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Gauge and parametrization dependence in higher derivative quantum gravity

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The structure of counterterms in higher derivative quantum gravity is reexamined. Nontrivial dependence of charges on the gauge and parametrization is established. Explicit calculations of two-loop contributions are carried out with the help of the generalized renormalization group method, demonstrating consistency of the results obtained. [S0556-2821(99)04904-8]

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I. INTRODUCTION

As is well known, not all of the problems of quantum field theory are exhausted by the construction of the *S* matrix. The investigation of the evolution of the Universe, the behavior of quarks in quantum chromodynamics, etc., requires the introduction of a more general object—the so-called effective action. In addition to that, the program of the renormalization of the *S* matrix itself has not yet been carried out in terms of the *S* matrix alone. Renormalization of the Green functions, therefore, is the central point of the whole theory. Given these functions one can obtain the *S*-matrix elements with the help of the reduction formulas. In this respect those properties of the generating functionals which remain valid after the transition to the *S* matrix is made are of special importance.

We mean first of all the properties of the so-called ''essential'' coupling constants in the sense of Weinberg [1]. They are defined as those independent of any redefinition of the fields. In the context of quantum theory one can say that the renormalization of ''essential'' charges is independent of renormalizations of the fields. Separation of quantities into ''essential'' and ''inessential'' ones is convenient and we use it below.

In this paper we shall consider the problem of the gauge and parametrization dependence of the effective action of R^2 gravity.

There are two general and powerful methods of investigation of the gauge dependence in quantum field theory. The first of them [2] uses the Batalin-Vilkovisky formalism [3–5] and is based on the fact that any change of gauge condition can be presented as a (local) canonical (in the sense of ''antibrackets'' [4]) transformation of the effective action. This canonical transformation induces a corresponding renormalized canonical transformation of the renormalized effective action. This leads to the following result: the renormalization boils down to the redefinition of the coupling constants (which are the coefficients of independent gauge invariant structures entering the Lagrangian) and some canonical transformations of the fields and sources of Becchi-Rouet-Stora-Tyutin (BRST) transformations. The second approach [6–8] consists of the introduction of some additional anticommuting source to the effective action in such a way that the Slavnov identities for the corresponding generating functional of proper vertices connect its derivatives with respect to the gauge-fixing parameter and to the mean fields (and sources of BRST transformations).

The second method was used in [8] to prove the gauge independence of the gauge-invariant divergent parts of the effective action up to terms proportional to the classical equations of motion of the gravitational field. Together with the general result of the first approach mentioned above this would imply some far-reaching consequences concerning the renormalization of the fields. For example, one could conclude that the canonical transformation corresponding to a change of the gauge condition should not be renormalized. Unfortunately, this is not the case. We will show in this paper that the aforesaid result of [8] holds at the one-loop level only. Introduction of the additional source mentioned above requires also introducing some additional terms needed to compensate divergences which arise because of the presence of the new source. As a result the corresponding Slavnov identities impose only some constraints on the divergent structures from which a nontrivial dependence[1] on the gauge follows already at the two-loop level.

Our paper is organized as follows. In Sec. II we determine possible divergent structures which are originated due to the presence of the new source and obtain the correct Slavnov identities. In Sec. III we calculate explicitly the divergent part of the effective action at the one-loop level in arbitrary (linear) gauge and the special class of parametrizations. In Sec. IV we calculate the divergence as the $1/\varepsilon^2$ (ε being the dimensional regulator[2]) part at the two-loop level with the help of the generalized renormalization group method and show that the results obtained in Secs. III and IV satisfy the relations derived in Sec. II.

We use the highly condensed notation of DeWitt throughout this paper. Also left derivatives with respect to anticommuting variables are used. The dimensional regularization of all divergent quantities is supposed.

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[1] I.e., a dependence which cannot be presented as proportional to the equations of motion.

[2] We set $2\varepsilon = d-4$, d being the dimensionality of the space-time.

II. GENERATING FUNCTIONALS AND SLAVNOV IDENTITIES

A. Action, gauge fixing, and parametrization

Let us consider higher derivative quantum gravity described by an action which includes the minimal set of terms added to the usual ones of Einstein to ensure the power-counting renormalizability of the theory:[3]

$$S_0=\int d^4x\sqrt{-g}\left(\alpha_1 R^2+\beta R_{\mu\nu}R^{\mu\nu}-\frac{1}{k^2}(R-2\Lambda)\right), \tag{1}$$

where α_1 and β are arbitrary constants satisfying only $\beta\neq 0$, $3\alpha_1+\beta\neq 0$, which imply that the graviton propagator behaves like p^{-4} for large momenta (see [8]); k is the gravitational constant and Λ is the cosmological term.

The corresponding equations of motion are

$$\frac{1}{2}\alpha_1 R^2 g^{\alpha\beta}+\frac{1}{2}\beta R^{\mu\nu}R_{\mu\nu}g^{\alpha\beta}-2\alpha_1 RR^{\alpha\beta} -2\beta R_{\mu\nu}R^{\mu\alpha\nu\beta}-\left(2\alpha_1+\frac{1}{2}\beta\right)\Box R g^{\alpha\beta}-\beta\Box R^{\alpha\beta} +(2\alpha_1+\beta)R^{;\alpha\beta}-\frac{1}{2k^2}Rg^{\alpha\beta}+\frac{1}{k^2}\Lambda g^{\alpha\beta}+\frac{1}{k^2}R^{\alpha\beta}=0. \tag{2}$$

Renormalizability of this theory was proved in [8] in the case of the so-called unweighted (or weighted with a functional containing fourth or higher derivatives) harmonic gauge condition. The proof in the more general case boils down to the proof of the so-called locality hypotheses. In [10] its validity was shown most generally.

For our purposes it is sufficient to consider the harmonic gauge[4] following Stelle [8]:

$$F_\mu\equiv F_\mu^{\alpha\beta}h_{\alpha\beta}\equiv\partial^\nu h_{\mu\nu}=0, \tag{3}$$

where $h_{\mu\nu}$ denotes some set of dynamical variables describing the gravitational field. We recall that in the theory of gravity a natural ambiguity in the choice of such a set exists because the generators $D^\alpha_{\mu\nu}$ of gauge transformations of variables constructed from the metric $g_{\mu\nu}$ (or $g^{\mu\nu}$) and its determinant $g=\det g_{\mu\nu}$ in any combination have a simple form linear in fields and their derivatives. For general constructions of this section it does not matter what choice we make. We only note that the gauge in the form of Eq. (3) will always correspond to the set of dynamical variables[5] which enter the so-called reduced expression of the metric expansion [see Sec. III A, Eq.(40)]. Thus the BRST transformations of the Faddeev-Popov effective action with the gauge-fixing term being $(-1/2\Delta)F^\alpha\Box F_\alpha$, expressed in terms of these standard variables, are

$$\delta h_{\mu\nu}=D^\alpha_{\mu\nu}C_\alpha\lambda,$$
$$\delta C_\alpha=-\partial^\beta C_\alpha C_\beta\lambda,$$
$$\delta\bar{C}^\tau=-\Delta^{-1}\Box F^\tau\lambda, \tag{4}$$

where λ is an anticommuting constant parameter.

B. Green functions

We write the generating functional of Green functions in the extended form of Zinn-Justin [9] modified by Kluberg-Stern and Zuber [6,7]:

$$Z[T^{\mu\nu},\bar{\beta}^\sigma,\beta_\tau,K^{\mu\nu},L^\sigma] =\int dh_{\mu\nu}dC_\sigma d\bar{C}^\tau\exp\{i[\tilde{\Sigma}(h_{\mu\nu},C_\sigma,\bar{C}^\tau,K^{\mu\nu},L^\sigma) +YF_\sigma\bar{C}^\sigma+\bar{\beta}^\sigma C_\sigma+\bar{C}^\tau\beta_\tau+T^{\mu\nu}h_{\mu\nu}]\}, \tag{5}$$

where

$$\tilde{\Sigma}=S_0-\frac{1}{2\Delta}F^\alpha\Box F_\alpha+\bar{C}^\tau F_\tau^{\mu\nu}D^\alpha_{\mu\nu}C_\alpha +K^{\mu\nu}D^\alpha_{\mu\nu}C_\alpha+L^\sigma\partial^\beta C_\sigma C_\beta;$$

$K^{\mu\nu}(x)$ (anticommuting), $L^\sigma(x)$ (commuting) are the BRST transformation sources and Y is a constant anticommuting parameter.

Let us first consider the structure of divergences which correspond to the extra source Y. Power counting gives, for the degree of divergence D of an arbitrary diagram,

$$D=4-2n_2-2n_K-n_L-2n_Y-E_C-2E_{\bar{C}}, \tag{6}$$

where n_2 = number of graviton vertices with two derivatives, $n_{K,L,Y}$ = numbers of K,L,Y-source lines, respectively; E_C and $E_{\bar{C}}$ = numbers of external ghost and antighost lines.

Also from the expression (5) we see that one can ascribe the following ghost numbers N_g to all the fields and sources:

$$N_g[h]=0,\quad N_g[C]=+1,\quad N_g[\bar{C}]=-1,$$
$$N_g[K]=-1,\quad N_g[L]=-2,\quad N_g[Y]=+1. \tag{7}$$

Now from Eqs. (6) and (7) one can see that there are three types of divergent structures involving the Y vertex: YK, $Y\bar{C}$, and YLC, each of which may have arbitrary number of external graviton lines. As far as we have adopted the standard covariant approach thus only Lorentz-covariant quantities may appear and therefore we have, for the general form of the above structures,

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[3] Our notation is $R_{\mu\nu}\equiv R^\alpha_{\mu\alpha\nu}=\partial_\alpha\Gamma^\alpha_{\mu\nu}-\cdots$, $R\equiv R_{\mu\nu}g^{\mu\nu}$, $g_{\mu\nu}=\mathrm{sgn}(+,-,-,-)$.

[4] We use the flat-space metric tensor

$$\eta^{\mu\nu}=\mathrm{diag}\,(+1,-1,-1,-1)$$

to raise Lorentz indices.

[5] They will be referred to as *standard variables*.

$$YK^{\mu\nu}P_{\mu\nu}, \tag{8}$$

$$Y\bar{C}^{\nu}\partial^{\mu}Q_{\mu\nu}, \tag{9}$$

and

$$YL^{\sigma}C^{\tau}M_{\sigma\tau}, \tag{10}$$

where P, Q, and M are some Lorentz-covariant tensors depending on $h_{\mu\nu}$ alone.

Thus to renormalize the Green functions we must introduce corresponding counterterms and consider the new generating functional[6]

$$\begin{aligned}Z[T^{\mu\nu},\bar{\beta}^{\sigma},\beta_{\tau},K^{\mu\nu},L^{\sigma}] &= \int dh_{\mu\nu}dC_{\sigma}d\bar{C}^{\tau}\exp\{i[\tilde{\Sigma}(h_{\mu\nu},C_{\sigma},\bar{C}^{\tau},K^{\mu\nu},L^{\sigma})\\ &+YK^{\mu\nu}P_{\mu\nu}+Y\bar{C}^{\nu}\partial^{\mu}Q_{\mu\nu}\\ &+YL^{\sigma}C^{\tau}M_{\sigma\tau}+\bar{\beta}^{\sigma}C_{\sigma}+\bar{C}^{\tau}\beta_{\tau}+T^{\mu\nu}h_{\mu\nu}]\}\end{aligned} \tag{11}$$

instead of Eq. (5).

C. Slavnov identities

Let us proceed to successive renormalization of Green functions corresponding to Eq. (11). We will first consider the case when $g^{*}_{\mu\nu}=g_{\mu\nu}$ are chosen as a parametrization of the gravitational field. Then the general result will be clear.

To ensure renormalizability we work with a generating functional (11) from the very beginning. We will see below that Slavnov identities determine the structure of the polynomials P and Q completely. They turn out to be

$$P_{\mu\nu}=a(\eta_{\mu\nu}+h_{\mu\nu}), \tag{12}$$

$$Q_{\mu\nu}=ah_{\mu\nu}, \tag{13}$$

a being some divergent constant. Thus we set

$$P_{\mu\nu}=(\eta_{\mu\nu}+h_{\mu\nu}),Q_{\mu\nu}=h_{\mu\nu}$$

at the zero order. Then inclusion of the counterterms (12),(13) is just a multiplicative renormalization of the source Y.

1. One-loop order

To obtain Slavnov identities at this order we perform a BRST shift (4) of integration variables in Eq. (11):

$$\begin{aligned}\int dh_{\mu\nu}dC_{\sigma}d\bar{C}^{\tau}&\Bigg[\left(T^{\mu\nu}+YL^{\sigma}C^{\tau}\frac{\delta M^{(0)}_{\sigma\tau}}{\delta h_{\mu\nu}}+YK^{\mu\nu}\right)\left(\frac{\delta}{\delta K^{\mu\nu}}+iY(\eta_{\mu\nu}+h_{\mu\nu})\right)\\ &-(\bar{\beta}_{\sigma}+YL^{\tau}M^{(0)}_{\sigma\tau})\left(\frac{\delta}{\delta L^{\sigma}}+iYC^{\tau}M^{(0)}_{\sigma\tau}\right)+\frac{1}{\Delta}\beta_{\tau}\Box F^{\tau,\mu\nu}\frac{\delta}{\delta T^{\mu\nu}}-2Y\Delta\frac{d}{d\Delta}+iY\bar{C}^{\sigma}F^{\mu\nu}_{\sigma}D^{\alpha}_{\mu\nu}C_{\alpha}\Bigg]\\ &\times\exp\{i[\tilde{\Sigma}+YF_{\sigma}\bar{C}^{\sigma}+YK^{\mu\nu}(\eta_{\mu\nu}+h_{\mu\nu})+YL^{\sigma}C^{\tau}M^{(0)}_{\sigma\tau}+\bar{\beta}^{\sigma}C_{\sigma}+\bar{C}^{\tau}\beta_{\tau}+T^{\mu\nu}h_{\mu\nu}]\}=0.\end{aligned} \tag{14}$$

Our aim is to find the Δ dependence of the gauge-invariant terms only. Terms containing $M^{(0)}_{\sigma\tau}$ in Eq. (14) depending on anticommuting fields C_{σ} and source L^{σ} are unimportant in this respect and we replace them simply by "$+\cdots$" in what follows because these terms will be omitted in the end of the calculation anyway.

Using the ghost equation of motion

$$\begin{aligned}\int dh_{\mu\nu}dC_{\sigma}d\bar{C}^{\tau}&(F^{\mu\nu}_{\tau}D^{\alpha}_{\mu\nu}C_{\alpha}-YF_{\tau}+\beta_{\tau})\\ &\times\exp\{i[\tilde{\Sigma}+YF_{\sigma}\bar{C}^{\sigma}+YK^{\mu\nu}(\eta_{\mu\nu}+h_{\mu\nu})+\cdots\\ &+\bar{\beta}^{\sigma}C_{\sigma}+\bar{C}^{\tau}\beta_{\tau}+T^{\mu\nu}h_{\mu\nu}]\}=0,\end{aligned} \tag{15}$$

introducing the generating functional of proper vertices $\tilde{\Gamma}$,

$$\begin{aligned}\tilde{\Gamma}[h_{\mu\nu},C_{\sigma},\bar{C}^{\tau},K^{\mu\nu},L^{\sigma},Y]&=W[T^{\mu\nu},\bar{\beta}^{\sigma},\beta_{\tau},K^{\mu\nu},L^{\sigma},Y],\\ &-\bar{\beta}^{\sigma}C_{\sigma}-\bar{C}^{\tau}\beta_{\tau}-T^{\mu\nu}h_{\mu\nu},\quad W\equiv -i\ln Z,\end{aligned} \tag{16}$$

$$h_{\mu\nu}=\frac{\delta W}{\delta T^{\mu\nu}},\quad C_{\sigma}=\frac{\delta W}{\delta\bar{\beta}^{\sigma}},\quad \bar{C}^{\tau}=-\frac{\delta W}{\delta\beta_{\tau}}, \tag{17}$$

and noting that

$$\frac{d\tilde{\Gamma}}{d\Delta}=\frac{dW}{d\Delta}, \tag{18}$$

we rewrite Eq. (14) as the Slavnov identity for $\tilde{\Gamma}$:

[6]It is easy to see that inclusion of additional structures (8)–(10) into the action does not alter the expression (6) for D.

$$\frac{\delta\tilde{\Gamma}}{\delta h_{\mu\nu}}\left[\frac{\delta\tilde{\Gamma}}{\delta K^{\mu\nu}}+Y(\eta_{\mu\nu}+h_{\mu\nu})\right]+\frac{\delta\tilde{\Gamma}}{\delta C_\sigma}\frac{\delta\tilde{\Gamma}}{\delta L^\sigma}+\frac{1}{\Delta}\Box F_\tau\frac{\delta\tilde{\Gamma}}{\delta\overline{C}_\tau}+2Y\Delta\frac{d\tilde{\Gamma}}{d\Delta}+Y\frac{\delta\tilde{\Gamma}}{\delta\overline{C}_\tau}\overline{C}_\tau+\cdots=0. \tag{19}$$

To simplify Eq. (19) we introduce the reduced generating functional

$$\Gamma=\tilde{\Gamma}+\frac{1}{2\Delta}F_\alpha\Box F^\alpha-YK^{\mu\nu}(\eta_{\mu\nu}+h_{\mu\nu})-YF_\sigma\overline{C}^\sigma.$$

Then Eq. (19) reduces to

$$\frac{\delta\Gamma}{\delta h_{\mu\nu}}\frac{\delta\Gamma}{\delta K^{\mu\nu}}+\frac{\delta\Gamma}{\delta C_\sigma}\frac{\delta\Gamma}{\delta L^\sigma}+2Y\Delta\frac{d\Gamma}{d\Delta}+YK^{\mu\nu}\frac{\delta\Gamma}{\delta K^{\mu\nu}}+\cdots=0. \tag{20}$$

The ghost equation of motion written in terms of Γ is

$$F^{\mu\nu}_\tau\frac{\delta\Gamma}{\delta K^{\mu\nu}}-\frac{\delta\Gamma}{\delta\overline{C}^\tau}=0. \tag{21}$$

Now let us separate the Y-independent part of Γ from the part linear in Y:

$$\Gamma=\Gamma_1+Y\Gamma_2. \tag{22}$$

Then Eq. (20) gives an ordinary Slavnov identity for Γ_1,

$$\frac{\delta\Gamma_1}{\delta h_{\mu\nu}}\frac{\delta\Gamma_1}{\delta K^{\mu\nu}}+\frac{\delta\Gamma_1}{\delta C_\sigma}\frac{\delta\Gamma_1}{\delta L^\sigma}=0, \tag{23}$$

and an equation involving Γ_2:

$$-\frac{\delta\Gamma_1}{\delta h_{\mu\nu}}\frac{\delta\Gamma_2}{\delta K^{\mu\nu}}+\frac{\delta\Gamma_2}{\delta h_{\mu\nu}}\frac{\delta\Gamma_1}{\delta K^{\mu\nu}}-\frac{\delta\Gamma_1}{\delta C_\sigma}\frac{\delta\Gamma_2}{\delta L^\sigma}-\frac{\delta\Gamma_2}{\delta C_\sigma}\frac{\delta\Gamma_1}{\delta L^\sigma}+2\Delta\frac{d\Gamma_1}{d\Delta}+K^{\mu\nu}\frac{\delta\Gamma_1}{\delta K^{\mu\nu}}+\cdots=0. \tag{24}$$

Finally, we omit all but the terms depending on $h_{\mu\nu}$ only and obtain, in the first order,

$$-\frac{\delta S_0}{\delta h_{\mu\nu}}\frac{\delta\Gamma_2^{div(1)}}{\delta K^{\mu\nu}}+2\Delta\frac{d\Omega^{div(1)}}{d\Delta}+\cdots=0, \tag{25}$$

where Ω denotes the gauge invariant part of Γ_1 and the superscript $div(1)$ denotes the one-loop divergent part of the corresponding quantities.

As we know $\Gamma_2^{div(1)}=K^{\mu\nu}P^{(1)}_{\mu\nu}$, $P^{(1)}$ being some divergent polynom in $h_{\mu\nu}$.

Thus, dropping the terms proportional to $K^{\mu\nu}$ again and the symbol "$+\cdots$" we obtain the following equation for the gauge invariant terms $\Omega^{div(1)}$ of the effective action:[7]

$$2\Delta\frac{d\Omega^{div(1)}}{d\Delta}=\frac{\delta S_0}{\delta h_{\mu\nu}}P^{(1)}_{\mu\nu}. \tag{26}$$

The left hand side of this equation is gauge invariant and thus so is the right hand side. Therefore $P^{(1)}_{\mu\nu}$ has the form mentioned above. The corresponding form of $Q^{(1)}_{\mu\nu}$ follows from Eq. (21).

To make the Green functions finite at the one-loop level we must redefine the initial effective action Σ,

$$\Sigma\to\Sigma^{(1)}=\Sigma-\Gamma_1^{div(1)}, \tag{27}$$

and the source Y,[8]

$$Y\to Y(1-a^{(1)}). \tag{28}$$

As explained in [8] subtraction of $\Gamma_1^{div(1)}$ boils down to a redefinition of all the fields in such a way that $\Sigma^{(1)}$ is invariant under renormalized set of BRST transformations for which we do not introduce new notation.

2. Two-loop order

We perform a renormalized BRST transformation of integration variables in the generating functional of Green functions finite at the one-loop level,

$$Z^{[1]}[T^{\mu\nu},\overline{\beta}^\sigma,\beta_\tau,K^{\mu\nu},L^\sigma,Y]=\int dh_{\mu\nu}dC_\sigma d\overline{C}^\tau\exp\{i[\tilde{\Sigma}^{(1)}+Y(1-a^{(1)})F_\sigma\overline{C}^\sigma+(1-a^{(1)})YK^{\mu\nu}(\eta_{\mu\nu}+h_{\mu\nu})+\cdots+\overline{\beta}^\sigma C_\sigma+\overline{C}^\tau\beta_\tau+T^{\mu\nu}h_{\mu\nu}]\}, \tag{29}$$

and obtain the following Slavnov identity:

[7]We will see in Sec. III that the non-gauge-invariant terms in Γ_1^{div} depending on $h_{\mu\nu}$ only are absent.

[8]We should also include counterterms of the type YLC, but they are irrelevant to the issue and replaced by "$+\cdots$" as we have mentioned above.

$$\int dh_{\mu\nu}dC_\sigma d\overline{C}^\tau\bigg[[T^{\mu\nu}+Y(1-a^{(1)})K^{\mu\nu}]\bigg(\frac{\delta}{\delta K^{\mu\nu}}+iY(1-a^{(1)})(\eta_{\mu\nu}+h_{\mu\nu})\bigg)$$
$$-\overline{\beta}^\sigma\frac{\delta}{\delta L^\sigma}+\frac{1}{\Delta}\beta_\tau\Box F^{\tau,\mu\nu}\frac{\delta}{\delta T^{\mu\nu}}-2Y(1-a^{(1)})\Delta\bigg(\frac{d}{d\Delta}+i\frac{d\Gamma_1^{div(1)}}{d\Delta}\bigg)+iY(1-a^{(1)})\overline{C}^\sigma F^{\mu\nu}_\sigma D^\alpha_{\mu\nu}C_\alpha+\cdots\bigg]$$
$$\times\exp\{i[\tilde{\Sigma}^{(1)}+Y(1-a^{(1)})F_\sigma\overline{C}^\sigma+Y(1-a^{(1)})K^{\mu\nu}(\eta_{\mu\nu}+h_{\mu\nu})+\cdots+\overline{\beta}^\sigma C_\sigma+\overline{C}^\tau\beta_\tau+T^{\mu\nu}h_{\mu\nu}]\}=0. \tag{30}$$

To evaluate the term[9]

$$\int dh_{\mu\nu}dC_\sigma d\overline{C}^\tau Y\Delta\frac{d\Gamma_1^{div(1)}}{d\Delta}\exp\{i[\tilde{\Sigma}^{(1)}+Y(1-a^{(1)})F_\sigma\overline{C}^\sigma$$
$$+(1-a^{(1)})YK^{\mu\nu}(\eta_{\mu\nu}+h_{\mu\nu})+\cdots+\overline{\beta}^\sigma C_\sigma+\overline{C}^\tau\beta_\tau+T^{\mu\nu}h_{\mu\nu}]\}, \tag{31}$$

we use Eq. (26) and equation of motion of the h field which is obtained from (29),[10]

$$Y\int dh_{\mu\nu}dC_\sigma d\overline{C}^\tau\bigg(\frac{\delta\tilde{\Sigma}}{\delta h_{\mu\nu}}-\frac{\delta\Gamma_1^{div(1)}}{\delta h_{\mu\nu}}+T^{\mu\nu}\bigg)\exp\{i(\tilde{\Sigma}^{(1)}+Y(1-a^{(1)})F_\sigma\overline{C}^\sigma$$
$$+Y(1-a^{(1)})K^{\mu\nu}(\eta_{\mu\nu}+h_{\mu\nu})+\cdots+\overline{\beta}^\sigma C_\sigma+\overline{C}^\tau\beta_\tau+T^{\mu\nu}h_{\mu\nu})\}=0. \tag{32}$$

In the two-loop approximation we may write

$$a^{(1)}\int dh_{\mu\nu}dC_\sigma d\overline{C}^\tau\frac{\delta\Gamma_1^{div(1)}}{\delta h_{\mu\nu}}\exp\{i\cdots\}=a^{(1)}\frac{\delta\Gamma_1^{div(1)}}{\delta h_{\mu\nu}}Z^{[1]}.$$

Finally, using the ghost equation of motion

$$F^{\mu\nu}_\tau\frac{\delta\Gamma^{[1]}}{\delta K^{\mu\nu}}-\frac{\delta\Gamma^{[1]}}{\delta\overline{C}^\tau}=0, \tag{33}$$

written in terms of the one-loop finite reduced generating functional of proper vertices,

$$\Gamma^{[1]}=\tilde{\Gamma}^{[1]}+\frac{1}{2\Delta}F_\alpha\Box F^\alpha-YK^{\mu\nu}(\eta_{\mu\nu}+h_{\mu\nu})-YF_\sigma\overline{C}^\sigma,$$

we rewrite the rest of Eq. (30) as in Sec. II C 1 and obtain the following Slavnov identity for one-loop finite proper vertices, valid up to two-loop order:

$$\frac{\delta\Gamma^{[1]}}{\delta h_{\mu\nu}}\frac{\delta\Gamma^{[1]}}{\delta K^{\mu\nu}}+\frac{\delta\Gamma^{[1]}}{\delta C_\sigma}\frac{\delta\Gamma^{[1]}}{\delta L^\sigma}+2Y\Delta\frac{d\Gamma^{[1]}}{d\Delta}$$
$$+Ya^{(1)}\frac{\delta\Gamma_1^{div(1)}}{\delta h_{\mu\nu}}(\eta_{\mu\nu}+h_{\mu\nu})+\cdots=0, \tag{34}$$

where terms explicitly dependent on $K^{\mu\nu}$ and $\overline{C}_\sigma$ are included in ‘‘$+\cdots$’’ for simplicity. Again the separation

$$\Gamma^{[1]}=\Gamma_1^{[1]}+Y\Gamma_2^{[1]} \tag{35}$$

gives an ordinary Slavnov identity

$$\frac{\delta\Gamma_1^{[1]}}{\delta h_{\mu\nu}}\frac{\delta\Gamma_1^{[1]}}{\delta K^{\mu\nu}}+\frac{\delta\Gamma_1^{[1]}}{\delta C_\sigma}\frac{\delta\Gamma_1^{[1]}}{\delta L^\sigma}=0 \tag{36}$$

and an identity involving $\Gamma_2^{[1]}$:

$$-\frac{\delta\Gamma_1^{[1]}}{\delta h_{\mu\nu}}\frac{\delta\Gamma_2^{[1]}}{\delta K^{\mu\nu}}+\frac{\delta\Gamma_2^{[1]}}{\delta h_{\mu\nu}}\frac{\delta\Gamma_1^{[1]}}{\delta K^{\mu\nu}}-\frac{\delta\Gamma_1^{[1]}}{\delta C_\sigma}\frac{\delta\Gamma_2^{[1]}}{\delta L^\sigma}-\frac{\delta\Gamma_2^{[1]}}{\delta C_\sigma}\frac{\delta\Gamma_1^{[1]}}{\delta L^\sigma}$$
$$+2\Delta\frac{d\Gamma_1^{[1]}}{d\Delta}+a^{(1)}\frac{\delta\Gamma_1^{div(1)}}{\delta h_{\mu\nu}}(\eta_{\mu\nu}+h_{\mu\nu})+\cdots=0. \tag{37}$$

Thus for the two-loop gauge-invariant divergent part $\Omega^{[1]div(2)}$ of the one-loop finite generating functional of proper vertices we have[11]

$$-\frac{\delta S_0}{\delta h_{\mu\nu}}P^{(2)}_{\mu\nu}+2\Delta\frac{d\Omega^{[1]div(2)}}{d\Delta}$$
$$+a^{(1)}\frac{\delta\Omega^{div(1)}}{\delta h_{\mu\nu}}(\eta_{\mu\nu}+h_{\mu\nu})=0. \tag{38}$$

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[9]Again evaluation of the gauge invariant part of $\Delta\, d\Gamma_1^{div(1)}/d\Delta$ is needed only.

[10]We use the property $Y^2=0$.

[11]Strictly speaking, a non-gauge-invariant term corresponding to a nonlinear reparametrization of the field h should appear in the two-loop approximation. However, on account of the well-known structure of this term [8] it does not change the final result (39), as one can easily verify. If we were to carry out the renormalization procedure to all orders, we would deal with it more carefully.

Again it follows from Eqs. (38),(33) that $P^{(2)}_{\mu\nu}=a^{(2)}(\eta_{\mu\nu}+h_{\mu\nu})$, $Q^{(2)}_{\mu\nu}=a^{(2)}h_{\mu\nu}$.

Thus we obtain the following identity for the one- and two-loop divergent gauge-invariant parts of the effective action:

$$2\Delta\frac{d\Omega^{[1]div(2)}}{d\Delta}=-a^{(1)}\frac{\delta\Omega^{div(1)}}{\delta h_{\mu\nu}}(\eta_{\mu\nu}+h_{\mu\nu})+a^{(2)}\frac{\delta S_0}{\delta h_{\mu\nu}}(\eta_{\mu\nu}+h_{\mu\nu}). \quad (39)$$

Had we used any other parametrization of the gravitational field, $P^{(1)}$ and $P^{(2)}$ would have such a form that provides the gauge invariance of the product $(\delta S_0/\delta h_{\mu\nu})P_{\mu\nu}$, where $h_{\mu\nu}$ denotes the set of standard variables. Therefore the result (39) holds in general if $h_{\mu\nu}$ denotes a quantum part of the covariant components of the metric field.

Thus we see that in the presence of the new source Y the renormalization procedure differs from the usual one substantially. Although the renormalized Green functions satisfy the same Slavnov identities as the bare ones, the renormalization equation [of the type (39)] for their divergent parts in $(n+1)$ th-loop order cannot be obtained by a simple omitting of the finite parts of the Slavnov identities for the Green functions renormalized up to n th-loop order. The correct procedure presented above leads to the Slavnov identities which just impose some nontrivial constraints on the form of the gauge-dependent divergent structures of the Green functions.

III. CALCULATION OF THE ONE-LOOP DIVERGENT PART OF Ω

In the previous section we have obtained the relation (39) which identifies (*modulo* terms proportional to the equations of motion of h field) the Δ derivatives of the two-loop gauge-invariant divergent part of the effective action with the variational derivatives of the corresponding one-loop part up to some coefficient being defined by divergent parts of diagrams with one insertion of the Y vertex. To prove this coefficient is not zero we present explicit calculations of the values $\Gamma_1^{div(1)}$ and $\Gamma_1^{[1]div(2)}$ in an arbitrary gauge of the type (3) and an arbitrary parametrization with the only restriction being the linearity of group generators. We prefer this way to direct computation of diagrams with a Y insertion because it allows us to verify the relation (39).

A. Arbitrary parametrizations

In general the metric is an arbitrary function of the dynamical variables. The only restriction is that this function must be nondegenerate. For example, if dynamical variables are chosen as $g^{*}_{\mu\nu}=g_{\mu\nu}(-g)^r, g=\det g_{\mu\nu}$, then we should avoid the case of $r=-\frac{1}{4}$; otherwise, $\det g^{*}_{\mu\nu}=1$ and one more independent variable must be introduced in addition to the set of $g^{*}_{\mu\nu}$.

To calculate one-loop divergences the background field method is used [11–13]. Accordingly, we should first find expansions of all the quantities entering Eq. (1) in powers of the dynamical quantum variables $h_{\mu\nu}$ around the background field $g_{\mu\nu}$ up to second order.

Now we note that the form of the graviton propagator is, of course, parametrization dependent and this dependence complicates all calculations considerably. However, it is fictitious in the sense that it always can be removed by a *linear* redefinition of the quantum variables. Such a change does not mix different orders in the loop expansion and therefore does not alter the values of the one-loop divergent part in particular.[12] We will show below that our calculations are highly simplified if the linear part of the metric expansion is chosen to have the simplest form $h_{\mu\nu}$:

$$\underline{g}_{\mu\nu}=g_{\mu\nu}+h_{\mu\nu}+ahh_{\mu\nu}+bh_{\mu\alpha}h^{\alpha}_{\nu}+ch^2g_{\mu\nu}+dh_{\alpha\beta}h^{\alpha\beta}g_{\mu\nu}+O(h^3), \quad (40)$$

where a,b,c,d are arbitrary constants, $\underline{g}_{\mu\nu}$ denotes the full metric field, and all raising of indices is done by means of the inverse background metric $g^{\mu\nu}$: $g_{\mu\alpha}g^{\alpha\nu}=\delta^{\nu}_{\mu}$, $h\equiv h_{\mu\nu}g^{\mu\nu}$. Any parametrization $g^{*}_{\mu\nu}$ somehow constructed from the metric $g_{\mu\nu}$ and its determinant has a background expansion reducible to Eq. (40).

To show the advantage of such a choice of the background metric expansion we note that it is only the linear part of this expansion which in fact contributes to the curvature $R^{\mu}_{\nu\alpha\beta}$ expansion. Really, this tensor has the following structure:

$$\underline{R}=\partial\underline{\Gamma}-\partial\underline{\Gamma}+\underline{\Gamma}\underline{\Gamma}-\underline{\Gamma}\underline{\Gamma}. \quad (41)$$

Suppose at the moment that we have chosen our coordinate system in such a way that $\Gamma=0$ at any fixed point of space-time.[13] Then

$$\underline{R}=\partial\Gamma-\partial\Gamma+\Gamma_1\Gamma_1-\Gamma_1\Gamma_1+\partial(\Gamma_1+\Gamma_2)-\partial(\Gamma_1+\Gamma_2)+O(h^3),$$

where subscripts 1 and 2 denote parts of $\underline{\Gamma}$ of the corresponding powers in h. We may rewrite this expression in explicitly covariant form as[14]

$$\underline{R}=R+\nabla(\Gamma_1+\Gamma_2)-\nabla(\Gamma_1+\Gamma_2)+\Gamma_1\Gamma_1-\Gamma_1\Gamma_1+O(h^3),$$

valid therefore in every coordinate system. In terms of the Lagrangian linear in curvature scalar all full derivatives of second order may be dropped out. In quadratic terms these derivatives are multiplied by the zeroth-order quantities $R_{\mu\nu}$, R, etc., when the second variation of the action is being calculated. Integrating by parts one can easily verify

[12]The corresponding Jacobian is $\exp[\delta(0)\cdots]=1$ in the dimensional regularization.

[13]Recall that $\underline{\Gamma}$ and Γ are constructed from $\underline{g}_{\mu\nu}$ and $g_{\mu\nu}$, respectively.

[14]Note that $\Gamma_{1,2}$ are tensors.

by means of power counting that these terms do not contribute to the one-loop divergent part of the effective action.

It follows from the above discussion that the dependence on the parametrization appears only in terms without h derivatives in the second variation of the action if the reduced expansion (40) is used.

B. One-loop invariance on shell

To calculate the one-loop divergent part of the effective action in arbitrary gauge and parametrization we shall use the fact that the dependence on parameters Δ,a,b,c,d appears in terms proportional to the equations of motion only. As far as the Δ dependence is concerned the corresponding result follows directly from Eq. (26).

To prove the on-shell independence of a,b,c,d we note first of all that these parameters appear in the second variation of the action only in terms having the form

$$\int \frac{\delta S}{\delta g_{\mu\nu}}[g^*_{\mu\nu}]_2,$$

where $[g^*_{\mu\nu}]_2$ denotes the second order part of the reduced metric expansion (40).

Next, calculating generators of the gauge transformations of dynamical variables belonging to the class of parametrizations described above and passing to the set of standard variables again one easily sees that these generators just coincide with the ordinary ones of the metric field transformations; i.e., they are a,b,c,d independent and therefore so is the ghost contribution.

Thus the on-shell invariance is proved.

C. Background field method

According to the background field method we separate the quantum field part $h^*_{\mu\nu}$ from the external field $g^*_{\mu\nu}$:

$$\underline{g}^*_{\mu\nu}=g^*_{\mu\nu}+h^*_{\mu\nu}.$$

Then we expand the metric field $g_{\mu\nu}$ in powers of $h^*_{\mu\nu}$ and perform a linear transformation on $h^*_{\mu\nu}$ bringing this expansion to the form of Eq. (40).

Imposing the background Lorentz gauge on the quantum field $h_{\mu\nu}$,

$$F_\mu(g)\equiv F^{\alpha\beta}_\mu(g)h_{\alpha\beta}\equiv\nabla^\nu h_{\mu\nu},$$

we have, for the generating functional of Green functions,

$$Z[T^{\mu\nu}]=\int dh_{\mu\nu}dC_\sigma d\bar{C}^\tau\{\det g_{\mu\nu}\nabla^2\}^{1/2} \times\exp\left\{i\left(S_0(g,h)-\frac{1}{2\Delta}F^\alpha(g)\nabla^2F_\alpha(g) \times\sqrt{-g}+\bar{C}^\tau F^{\mu\nu}_\tau(g)D^\alpha_{\mu\nu}C_\alpha+T^{\mu\nu}h_{\mu\nu}\right)\right\}.$$

We suppose that the background field $g_{\mu\nu}-\eta_{\mu\nu}$ and the source $T^{\mu\nu}$ are absent out of some finite region of space-time. Integration is carried out in all fields $h_{\mu\nu}$ tending to zero at infinity. We do not introduce background ghost fields or their sources because renormalization of these fields plays no role in this section or in Sec. IV.

In the one-loop approximation we expand the gauge fixed action

$$S_{gf}\equiv S_0(g,h)-\frac{1}{2\Delta}F^\alpha(g)\nabla^2F_\alpha(g)\sqrt{-g}$$

around the extremal $\tilde{h}$, satisfying the classical equations of motion

$$\frac{\delta S_{gf}(g,h)}{\delta h_{\mu\nu}}+T^{\mu\nu}=0 \tag{42}$$

up to the second order, and obtain

$$Z[T^{\mu\nu}]=\exp\{i[S_{gf}(g,\tilde{h})+T^{\mu\nu}\tilde{h}^{\mu\nu}]\} \times\int dh_{\mu\nu}dC_\sigma d\bar{C}^\tau\{\det g_{\mu\nu}\nabla^2\}^{1/2} \times\det F^{\mu\nu}_\beta(g)D^\alpha_{\mu\nu}(g,h) \times\exp\left\{\frac{i}{2}\frac{\delta^2S_{gf}(g,\tilde{h})}{\delta h_{\mu\nu}\delta h_{\alpha\beta}}(h_{\mu\nu}-\tilde{h}_{\mu\nu})(h_{\alpha\beta}-\tilde{h}_{\alpha\beta})\right\}.$$

As far as we have supposed the background field $g-\eta$ and the source T to disappear out of some finite region of space-time one can choose a solution $\tilde{h}$ of Eq. (42) to be zero at infinity. Thus the shift of integration variables $h\to h+\tilde{h}$ does not change boundary conditions for h and we have, for the generating functional of connected Green functions,

$$W=S_{gf}(g,\tilde{h})+T^{\mu\nu}\tilde{h}_{\mu\nu}+\frac{i}{2}\mathrm{Tr}\ln\frac{\delta^2S_{gf}(g,\tilde{h})}{\delta h_{\mu\nu}\delta h_{\alpha\beta}} -i\,\mathrm{Tr}\ln F^{\mu\nu}_\tau(g)D^\alpha_{\mu\nu}(g,\tilde{h})-\frac{i}{2}\mathrm{Tr}\ln g_{\mu\nu}\nabla^2.$$

To perform a Legendre transformation we calculate

$$h_{\mu\nu}\equiv\frac{\delta W}{\delta T^{\mu\nu}} =\tilde{h}_{\mu\nu}+\frac{\delta}{\delta T^{\mu\nu}}\left\{\frac{i}{2}\mathrm{Tr}\ln\frac{\delta^2S_{gf}(g,\tilde{h})}{\delta h_{\mu\nu}\delta h_{\alpha\beta}} -i\,\mathrm{Tr}\ln F^{\mu\nu}_\tau(g)D^\alpha_{\mu\nu}(g,\tilde{h})\right\}$$

and

$$\begin{aligned}\Gamma(g,h)&=W(g,h)-h_{\mu\nu}T^{\mu\nu}\\&=S_{gf}(g,\tilde{h})+\frac{i}{2}\,\mathrm{Tr}\ln\frac{\delta^2 S_{gf}(g,\tilde{h})}{\delta h_{\mu\nu}\delta h_{\alpha\beta}}-i\,\mathrm{Tr}\ln F^{\mu\nu}_{\tau}(g)D^{\alpha}_{\mu\nu}(g,\tilde{h})\\&\quad-\frac{i}{2}\,\mathrm{Tr}\ln g_{\mu\nu}\nabla^2-T^{\mu\nu}\frac{\delta}{\delta T^{\mu\nu}}\left\{\frac{i}{2}\;\mathrm{Tr}\;\ln\frac{\delta^2 S_{gf}(g,\tilde{h})}{\delta h_{\mu\nu}\delta h_{\alpha\beta}}-i\,\mathrm{Tr}\ln F^{\mu\nu}_{\tau}(g)D^{\alpha}_{\mu\nu}(g,\tilde{h})\right\}\\&=S_{gf}(g,h)+\frac{i}{2}\,\mathrm{Tr}\ln\frac{\delta^2 S_{gf}(g,h)}{\delta h_{\mu\nu}\delta h_{\alpha\beta}}-i\,\mathrm{Tr}\ln F^{\mu\nu}_{\tau}(g)D^{\alpha}_{\mu\nu}(g,h)-\frac{i}{2}\,\mathrm{Tr}\ln g_{\mu\nu}\nabla^2;\end{aligned}\tag{43}$$

Eq. (42) was used in the last passage.

Obtaining the relation (39) in Sec. II we used the flat background $\eta_{\mu\nu}$. Had we started with an arbitrary background metric $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$ we would have modulo terms proportional to the equations of motion

$$2\Delta\frac{d\Omega^{[1]div(2)}}{d\Delta}=-a^{(1)}\frac{\delta\Omega^{div(1)}}{\delta g_{\mu\nu}}g_{\mu\nu}.\tag{44}$$

We wrote $g_{\mu\nu}$ instead of $\underline{g}_{\mu\nu}$ in Eq. (44) because it is sufficient to verify this relation in the case $h_{\mu\nu}=0$.

D. Calculation of $\Omega^{div(1)}$

Let us first reveal some ''essential'' properties of charges.

As R^2 gravity is renormalizable we can write $\Omega^{div(1)}$ in the form

$$\Omega^{div(1)}=\frac{1}{32\pi^2\varepsilon}\int d^4x\sqrt{-g}\times(c_1R^{\mu\nu}R_{\mu\nu}+c_2R^2+c_3R+c_4\Lambda+c_5),\tag{45}$$

where $c_i, i=1,\dots,5$ are some gauge- and parametrization-dependent coefficients.

As we know from Sec. III B, $\Omega^{div(1)}$ is gauge and parametrization independent on shell. It is obvious that the only scalar which can be constructed from Eq. (2) to transform $\Omega^{div(1)}$ Eq. (45), is

$$\frac{1}{k^2}(R-4\Lambda)=-2(3\alpha_1+\beta)\nabla^2R.\tag{46}$$

It follows from these simple facts that $c_i, i=1,2,5$ and the combination $4c_3+c_4$ do not depend on Δ,a,b,c,d.

Thus we may simplify the calculation of $\Omega^{div(1)}$ in arbitrary gauge and parametrization if divide it into two parts.

(1) Calculation of $\Omega^{div(1)}$ in the case of the simplest gauge and parametrization. We choose $g^{*}_{\mu\nu}=g_{\mu\nu}$ and the minimal gauge.

(2) Calculation of the coefficient c_4 alone in arbitrary gauge and parametrization. In this part we may obviously consider the space-time as flat.

The correct result of the first part of our program was obtained in [14]:

$$\Omega^{div(1)}=\frac{1}{32\pi^2\varepsilon}\int d^4x\times\sqrt{-g}(c_1R^{\mu\nu}R_{\mu\nu}+c_2R^2+c_3R+c_4\Lambda+c_5),$$

where

$$c_1=\frac{133}{10},\quad c_2=\frac{10\alpha_1^2}{\beta^2}+\frac{10\alpha_1}{6\beta}-\frac{291}{60},$$

$$c_3=-\frac{1}{(3\alpha_1+\beta)k^2}\left[\frac{30\alpha_1^2}{\beta^2}+\frac{53\alpha_1}{2\beta}+\frac{21}{4}\right],$$

$$c_4=\frac{1}{(3\alpha_1+\beta)k^2}\left[\frac{28\alpha_1}{\beta}+9\right],$$

$$c_5=\frac{3}{(3\alpha_1+\beta)^2k^4}\left[\frac{15\alpha_1^2}{2\beta^2}+\frac{5\alpha_1}{\beta}+\frac{7}{8}\right].$$

Calculation of c_4 in the flat space-time is presented in Appendix A. Combination of the two results gives

$$\Omega^{div(1)}=\frac{1}{32\pi^2\varepsilon}\int d^4x\times\sqrt{-g}(c_1R^{\mu\nu}R_{\mu\nu}+c_2R^2+c_3R+c_4\Lambda+c_5),\tag{47}$$

where

$$c_1=\frac{133}{10},\quad c_2=\frac{10}{9}\alpha^2-\frac{5}{3}\alpha-\frac{773}{180},$$

$$4c_3+c_4=\frac{1}{\beta k^2}\left[\frac{1}{3\alpha}+10-\frac{40\alpha}{3}\right],$$

$$c_4=-\frac{1}{\beta k^2}\left[u\left(2\delta+\frac{3}{\alpha}\right)+v\left(14\delta+\frac{1}{\alpha}+20\right)\right],$$

$$c_5=\frac{1}{\beta^2k^4}\left(\frac{5}{2}+\frac{1}{8\alpha^2}\right),$$

$$\alpha = \frac{3\alpha_1}{\beta} + 1, \quad u = a + 4c + \frac{1}{4},$$

$$v = b + 4d - \frac{1}{2}, \quad \delta \equiv \Delta\beta.$$

IV. CALCULATION OF THE TWO-LOOP DIVERGENT PART OF Ω

As follows from Eq. (39) the nontrivial dependence on the gauge parameter Δ (i.e., which is not zero modulo equations of motion) is contained in terms proportional to $1/\varepsilon^2$. To calculate the latter we use the renormalization group method. It is very convenient to apply the generalized version of the renormalization group equations given in [15,16]. For the sake of completeness we give an account of this method following [16].

A. Generalized renormalization group method

The idea of this approach is to obtain renormalization group equations without explicit distinguishing of different charges, i.e., in terms of the whole Lagrangian.

Let us consider the bare Lagrangian L^b as a functional of the initial Lagrangian L:

$$L^b = (\mu^2)^{\varepsilon}\left\{ L + \sum_{n=1}^{\infty} \frac{A_n(L)}{\varepsilon^n} \right\}, \tag{48}$$

where symbol $A_n(L)$ means that the corresponding counterpart is calculated for the Lagrangian L. Independence of the L^b from the mass scale implies

$$\beta(L) = \left(L\frac{\delta}{\delta L} - 1 \right) A_1(L), \tag{49}$$

$$\left(L\frac{\delta}{\delta L} - 1 \right) A_n(L) = \beta(L)\frac{\delta}{\delta L} A_{n-1}(L), \tag{50}$$

where the so-called generalized beta-function $\beta(L)$ is defined by

$$\mu^2 \frac{dL}{d\mu^2}\bigg|_{L^b} = -\varepsilon L + \beta(L),$$

We do not have to muse upon the concrete sense which the operation $\delta/\delta L$ possesses. Using the loop expansion of A_n,

$$A_n(L) = \sum_{k=n}^{\infty} A_{nk}(L),$$

and noting the homogeneity of functionals $A_{nk}(L)$,

$$A_{nk}(\lambda L) = \lambda^{1-k} A_{nk}(L),$$

(λ being a constant), we can express the operations $L\,\delta/\delta L$ and $\beta(L)\,\delta/\delta L$ in terms of the ordinary differentiation

$$L\frac{\delta}{\delta L} A_{nk}(L) = \frac{\partial}{\partial\lambda} A_{nk}(\lambda L)\big|_{\lambda=1} = (1-k) A_{nk}(L) \tag{51}$$

and

$$\beta(L)\frac{\delta}{\delta L} A_{nk}(L) = \frac{d}{dx} A_{nk}(L + x\beta(L))\big|_{x=0}, \tag{52}$$

whatever meaning has to be assigned to $\delta/\delta L$.

Thus we obtain

$$\beta(L) = \left(\frac{\partial}{\partial\lambda} - 1 \right) A_1(\lambda L)\bigg|_{\lambda=1} = \sum_{k=1}^{\infty} -k A_{1k}(L), \tag{53}$$

$$\left(\frac{\partial}{\partial\lambda} - 1 \right) \sum_{k=n}^{\infty} A_{nk}(\lambda L)\bigg|_{\lambda=1} = \frac{d}{dx} \sum_{k=n-1}^{\infty} A_{n-1,k}\left(L - x\sum_{l=1}^{\infty} l A_{1l}(L) \right)\bigg|_{x=0}. \tag{54}$$

To relate A_{nn} and $A_{n-1,n-1}$ we substitute $L \to \xi^{-1} L$ in Eq. (54), differentiate with respect to ξ $n-1$ times and set $\xi = 0$.

The result is

$$n A_{nn}(L) = \frac{d}{dx} A_{n-1,n-1}(L + x A_{11}(L))\bigg|_{x=0}. \tag{55}$$

B. Calculation of $\Omega^{[1]div(2)}_{1/\varepsilon^2}$

To apply the relation

$$-\varepsilon^2 \Omega^{[1]div(2)}_{1/\varepsilon^2} = \frac{1}{2}\frac{d}{dx} A_{11}[L + x A_{11}(L)]\bigg|_{x=0} \tag{56}$$

to the case of

$$L = L_{gf} + L_{fp} = \sqrt{-g}\left(\alpha_1 R^2 + \beta R_{\mu\nu} R^{\mu\nu} - \frac{1}{k^2}(R - 2\Lambda) \right) - \frac{1}{2\Delta} F_{\alpha} \Box F^{\alpha} + \bar{C}^{\tau} F^{\mu\nu}_{\tau} D^{\alpha}_{\mu\nu} C_{\alpha} \tag{57}$$

we note first of all that the gauge-fixing term is not renormalized if the linear gauge is used (see, e.g., [8,10]). Also the renormalization of the ghost part of the effective action is immaterial as long as only the one-loop expression is needed in Eq. (56).

Thus to calculate $\Omega^{[1]div(2)}_{1/\varepsilon^2}$ we rewrite $L_0 + x A_{11}(L)$ as

$$\sqrt{-g}\left((\alpha_1+xc_2)R^2+(\beta+xc_1)R_{\mu\nu}R^{\mu\nu}-\frac{1}{\left(\frac{1}{k^2}-xc_3\right)^{-1}} \times\left\{R-2\left[\left(\frac{1}{k^2}-xc_3\right)^{-1}\left(\frac{\Lambda}{k^2}+\frac{xc_4\Lambda+xc_5}{2}\right)\right]\right\}\right), \tag{58}$$

apply the one-loop result (47), differentiate it with respect to x, and set $x=0$.

The result of this calculation is presented in Appendix B.

Now we are in position to verify the identity (44).

As follows from the result (B1) on the mass shell the left hand side of Eq. (44) is

$$2\Delta\frac{d\Omega_{1/\varepsilon^2}^{[1]div(2)}}{d\Delta}=\frac{1}{(32\pi^2\varepsilon)^2}\int d^4x\sqrt{-g}(A\Lambda+B),$$

$$A=\frac{-1}{\beta^2k^2}\{2\delta^2w^2+\delta w(-20v\alpha^{-1}+20v+3w\alpha^{-1}+40/3\alpha-1/3\alpha^{-1}-10)\},$$

$$B=\frac{\delta w}{\beta^3k^4}(1/4\alpha^{-2}+5),\quad w\equiv u+7v,$$

while Eq. (47) gives, for the right hand side,

$$-a^{(1)}g_{\mu\nu}\frac{\delta\Omega^{div(1)}}{\delta g_{\mu\nu}}=\frac{a^{(1)}}{32\pi^2\varepsilon}\int d^4x\ \sqrt{-g}(\tilde{A}\Lambda+\tilde{B}),$$

$$\tilde{A}=\frac{1}{\beta k^2}\{2\delta w-20v\alpha^{-1}+20v+3w\alpha^{-1}+40/3\alpha-1/3\alpha^{-1}-10\},$$

$$\tilde{B}=-\frac{1}{\beta^2k^4}\{1/4\alpha^{-2}+5\}.$$

We see that Eq. (44) is really satisfied[15] and the coefficient $a^{(1)}$ turns out to be equal to $-\Delta w/32\pi^2\varepsilon$.[16] Note that $d\Omega^{[1]div(2)}/d\Delta$ is not zero even if the unweighted (Landau) gauge condition $\Delta\rightarrow 0$ is used.

V. CONCLUSION

We have shown in this paper that generally the divergent parts of the effective action of R^2 gravity depend on the gauge and parametrization nontrivially—this dependence can not be presented as proportional to the equations of motion. The renormalization procedure in the presence of the new anticommuting source Y turned out to be more complicated than the usual one: the renormalization equation corresponding to the modified generating functional cannot be obtained by a naive extracting of divergent terms in Slavnov identities. We have considered the renormalization of modified Green functions at one- and two-loop levels and obtained renormalization equations corresponding to the insertion of the Y source [Eqs. (26),(39)]. Also explicit calculation of the one- and two-loop divergent parts has been carried out, confirming our results and demonstrating that the nontrivial gauge dependence of the divergent parts of the effective action actually exists in arbitrary (Lorentz) gauge and arbitrary parametrizations except those satisfying $w=0$.[17]

We emphasize that this nontrivial dependence is due to the presence of the Einstein term in the Lagrangian. Had we considered a theory with the Lagrangian containing the higher derivative terms only, we would not have had such a dependence.

Our conclusion does not contradict the equivalence theorem [20] in view of the general results of [2,21]. Their validity in the present case is verified in Appendix B. However, these results do not allow us to say that the renormalization of the coupling constants is independent of the renormalization of fields (as in the case of two-dimensional chiral theories [21], for example), because renormalization of the Newtonian gravitational constant k cannot be separated from renormalization of the gravitational field: one can always perform additional redefinitions of the constant k and the metric field which compensate each other. This is a consequence of the fact that k is an ''inessential'' coupling constant.

Finally, we note that our results are in agreement with the general statements of [22].

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APPENDIX A

In this appendix we present calculation of the one-loop divergent part of the effective action in the flat space-time.

According to algorithm derived in [17] we should first calculate the part of δ^2S_{gf} with four derivatives:

[15]One can easily verify that the results (47) and (B1) satisfy Eq. (44) exactly, i.e., even off mass shell. In other words, the functional $\varepsilon^2\Omega_{1/\varepsilon^2}^{[1]div(2)}$ is gauge independent on ''mass shell'' determined by the ''action'' $\varepsilon\Omega^{div(1)}$.

[16]Of course, this value of $a^{(1)}$ could be determined already from Eqs. (26),(47).

[17]For the construction of the parametrization satisfying $w=0$ see [18,19].

$$\delta^2 S_{gf}|_4 = \left(\alpha_1 + \frac{\beta}{4}\right) h\Box^2 h + \frac{\beta}{4} h_{\mu\nu}\Box h^{\mu\nu}$$
$$-2\left(\alpha_1 + \frac{\beta}{4}\right)(\nabla A)\Box h + \left(\alpha_1 + \frac{\beta}{2}\right)(\nabla A)^2$$
$$+\left(\frac{\beta}{2} - \frac{1}{2\Delta}\right) A_\nu \Box A^\nu, \quad A_\nu \equiv \nabla^\mu h_{\mu\nu}. \tag{A1}$$

Then we substitute $\nabla_\mu \to n_\mu$, n_μ being a vector with $n_\mu^2 = 1$, and calculate the ''propagator'' $(Kn^{-1})^{\alpha\beta,\gamma\delta}$ which is the inverse of the operator $(Kn)_{\mu\nu,\alpha\beta} = \delta^2 S_{gf}/\delta h_{\mu\nu}\delta h_{\alpha\beta}$:

$$(Kn)_{\mu\nu,\alpha\beta}(Kn^{-1})^{\alpha\beta,\gamma\delta} = \delta^{\gamma\delta}_{\mu\nu},$$

$$(Kn^{-1})^{\alpha\beta,\gamma\delta}$$
$$= 1/2(g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}) + A_1 g^{\alpha\beta}g^{\gamma\delta}$$
$$+ B_1(g^{\alpha\gamma}n^\beta n^\delta + g^{\alpha\delta}n^\beta n^\gamma + g^{\beta\gamma}n^\alpha n^\delta + g^{\beta\delta}n^\alpha n^\gamma)$$
$$+ C_1(g^{\alpha\beta}n^\gamma n^\delta + g^{\gamma\delta}n^\alpha n^\beta) + D_1 n^\alpha n^\beta n^\gamma n^\delta,$$

where

$$A_1 = -(A + 4AB + AD - C^2)/Z,$$

$$B_1 = -B/(1+2B),$$

$$C_1 = (4AB + AD - C - C^2)/Z,$$

$$D_1 = -(16AB + 4AD + D + 4B - 4C^2)/Z + 4B/(1+2B),$$

$$Z = 1 + 4A + 4B + 2C + D + 3(4AB + AD - C^2),$$

the coefficients A,B,C,D being defined from L_{gf}:

$$A = 1 + \frac{4\alpha_1}{\beta},$$

$$B = \frac{1}{2}\left(\frac{1}{\beta\Delta} - 1\right),$$

$$C = -1 - \frac{4\alpha_1}{\beta},$$

$$D = 2 + \frac{4\alpha_1}{\beta}.$$

We have multiplied the initial Lagrangian by $4/\beta$ for convenience.

Second, we calculate the part W of $\delta^2 S_{gf}$ containing two derivatives substituting $\nabla_\mu \to n_\mu$ again,

$$(Wn)_{\mu\nu,\alpha\beta} = \frac{1}{k^2\beta}\{g_{\mu\nu}g_{\alpha\beta} - (g_{\mu\nu}n_\alpha n_\beta + g_{\alpha\beta}n_\mu n_\nu)$$
$$- g_{\mu\alpha}g_{\nu\beta} + (g_{\mu\beta}n_\nu n_\alpha + g_{\nu\alpha}n_\mu n_\beta)\}, \tag{A2}$$

and the part M without derivatives,

$$M_{\mu\nu,\alpha\beta} = \frac{4\Lambda u}{\beta k^2} g_{\mu\nu}g_{\alpha\beta} + \frac{4\Lambda v}{\beta k^2} g_{\mu\alpha}g_{\nu\beta},$$

where $u = a + 4c + \frac{1}{4}$, $v = b + 4d - \frac{1}{2}$.

The one-loop divergent part of the effective action has the form[18]

$$\Omega^{div(1)} = \frac{1}{32\pi^2\varepsilon} tr \int d^4x \sqrt{-g}$$
$$\times \left(\frac{1}{2}(Kn^{-1})(Wn)(Kn^{-1})(Wn) - (Kn^{-1})(M)\right), \tag{A3}$$

where the matrix product of $(Kn^{-1})_{\mu\nu,\alpha\beta}$, $(Wn)_{\mu\nu,\alpha\beta}$, $M_{\mu\nu,\alpha\beta}$ is supposed.

A simple calculation gives

$$\Omega^{div(1)}_{flat} = \frac{1}{32\pi^2\varepsilon}\int d^4x \sqrt{-g}(c_4\Lambda + c_5), \tag{A4}$$

where

$$c_4 = -\frac{1}{\beta k^2}\left[u\left(2\delta + \frac{3}{\alpha}\right) + v\left(14\delta + \frac{1}{\alpha} + 20\right)\right],$$

$$c_5 = \frac{1}{\beta^2 k^4}\left(\frac{5}{2} + \frac{1}{8\alpha^2}\right), \quad \delta \equiv \Delta\beta.$$

APPENDIX B

In this appendix the result of the calculation of the two-loop divergent as the $1/\varepsilon^2$ part of the effective action is presented. Also, the validity of the general statements of [2,21] is verified.

Following the algorithm derived in Sec. IV B we obtain

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[18] Since the space-time is flat, the contributions of the Faddeev-Popov ghosts and of the ''third'' ghost are equal to zero.

$$-\Omega^{[1]div(2)}_{1/\varepsilon^2}=\frac{1}{2(32\pi^2\varepsilon)^2}\int d^4x\sqrt{-g}(c_{22}R^2+c_{32}R+c_{42}\Lambda+c_{52}), \tag{B1}$$

$$c_{22}=\frac{1}{\beta}(200/27\alpha^3-416/9\alpha^2+1697/54\alpha-25/36),$$

$$c_{32}=\frac{1}{\beta^2k^2}\{\delta^2(-1/4u^2-7/2uv-49/4v^2)+\delta(-3/4u^2\alpha^{-1}-11/2uv\alpha^{-1}-5uv+10/3u\alpha$$
$$-1/12u\alpha^{-1}-5/2u-7/4v^2\alpha^{-1}-35v^2+70/3v\alpha-7/12v\alpha^{-1}-35/2v)-9/16u^2\alpha^{-2}$$
$$-15/2uv\alpha^{-1}-3/8uv\alpha^{-2}-7/16u\alpha^{-2}+5/2u-5/2v^2\alpha^{-1}-1/16v^2\alpha^{-2}-25v^2+100/3v\alpha$$
$$-5/6v\alpha^{-1}-7/48v\alpha^{-2}-272/3v-200/9\alpha^2+122\alpha-1/24\alpha^{-2}-731/18\},$$

$$c_{42}=\frac{1}{\beta^2k^2}\{2\delta^2(u^2+14uv+49v^2)+\delta(6u^2\alpha^{-1}+44uv\alpha^{-1}+40uv+14v^2\alpha^{-1}+280v^2)$$
$$+9/2u^2\alpha^{-2}+60uv\alpha^{-1}+3uv\alpha^{-2}-15u\alpha^{-1}+5/4u\alpha^{-2}+10u+20v^2\alpha^{-1}+1/2v^2\alpha^{-2}$$
$$+200v^2-5v\alpha^{-1}+5/12v\alpha^{-2}+808/3v\},$$

$$c_{52}=\frac{1}{\beta^3k^4}\{\delta(-1/4u\alpha^{-2}-5u-7/4v\alpha^{-2}-35v)-15/2u\alpha^{-1}-3/8u\alpha^{-3}-5/2v\alpha^{-1}-5/2v\alpha^{-2}$$
$$-1/8v\alpha^{-3}-50v+50/3\alpha-5/12\alpha^{-1}+5/8\alpha^{-2}-1/8\alpha^{-3}-79\}.$$

To show that the gauge and parametrization dependence can be absorbed by a field renormalization we first remove the one-loop divergences (47) by the following redefinition of charges and fields,

$$g_{\mu\nu}\to g_{\mu\nu}(1+\delta_1 Z),$$

$$\lambda\to\lambda(1+\delta_1\lambda),\quad \frac{1}{k^2}\to\frac{1}{k^2}\left(1+\delta_1\frac{1}{k^2}\right),$$

$$\alpha_1\to\alpha_1(1+\delta_1\alpha_1),\quad \beta\to\beta(1+\delta_1\beta),$$

where

$$\delta_1\frac{1}{k^2}+\delta_1 Z=\frac{c_3}{32\pi^2\varepsilon},$$

$$\delta_1\lambda=-\frac{2c_3^i+c_5/2\lambda}{32\pi^2\varepsilon},$$

$$\delta_1\alpha_1=\frac{-c_2}{32\pi^2\varepsilon\alpha_1},\quad \delta_1\beta=\frac{-c_1}{32\pi^2\varepsilon\beta},$$

and we have introduced a notation c_3^i for the gauge- and parametrization-independent part of the coefficient c_3:

$$c_3\equiv c_3^i-\frac{c_4}{4}.$$

As seen from the above equations renormalizations of the gravitational constant and of the metric field cannot be separated from each other. This property is inherent to any metrical theory of gravity with the Lagrangian containing terms linear in curvature and holds at any order of perturbation theory.

To make the theory finite at the two-loop level we should take into account counterterms which arise in second order from the one-loop redefinitions of the charges and fields which were made above. Correspondingly, we extract these counterterms from the two-loop order result (B1), rewriting coefficients c_{32}, c_{42}, and c_{52} as[19]

$$c_{32}=\frac{k^2}{2}c_3^i c_4+\frac{1}{\beta^2k^2}\{\delta^2(-1/4u^2-7/2uv-49/4v^2)+\delta(-3/4u^2\alpha^{-1}-11/2uv\,\alpha^{-1}-5uv-7/4v^2\alpha^{-1}-35v^2)$$
$$-9/16u^2\alpha^{-2}-15/2uv\,\alpha^{-1}-3/8uv\,\alpha^{-2}+15/4u\,\alpha^{-1}-5/16u\,\alpha^{-2}-5/2u-5/2v^2\alpha^{-1}-1/16v^2\alpha^{-2}$$
$$-25v^2+5/4v\,\alpha^{-1}-5/48v\,\alpha^{-2}-202/3v-200/9\alpha^2+122\alpha-1/24\alpha^{-2}-731/18\},$$

$$c_{42}=k^2\{c_4^2/4-12c_{3i}^2\}+\frac{1}{\beta^2k^2}\{\delta^2(u^2+14uv+49v^2)+\delta(3u^2\alpha^{-1}+22uv\,\alpha^{-1}+20uv+7v^2\alpha^{-1}+140v^2)$$
$$+9/4u^2\alpha^{-2}+30uv\,\alpha^{-1}+3/2uv\,\alpha^{-2}-15u\,\alpha^{-1}+5/4u\,\alpha^{-2}+10u+10v^2\alpha^{-1}+1/4v^2\alpha^{-2}+100v^2$$
$$-5v\,\alpha^{-1}+5/12v\,\alpha^{-2}+808/3v+400/3\alpha^2-200\alpha+5\alpha^{-1}+1/12\alpha^{-2}+205/3\},$$

$$c_{52}=k^2c_5(c_4-4c_3^i)-\frac{1}{\beta^3k^4}\{50/3\alpha+5/4\alpha^{-1}-15/8\alpha^{-2}+1/12\alpha^{-3}+54\}.$$

Now it is easy to verify that

$$4c_{32}+c_{42}-k^2(2c_{3i}c_4+c_4^2/4-12c_{3i}^2)=\frac{1}{\beta^2k^2}\{400/9\alpha^2+288\alpha+5\alpha^{-1}-1/12\alpha^{-2}-847/9\},$$

which means that after subtraction of the counterterms corresponding to the one-loop renormalization of charges and fields is made the two-loop divergent part of the effective action becomes gauge and parametrization independent on shell. Therefore the gauge and parametrization dependence can be absorbed by a field renormalization or by renormalization of the Newtonian constant.

[19]The one-loop redefinitions do not affect the coefficient c_{22}.

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