline{\tilde{\varepsilon}}$ and $\underline{\tilde{\jmath}}_\alpha$ are defined in the same way as before except that this time they are evaluated for the surface Ω embedded in the reference spacetime. Thus, the net effect of including $\underline{I}$ is to change $\tilde{\varepsilon} \to \tilde{\varepsilon} - \underline{\tilde{\varepsilon}}$ and $\tilde{\jmath}_\alpha \to \tilde{\jmath}_\alpha - \underline{\tilde{\jmath}}_\alpha$.

Physically these conditions correspond to demanding that an observer living in the surface Ω and observing only quantities intrinsic to that surface (as it evolves through time) cannot tell whether she is living in the original spacetime or in the reference spacetime. From another point of view the observers have calibrated their instruments so that they will always measure the quasilocal quantities to be zero in the reference spacetime—no matter what kind of motion they undergo.

This definition of $\underline{I}$ differs slightly from both the one used in [3] and the one used in [11]. In the former case Ω was embedded in a reference three dimensional space and no demand was made of $\pounds_T \sigma_{\alpha\beta}$; however, in all examples considered in Ref. [3] (and indeed in the subsequent work) $\pounds_T \sigma_{\alpha\beta}=0$, and so insofar as that formalism has been pursued within the literature, it agrees with the formalism considered here. Note that if we do not include the conditions on $\pounds_T \sigma_{\alpha\beta}$, then boosted observers in Minkowski space will observe nonzero quasilocal energies which is clearly an undesirable situation.

In Ref. [11] $(B,\gamma_{\alpha\beta})$ as a whole is embedded in the reference four-space $(\underline{M},\underline{g}_{\alpha\beta})$. That requirement is essentially the global version of our definition of $\underline{I}$ and as such will locally yield the same results as our definition, though it is somewhat harder to apply computationally. Beyond that condition they further require that the reference spacetime be foliated in such a way that η in the reference spacetime is the same as in the original spacetime. Such a condition is neither necessary nor desirable in our approach which does not concern itself with the foliation of the spacetime as a whole. Finally we note that in the approach used in Ref. [11] the inclusion of this background term is necessary to remove an η dependence in the Hamiltonian—this dependence does not occur in our approach.

[3] We leave aside the issue as to whether this is possible in all cases. We will consider several examples where it is, but in general an arbitrary surface cannot be embedded in an arbitrary higher dimensional space.

IV. EXAMPLES

We now consider some sample calculations. For simplicity we will work with static spherically symmetric spacetimes parametrized with the natural spherical coordinates $\{t,r,\theta,\phi\}$ and therefore with metric

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{28}$$

In each case we will consider a surface of observers Ω defined by $r=r_0$ and $t=t_0$. Then geometrically Ω it is a two-sphere with metric

$$ds^2 = r_0^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{29}$$

(where we have parametrized Ω with the same θ and ϕ as the full space). If we then consider the timelike unit vector field

$$T^\alpha = \left[\frac{\tilde{N}}{\sqrt{F}}\sqrt{1+\frac{R^2}{\tilde{N}^2 F}}, R, \Theta, \Phi\right], \tag{30}$$

where $R=R(r,\theta,\phi)$, $\Theta=\Theta(r,\theta,\phi)$, and $\Phi=\Phi(r,\theta,\phi)$ are general functions of r, θ, and ϕ, the boundary lapse and boundary shift functions are easily found to be

$$\tilde{N}^2 = 1 + r^2(\Theta^2 + \sin^2\theta\Phi^2), \tag{31}$$

and

$$\tilde{V}^\alpha = [0,0,\Theta,\Phi], \tag{32}$$

while

$$\tilde{u}^\alpha = \left[\frac{1}{\sqrt{F}}\sqrt{1+\frac{R^2}{\tilde{N}^2 F}}, \frac{R}{\tilde{N}}, 0, 0\right], \tag{33}$$

and

$$n_\alpha = \left[-\frac{R}{\tilde{N}}, \frac{1}{\sqrt{F}}\sqrt{1+\frac{R^2}{\tilde{N}^2 F}}, 0, 0\right]. \tag{34}$$

Then, a straightforward calculation yields

$$\tilde{\varepsilon} = -\frac{2}{\kappa r}\sqrt{F+\frac{R^2}{\tilde{N}^2}} \quad (35)$$

and

$$\tilde{\jmath}_\alpha = -\frac{2}{\kappa\sqrt{F+\frac{R^2}{\tilde{N}^2}}}\left[0,0,\frac{\partial}{\partial\theta}\left(\frac{R}{\tilde{N}}\right),\frac{\partial}{\partial\phi}\left(\frac{R}{\tilde{N}}\right)\right]. \quad (36)$$

Finally, we calculate the Ω components of $\pounds_T\sigma_{\alpha\beta}$ as

$$(\pounds_T\sigma_{\alpha\beta})_{\theta\theta} = 2r\left(R + r\frac{\partial\Theta}{\partial\theta}\right), \quad (37)$$

$$(\pounds_T\sigma_{\alpha\beta})_{\theta\phi} = r^2\left(\frac{\partial\Theta}{\partial\phi} + \sin^2\theta\frac{\partial\Phi}{\partial\theta}\right), \quad (38)$$

$$(\pounds_T\sigma_{\alpha\beta})_{\phi\phi} = 2r\sin\theta\left(R\sin\theta + \Theta r\cos\theta + r\sin\theta\frac{\partial\Phi}{\partial\phi}\right). \quad (39)$$

Note that there is no F dependence in any of these components. Thus if we wish to calculate quasilocal quantities for observers moving through Schwarzschild space using Minkowksi space as a reference spacetime (as in the following examples), on embedding Ω (which is trivial for a sphere) and setting

$$\underline{T}^\alpha = \left[\tilde{N}\sqrt{1+\frac{R^2}{\tilde{N}^2}}, R, \Theta, \Phi\right], \quad (40)$$

the metric $\sigma_{\alpha\beta}$ and its derivative $\pounds_T\sigma_{\alpha\beta}$ will be the same for both Schwarzschild and Minkowski space.

We now specialize to specific examples using the system of units where $\kappa = 8\pi$.

A. Static observers

For our first example, we will consider a spherical set observers holding themselves static with respect to a Schwarzschild black hole ($F = 1 - 2m/r$) and take flat Minkowski space ($F=1$) as our reference spacetime. Then for both spacetimes $R=\Theta=\Phi=0$, $\tilde{N}=1$, and $\tilde{V}^\alpha=0$. Substituting these data into the above expressions we obtain

$$\tilde{\varepsilon} - \underline{\tilde{\varepsilon}} = \frac{1}{4\pi r}\left(1 - \sqrt{1-\frac{2m}{r}}\right). \quad (41)$$

Then, the total measured energy is

$$E = \int_\Omega d^2x\sqrt{\sigma}(\tilde{\varepsilon} - \underline{\tilde{\varepsilon}}) = r\left(1 - \sqrt{1-\frac{2m}{r}}\right). \quad (42)$$

This is the standard result as obtained in [3]. Taking the limit as $r\to\infty$ we obtain $E\to m$ as would be expected, while as $r\to 2m$ (the Schwarzschild horizon), $E\to 2m$.

With $\tilde{N}=1$ and the shift vector 0, the Hamiltonian $H = E$. Since T^α is a Killing vector in this case, H is a conserved charge. There is also a conserved angular momentum associated with each of the regular three spherical Killing vectors. $\tilde{\jmath}_\alpha$ is zero, however, and so each of these charges vanishes.

B. Radially infalling observers

A more interesting example is the case of observers taking their measurements as they fall radially along geodesics towards a Schwarzschild hole. Such motion is described by solutions to the geodesic equation. For observers who were stationary as they started falling in from infinity, the geodesic equation reduces to $dr/d\tau = -\sqrt{2m/r}$, where τ is the proper time coordinate. Then, $R = -\sqrt{2m/r}$, $\Theta=\Phi=0$, $\tilde{N}=1$, again $\tilde{V}^\alpha=0$, and

$$\tilde{\varepsilon} - \underline{\tilde{\varepsilon}} = \frac{1}{4\pi r}\left(\sqrt{1+\frac{2m}{r}} - 1\right). \quad (43)$$

The total measured energy is

$$E = \int_\Omega d^2x\sqrt{\sigma}(\tilde{\varepsilon} - \underline{\tilde{\varepsilon}}) = r\left(\sqrt{1+\frac{2m}{r}} - 1\right). \quad (44)$$

As for static observers, as $r\to\infty, E\to m$. Of course, this is not really surprising since the radially infalling observers at infinity actually are static. Over the rest of the range the two energy measures are not the same. In particular, as $r\to 2m$, $E\to 2m(\sqrt{2}-1)$.

As for the first example, the momentum terms are zero and the $\tilde{N}=1$, and so $H=E$. Here T^α is no longer a Killing vector, however, and so this is no longer a conserved charge. Physically of course this is to be expected since the observers are moving radially inwards and therefore through the gravitational field. As time passes therefore the amount of gravitational field energy contained within Ω_t changes. Again the three angular momenta are conserved but each has the uninteresting value of zero.

C. Radially boosted observers

We next consider a set of observers who are boosted to travel radially with ‘‘constant’’ velocity v. By constant velocity here we mean that a second set of observers dwelling on a $t=$const surface and being evolved by the timelike vector field [1,0,0,0] will measure the first set as having velocity v and acceleration 0.

Then $R = \gamma v\sqrt{1-2m/r}$ [$\gamma = 1/(\sqrt{1-v^2})$ the standard Lorentz factor from special relativity], $\Theta=\Phi=0$, $\tilde{N}=1$, and once more $\tilde{V}^\alpha=0$. A simple calculation then obtains,

$$\tilde{\varepsilon} - \underline{\tilde{\varepsilon}} = \frac{\gamma}{4\pi r}\left(\sqrt{1-\frac{2mv^2}{r}} - \sqrt{1-\frac{2m}{r}}\right), \quad (45)$$

and the total measured energy is

$$E=\int_\Omega d^2x\sqrt{\sigma}(\tilde{\varepsilon}-\underline{\tilde{\varepsilon}})=\gamma r\left(\sqrt{1-\frac{2mv^2}{r}}-\sqrt{1-\frac{2m}{r}}\right). \tag{46}$$

In this case as $r\to\infty$, $E\to\sqrt{1-v^2}m$, while as $r\to 2m$, $E\to 2m$.

Once more with $\tilde{N}=1$ and $\tilde{V}^\alpha=0,H=E$. Again T^α is not a Killing vector, and so this is not a conserved charge. The angular momenta are conserved charges though again each is zero.

At first glance it may seem unusual that in the $r\to\infty$ limit $E\propto m/\gamma$. Extrapolating from special relativity we would perhaps expect $E\propto\gamma m$. Physically, however, it is clear that there is a flow of gravitational field energy through the surface Ω. That is, there is a $j_\vdash$ component of the momentum. This momentum may be seen as ''drawing off'' some of the energy. We will not investigate the issue further in this paper, though it is addressed in the last example of [11] to a certain extent by the invariant quantities defined in [12].

D. z-boosted observers

Finally we consider a set of observers who are boosted to travel ''in the z-direction'' with ''constant'' velocity v. By constant we again mean with respect to other observers who are dwelling on $t=$const surfaces and being evolved by the timelike vector field [1,0,0,0]. Then, $R=\gamma v\cos\theta\sqrt{1-2m/r}$, $\Theta=\gamma v\sin\theta/r$, $\Phi=0$, $\tilde{N}=\sqrt{1+\gamma^2v^2\sin^2\theta}$, and $V^\alpha=[0,0,\gamma v\sin\theta/r,0]$. We now have

$$\tilde{\varepsilon}-\underline{\tilde{\varepsilon}}=\frac{1}{4\pi r}\frac{1}{\sqrt{1-v^2\cos^2\theta}}\times\left(\sqrt{1-\frac{2mv^2\cos^2\theta}{r}}-\sqrt{1-\frac{2m}{r}}\right), \tag{47}$$

and while $\tilde{\jmath}_\phi=\underline{\tilde{\jmath}}_\phi=0$,

$$\tilde{\jmath}_\theta-\underline{\tilde{\jmath}}_\theta=\frac{v\sin\theta}{8\pi(1-v^2\cos^2\theta)}\times\left(\sqrt{\frac{1-\frac{2m}{r}}{1-\frac{2mv^2\cos^2\theta}{r}}}-1\right). \tag{48}$$

In this case, the $\tilde{\varepsilon}-\underline{\tilde{\varepsilon}}$ does not integrate into a nice tidy form as it did in previous examples. Instead we will consider the two usual limiting cases. For $r\to\infty$,

$$(\tilde{\varepsilon}-\underline{\tilde{\varepsilon}})_{r\to\infty}=\frac{m}{4\pi r^2}\sqrt{1-v^2\cos^2\theta},$$

$$(\tilde{\jmath}_\alpha-\underline{\tilde{\jmath}}_\alpha)_{r\to\infty}=\frac{mv\sin\theta}{4\pi}. \tag{49}$$

Then, integrating over the $\{t=0,r\to\infty\}$ two-surface we obtain

$$E_\infty=\int_\Omega d^2x\sqrt{\sigma}(\tilde{\varepsilon}-\underline{\tilde{\varepsilon}})=\frac{m}{2}\left(\sqrt{1-v^2}+\frac{\arcsin v}{v}\right), \tag{50}$$

and

$$H_\infty=\int_\Omega d^2x\sqrt{\sigma}[\tilde{N}(\tilde{\varepsilon}-\underline{\tilde{\varepsilon}})-\tilde{V}^\alpha(\tilde{\jmath}_\alpha-\underline{\tilde{\jmath}}_\alpha)]=\sqrt{1-v^2}m. \tag{51}$$

Meanwhile, for $r\to 2m$,

$$(\tilde{\varepsilon}-\underline{\tilde{\varepsilon}})_{r\to 2m}=\frac{1}{4\pi(2m)},$$

$$(\tilde{\jmath}_\alpha-\underline{\tilde{\jmath}}_\alpha)_{r\to 2m}=\frac{v\sin\theta}{4\pi(1-v^2\cos^2\theta)}. \tag{52}$$

Integrating over the $\{t=0,r=2m\}$ two surface we obtain

$$E_{2m}=\int_\Omega d^2x\sqrt{\sigma}(\tilde{\varepsilon}-\underline{\tilde{\varepsilon}})=2m, \tag{53}$$

and

$$H_{2m}=\int_\Omega d^2x\sqrt{\sigma}[\tilde{N}(\tilde{\varepsilon}-\underline{\tilde{\varepsilon}})-\tilde{V}^\alpha(\tilde{\jmath}_\alpha-\underline{\tilde{\jmath}}_\alpha)]=m\left(1-2\gamma+\frac{\gamma}{v}\arcsin v+\frac{2}{\gamma v}\operatorname{arctanh} v\right). \tag{54}$$

As in the previous cases T^α is not a Killing vector and so H is not a conserved charge, as we would physically expect. Note that in this situation we have a nonzero component of $\tilde{\jmath}_\alpha$. Despite this we still do not have any nonzero conserved angular momenta. To see this recall that the three linearly independent spherical Killing vectors are $\phi_1^\alpha=[0,0,0,1]$, $\phi_2^\alpha=[0,0,\sin\phi,\cos\phi\cot\theta]$, and $\phi_3^\alpha=[0,0,\cos\phi,-\sin\phi\cot\theta]$. Then $\phi_1^\alpha\tilde{\jmath}_\alpha=0$, $\phi_2^\alpha j_\alpha$ is proportional to $\sin\phi$, and ϕ_3^α is proportional to $\cos\phi$. Therefore, as would be expected physically and by symmetry, on integration $J_{\phi_1}=J_{\phi_2}=J_{\phi_3}=0$.

The comments made in the previous example regarding the fact that $H_\infty=m/\gamma$ apply here as well.

V. DISCUSSION

In this paper we have seen that the orthogonality restriction of [3] may be lifted without too much difficulty. Further, by concentrating on the foliation of the boundary B rather than the spacetime region M we avoid many of the technical complications of the nonorthogonal treatment of [11], and obtain definitions of quasilocal quantities that are manifestly independent of the intersection angle between the foliation of M and the boundary B independent of our choice of the background spacetime.

In our choice of how to calculate the reference term $\underline{I}$ on the background spacetime we have given local conditions that modify those of [3] in a way that is more appropriate if we are considering moving observers. These conditions at the same time remain simpler and easier to implement than those required in [11].

In general the Hamiltonian and quasilocal quantities such as angular momenta are dependent on the motion of the observers as we have seen in several examples. One somewhat counterintuitive observation is that the observed mass of a source decreases rather than increases with the motion of the observers who are measuring that mass. This is a consequence of choosing our observers in such a way that there is a net flow of gravitational field energy through the surface Ω.

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