Large N phases, gravitational instantons, and the nuts and bolts of AdS holography

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Recent results in the literature concerning holography indicate that the thermodynamics of quantum gravity (at least with a negative cosmological constant) can be modeled by the large N thermodynamics of quantum field theory. We emphasize that this suggests a completely unitary evolution of processes in quantum gravity, including black hole formation and decay, and even more extreme examples involving topology change. As concrete examples which show that this correspondence holds even when the space-time is only *locally* asymptotically AdS, we compute the thermodynamical phase structure of the AdS-Taub-NUT and AdS-Taub-bolt spacetimes, and compare them to a $(2+1)$-dimensional conformal field theory (at large N) compactified on a squashed three-sphere and on the twisted plane. [S0556-2821(99)06302-X]

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I. INTRODUCTION AND MOTIVATION

The holographic principle [1,2] asserts that all of the information contained in some region of space-time may be represented as a ''hologram'': a theory which lives on the boundary of the region. The principle also requires that the theory on the boundary should contain at most one degree of freedom per Planck area. It follows from these two simple assumptions that the maximum number of quantum degrees of freedom, which can be stored in a region bounded by a surface of area A, will never exceed $\exp(A/4G)$ (where G is Newton's constant). This dovetails nicely with the laws of black hole thermodynamics (which provided some of the inspiration for the holographic principle), leading some investigators to conclude that the holographic principle may be an essential ingredient in the construction of a complete quantum theory of gravity.

Recently, it has been conjectured [3,4,5] that information about the physics of superconformal field theories (in the large N limit[1]) may be obtained by studying the region near the horizon of certain p-branes, which yields a gauged supergravity compactification involving $(p+2)$-dimensional anti–de Sitter (AdS_{p+2}) space-time. The correspondence is holographic [5] because the conformal field theory (CFT) resides on the causal boundary of AdS space-time. This boundary is the ''horosphere'' at infinity [6]—it is a timelike hypersurface with the topology $S^1\times S^p$, where the circle S^1 is the (Euclideanized) timelike factor.

The key feature of this AdS-CFT correspondence is the fact that fields propagating in the bulk of AdS space-time are *uniquely* specified by their behavior at the boundary. This allows one to calculate correlation functions in the boundary theory by calculating the effective action in the bulk for field configurations which asymptotically approach the given boundary data [5].

Given this correspondence, one is naturally led to consider bulk supergravity space-times which are asymptotically equivalent to AdS space-time. Since the AdS-CFT correspondence asserts that the generating functional of (large N) superconformal field theory propagators on the boundary M of AdS space-time are equivalent to supergravity partition functions in the bulk, it is of some interest to understand how many such distinct bulk manifolds B_i, with boundary M, may exist.

A more complete version of the conjecture states that the full $1/N$ expansion of the field theory partition function $Z_{\mathrm{CFT}}(M)$, on M, must be expressed as a sum over the B_i:

$$Z_{\mathrm{CFT}}(M)=\sum_i Z(B_i), \tag{1.1}$$

where $Z(B_i)$ is the *string theory* (or *M-theory*) partition function on B_i. The stringy part of the story controls the short distance bulk physics (where gravity alone would fail). In the stricter large N limit, the string theory reduces to gravity, valid on space-times of low curvature [whose typical length scale l is of the order $N^{f(p)}$, where $f(p)$ is some positive function of p], and this is the regime we will focus on in this paper.

Recently [5,7], this relation has been employed to study the large N thermodynamics[2] of conformal field theories (de-

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[1]Here ''N'' refers to that of U(N) gauge theory in the simplest case of $p=3$, with suitable generalizations in the other cases of p.

[2]Of course, there is a thermodynamic limit even in finite volume if we take the number of degrees of freedom, here measured by some power of N, to infinity. So we may indeed have phase transitions [8].

fined at finite temperature by Euclideanizing to periodic time) on the boundary $S^1 \times S^p$. (Here, S^1 is the Euclidean time.) There are two known (asymptotically AdS) bulk solutions with this boundary. The more obvious one is AdS itself (with suitable identifications), while the other is the Euclidean AdS-Schwarzschild solution. It was shown that the former solution governs the low temperature phase of the boundary conformal field theory, while the latter controls the high temperature phase. Many qualitative features of the dynamics of the finite temperature field theory were reproduced with these space-times, including the geometric behavior of spatial and temporal Wilson lines, confirming that the high and low temperature phases have distinct physical characteristics. This is a dramatic demonstration of the properties (and uses of) a holographic relationship or ''duality'' between two theories.

We would like to emphasize that the arrow runs both ways in this relationship. While the existence of—and transition between—two different phases of a field theory are uncontroversial concepts to most theorists, this is not the same for many processes in quantum gravity. Indeed, as many of the transitions between different space-time solutions in gravity are not completely understood, there is still room to assume that—especially in cases involving the evaporation or formation of black holes—the quantum processes may be nonunitary. It is also of considerable technical interest as to how to describe completely such processes, as they often describe space-time topology change to relate the initial and final states.

Crucially, note that in having a holographic relation between field theory and gravity (at least with negative cosmological constant), we have a powerful laboratory for studying those bulk topology change processes which are still a matter of debate.[3] In particular, the relation to field theory (if proved) completely removes the possibility of a nonunitary nature of the processes governing space-time topology change in quantum gravity with negative cosmological constant, and we find it highly suggestive of a similar conclusion for all gravitational situations.

In the field theory examples of Refs. [5,7] (specializing to the case $p=2$), while the boundary field theory phase transition takes place, the dominant contribution on the right hand side of Eq. (1.1) shifts from AdS_4, with topology $\mathbb{R}^3 \times S^1$, to AdS_4-Schwarzschild, with topology $\mathbb{R}^2 \times S^2$. This transition was studied originally in Ref. [10]. The nature of this phase transition is intimately associated with the fact that the gravitational potential of AdS space-time behaves more or less like a large, perfectly insulating ''box.'' Massive particles are confined to the interior of AdS space-time, and while massless modes may escape to infinity, the fluxes for incoming and outgoing radiation in a thermal state at infinity are equal (the causal boundary acts like a mirror).

It was shown in Ref. [10] that there is a critical temperature T_c past which thermal radiation is unstable to the formation of a Schwarzschild black hole. [In fact, they found that for $T > T_c$ there are two values of the black hole mass at which the Hawking radiation can be in equilibrium with the thermal radiation of the background. The lesser of these two masses is a point of unstable equilibrium (it has negative specific heat), whereas the greater mass is a point of stable equilibrium.]

Since a phase transition in the field theory is a unitary process, this means that it would seem that there is no ''information loss,'' or loss of unitarity, in the bulk physics involving the nucleation and evaporation of black holes as one moves between the various phases. It would be certainly interesting to see if this unitary conformal field theory description extends to other transitions between instantons which involve space-time topology change. Clearly, this would then be in sharp contrast to the claims of recent authors [11,12], who have argued that whenever there is a topology changing transition (i.e., by black hole pair creation or some other process), the superscattering matrix will not factorize into an S matrix and its adjoint, and hence there will be a loss of quantum coherence.

It would therefore seem, at first glance, that the AdS version of the holographic principle has provided us with a *precise* argument which shows that information is not lost in black hole evolution or topology changing transitions, at least as long as the topology change occurs in a spacetime which is asymptotically AdS.

This suggests an interesting and vigorous program of revisiting the study of various space-time transitions between many instantons of interest, now in an AdS context.

In this paper, we will extend the holography laboratory to include examples with nontrivial topology and which are only *locally* asymptotically AdS. We discuss the Taub-Newman-Unti-Tamburino- (NUT-)AdS (TN-AdS) and Taub-bolt-AdS (TB-AdS) space-times. These space-times have a global nontrivial topology due to the fact that one of the Killing vectors has a zero-dimensional fixed point set (''nut'') or a two-dimensional fixed point set (''bolt''). Further, these four-dimensional space-times have Euclidean sections which cannot be exactly matched to AdS at infinity.

We show that it is possible to have a thermally triggered phase transition from TN-AdS to TB-AdS, which is the natural generalization of the Hawking-Page phase transition from AdS to Schwarzschild-AdS. We also notice that in the limits where we can use the naive field theory expectations, the results are in agreement with boundary field theory.

In the first case under study, where the bolt is an S^2, the presence of these nuts or bolts implies that the bulk supports a nontrivial NUT charge, which in turn implies that the boundary must be realized as an S^1 bundle over S^2 [i.e., the Chern number of this Hopf fibration (denoted C_1) is related to the NUT charge N in the bulk by the explicit relation [13] $N=(1/8\pi)\beta C_1$, where β is the period of the S^1 fiber at infinity]; the boundary at infinity is a ''squashed'' three-sphere.

This squashed three-sphere is the three-dimensional space on which the boundary conformal field theory will be compactified, with β identified with the inverse temperature, in analogy with the AdS–AdS-Schwarzschild system [10]. As

[3]See Ref. [9] for a recent discussion—with a different flavor—of space-time topology change in this context.

studied in Refs. [5,7], we see that the bulk behavior is consistent with the expected phase structure of the conformal field theory on the boundary.

In the second case, the bolt is an $\mathbb{R}^2$, and the resulting absence of a nontrivial fibration means that there is no link between the temperature at infinity and the squashing parameter. The squashing parameter describes a fixed deformation of the boundary as a twisted product of $\mathbb{R}^2$ and Euclidean time S^1. In this case, the phase structure found in the bulk again is consistent with that of conformal field theory on the boundary.

II. NUTS AND BOLTS OF AdS

We now turn our attention to a particular class of metrics which are locally asymptotically equivalent to anti–de Sitter space-time: the Taub-NUT-AdS (TN-AdS) and Taub-bolt-AdS (TB-AdS) metrics. The metric on the Euclidean section of this family of solutions may be written in the form [14]

$$ds^2 = V(r)(d\tau + 2n\cos\theta\, d\varphi)^2 + \frac{dr^2}{V(r)} + (r^2-n^2)(d\theta^2+\sin^2\theta\, d\varphi^2), \qquad (2.1)$$

where

$$V = \frac{(r^2+n^2) - 2mr + l^{-2}(r^4 - 6n^2r^2 - 3n^4)}{r^2-n^2} \qquad (2.2)$$

and we are working with the usual convention ($l^2 = -3/\Lambda$), with $\Lambda<0$ being the cosmological constant. Here m is a (generalized) mass parameter and r is a radial coordinate. Also, τ, the analytically continued time, parametrizes a circle S^1, which is fibered over the two-sphere S^2, with coordinates θ and φ. The nontrivial fibration is a result of a nonvanishing ''nut parameter'' n.

In the asymptotic region, the metric (2.1) becomes

$$ds^2 = \frac{l^2}{r^2}dr^2 + r^2\left[\frac{4n^2}{l^2}(d\psi + \cos\theta\, d\varphi)^2 + d\theta^2 + \sin^2\theta\, d\varphi^2\right], \qquad (2.3)$$

where $\psi = \tau/2n$. One can recognize the angular part of the metric as that of a ''squashed'' three-sphere, where $4n^2/l^2$ parametrizes the squashing. This finite amount of squashing contrasts with the standard Taub-NUT solution [15] with $\lambda = 0$. In the latter, a squashed three-sphere also arises in the asymptotic region, but $4n^2/l^2$ is replaced by $4n^2/r^2$ in the angular part of the metric [cf. Eq. (2.3)].

Remarkably, this asymptotic metric (2.3) is still maximally symmetric, to leading order, i.e., $R_{\mu\nu\alpha\beta} = -1/l^2(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})$. Hence we can still think of these solutions as locally asymptotically AdS_4.

A. Taub-NUT-AdS

To begin with, let us restrict our attention to nuts, the zero-dimensional fixed point set. For a regular nut to exist, we need to satisfy the following conditions.

(a) In order to ensure that the fixed point set is zero dimensional, it is necessary that the Killing vector ∂_τ has a fixed point which occurs precisely when the (θ,φ) two-sphere degenerates, i.e., $V(r=n)=0$.

(b) In order for the ''Dirac-Misner'' [16] string to be unobservable, it is necessary that the period of τ satisfy $\Delta\tau = 4n\Delta\varphi$. Since we want to avoid conical singularities at the poles of the angular spheres, then $\Delta\varphi = 2\pi$, and therefore $\Delta\tau = 8\pi n$.

(c) In general, these constraints will make the point $r=n$ look like the origin of $\mathbb{R}^4$ with a conical deficit. In order to avoid a conical singularity, the fiber has to close smoothly at $r=n$. This requires $\Delta\tau V'(r=n) = 4\pi$, i.e., $V'(r=n) = 1/2n$.

Now condition (a) requires that the numerator of V has a double zero at $r=n$. It is easy to see then that the ''mass'' parameter m must be

$$m_n = n - \frac{4n^3}{l^2} \qquad (2.4)$$

and then

$$V_n(r) = \frac{r-n}{r+n}[1 + l^{-2}(r-n)(r+3n)]. \qquad (2.5)$$

With this, condition (c) is automatically satisfied. This is due to the fact that the term that multiplies the cosmological constant vanishes at the nut and what remains is the same as in the familiar case with $\Lambda = 0$. This means that the presence of a cosmological constant does not affect the nut.

It is interesting to notice that here with $\Lambda<0$, m does not need to be positive in order for the nut to be regular. It is also worth remarking that n remains an arbitrary parameter, which will be assumed to be positive, without loss of generality. That is, as n varies in this family, we see that the squashing of the asymptotic three-spheres changes, and thus for fixed cosmological constant we have a one-parameter family of TN-AdS solutions.

Note that for the special case $n = l/2$ the squashing in Eq. (2.3) vanishes; i.e., the asymptotic spheres are round. In fact, in this special case, the geometry coincides precisely with the AdS_4 space. In order to see this, change τ to the more usual ψ coordinate in S^3, $\tau = 2n\psi$, so that the period of ψ is 4π. It is convenient to perform another coordinate change on Eq. (2.1) by shifting $r \to r+n$ to find

$$ds^2 = \frac{U(r)}{f(r)}dr^2 + 4n^2\frac{f(r)}{U(r)}(d\psi + \cos\theta\, d\varphi)^2 + r^2U(r)(d\theta^2 + \sin^2\theta\, d\varphi^2), \qquad (2.6)$$

with

$$f(r) = 1 + \frac{r^2}{l^2}\left(1 + \frac{4n}{r}\right)$$

and

$$U(r)=1+\frac{2n}{r}.$$

The nut is now at $r=0$.

Now start from the following form for the metric on AdS_4 space-time as the Poincaré ball [5]:

$$ds^2=4\frac{dy^2+y^2d\Omega_3^2}{(1-y^2/l^2)^2}. \tag{2.7}$$

The boundary is at $y=l$, and it is an S^3. Changing coordinates according to

$$\frac{y^2}{l^2}=\frac{r}{r+l}, \tag{2.8}$$

so that the boundary is now at $r\to\infty$, we find that the following metric for AdS_4 space-time:

$$ds^2=\frac{l^2}{r^2}\left(\frac{dr^2}{1+l/r}\right)+r^2\left(1+\frac{l}{r}\right)\times[(d\psi+\cos\theta\,d\varphi)^2+d\theta^2+\sin^2\theta\,d\varphi^2]. \tag{2.9}$$

This AdS metric coincides precisely with the TN-AdS metric (2.6) with $n=l/2$. At $r=0$ there is a coordinate singularity, but this is easily seen to be just like the origin of $\mathbb{R}^4$, i.e., a nut. It is not surprising to find a slicing where AdS_4 space-time contains a nut: given any point in the Poincaré ball, we can always choose coordinates such that it looks like the origin of $\mathbb{R}^4$.

One can confirm that in general the TN-AdS metric is distinct from AdS_4 by comparing curvature invariants, e.g., $R^{\mu\nu}R_{\mu\nu}$, on the two spaces.

B. Taub-bolt-AdS

We begin by casting the metric (2.1) in the form

$$ds^2=4n^2V(r)(d\psi+\cos\theta\,d\varphi)^2+\frac{dr^2}{V(r)}+(r^2-n^2)(d\theta^2+\sin^2\theta\,d\varphi^2), \tag{2.10}$$

with

$$V_b(r)=\frac{r^2-2mr+n^2+l^{-2}(r^4-6n^2r^2-3n^4)}{r^2-n^2}, \tag{2.11}$$

where as usual ψ has period 4π. In order to have a regular bolt at $r=r_b>n$, the following conditions must be met: (a) $V(r_b)=0$ and (b) $V'(r_b)=1/2n$. These are rather like the conditions for having a nut, but since $r_b>n$, the fixed point set of ∂_ψ is two dimensional, instead of zero dimensional. Moreover, the zero of the numerator of $V(r)$ at $r=r_b$ must now be a single one.

After some simple algebra, we find that condition (a) imposes

$$m=m_b=\frac{r_b^2+n^2}{2r_b}+\frac{1}{2l^2}\left(r_b^3-6n^2r_b-3\frac{n^4}{r_b}\right). \tag{2.12}$$

Then we find

$$V'(r_b)=\frac{3}{l^2}\left(\frac{r_b^2-n^2+l^2/3}{r_b}\right). \tag{2.13}$$

Now we require (b) to be satisfied. The ensuing equation yields r_b as a function of n and l:

$$r_{b\pm}=\frac{l^2}{12n}\left(1\pm\sqrt{1-48\frac{n^2}{l^2}+144\frac{n^4}{l^4}}\right). \tag{2.14}$$

For r_b to be real the discriminant must be non-negative. Furthermore, we must take the part of the solution which corresponds to $r_b>n$. This gives

$$n\leqslant\left(\frac{1}{6}-\frac{\sqrt{3}}{12}\right)^{1/2}l=n_{\max}. \tag{2.15}$$

It is only for this range of parameters that one can construct real Euclidean TB-AdS solutions. Notice, in particular, that the AdS value $l=2n$ lies outside this range.

It is worth noting that the properties of Taub-bolt in AdS space-time (for the upper branch r_{b+}) are very different from those of Taub-bolt space-time in an asymptotically locally flat (ALF) space. The reason is that these upper branch TB-AdS solutions do not go smoothly onto ALF-TB space-time as the cosmological constant is switched off. As l is taken to infinity, we can see that $r_{b+}\to\infty$. The ALF-TB limit can be achieved only with the r_{b-} branch TB-AdS solutions. In those cases, $r_{b-}\to 2n$ as the cosmological constant goes to zero, reproducing the ALF-TB value.

The lower branch family is more analogous to the Schwarzschild-AdS solutions. In the latter, when the bolt (the Euclidean horizon) is much smaller than the AdS scale, it resembles closely the corresponding asymptotically flat bolt. It is only when the black hole grows enough in size that the AdS structure shows up. By contrast, for the upper brach TB-AdS solutions, the fact that they live in anti–de Sitter space is always relevant.

Interestingly, the global topology of the TB-AdS solution is quite unlike that of TN-AdS. Arguments similar to those put forward in Ref. [17] lead to the conclusion that this solution has the topology of $\mathbb{CP}^2-\{0\}$, where the removed ''point'' $\{0\}$ corresponds to the squashed three-sphere at infinity. Furthermore, the bolt itself may be interpreted as the two-cycle in $\mathbb{CP}^2$ with odd self-intersection number; i.e., this space-time does not admit any spin structure.[4]

[4]This would suggest that there might be problems with interpreting this as a supergravity compactification. Recall, however, that there is the possibility of introducing a generalized spin structure [18], particularly in the case of $\mathbb{CP}^2$. Even without that possibility, we expect that holography in AdS_4 (and related space-times) is a property which exists independently of the possibility of supergravity compactifications.

Now that we have understood the structure of the TN-AdS and TB-AdS solutions, we need to examine the possibility of transitions between them. In order to understand the conditions for this phase transition, we need to calculate the actions for TN-AdS and TB-AdS space-times.

C. Action calculation

The Euclidean action is given by the formula [19,20]

$$I=-\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}(R-2\Lambda)-\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{\gamma}\Theta, \tag{2.16}$$

where $\mathcal{M}$ is a compact region of the spacetime, with boundary $\partial\mathcal{M}$ (which we will ultimately send to infinity). Here $\gamma_{\mu\nu}$ is the induced metric on $\partial\mathcal{M}$, and Θ is the trace of the extrinsic curvature of $\partial\mathcal{M}$ in $\mathcal{M}$. Of course, both of the terms above diverge as the boundary goes to infinity. Hence, to produce a finite and well-defined action as the boundary $\partial\mathcal{M}$ goes to infinity, we will subtract an infinite contribution from a background or reference space-time solution.

For a background to be suitable for a given space-time whose action we wish to compute, we must match the metric that it induces on $\partial\mathcal{M}$ to the metric induced by the space-time on $\partial\mathcal{M}$, to an order that is sufficient to ensure that the difference disappears in the limit where we take $\partial\mathcal{M}$ to infinity. Here this does not seem to be possible using AdS_4 as a reference solution. However, given the asymptotic structure of the TN-AdS and TB-AdS instantons, it is natural to use TN-AdS space-time as the background for solutions described by the metric (2.1) which have the same asymptotic behavior. It follows that the action of TN-AdS space-time is defined to be *zero*, because it is regarded as the ground state.

The calculation of the action of TB-AdS space-time relative to TN-AdS space-time is just the ''nutty'' generalization of the calculations [10,7] of the action of AdS-Schwarzschild space-time relative to AdS space-time. Just as with these previous calculations, the surface term in Eq. (2.16) does not make any contribution. It follows that we just need to focus on the bulk contribution. Since we are in four dimensions and working with solutions of the vacuum Einstein equations, it follows that the Ricci scalar is given as $R=4\Lambda$, and the bulk action term assumes the form

$$I=-\frac{\Lambda}{8\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}=\frac{3}{8\pi G l^2}\mathrm{Vol}(M). \tag{2.17}$$

We now need to compare the infinite volume contribution of TB-AdS space-time to the infinite contribution of TN-AdS space-time; this difference should give us a finite, physically meaningful answer. For both metrics, one calculates the determinant as

$$\sqrt{g}=2n(r^2-n^2)\sin\theta. \tag{2.18}$$

Taking as our hypersurface $\partial\mathcal{M}$ the fixed radius surface $r=R$, the volume contributions from TB-AdS and TN-AdS space-times thus take the explicit form

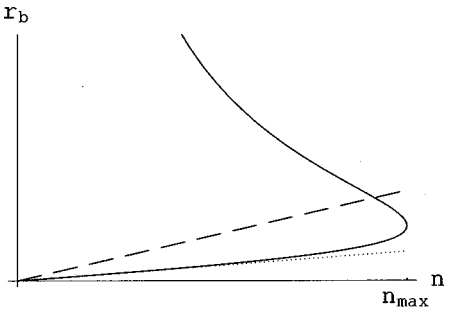

FIG. 1. Plot of r_b as a function of n, up to $n_{\max}$, which limits the existence of TB-AdS solutions. The straight lines are $r_b=2n$ (dotted line) and $r_b=(3+\sqrt{10})n$ (dashed line).

$$\mathrm{Vol}_b(R)=2n\int_0^{4\pi} d\psi\int_{r_b}^{R}(r^2-n^2)dr\int_0^{\pi}\int_0^{2\pi}\sin\theta\, d\theta\, d\varphi \tag{2.19}$$

and

$$\mathrm{Vol}_n(R)=2n\int_0^{4\pi} d\psi\int_{n}^{R}(r^2-n^2)dr\int_0^{\pi}\int_0^{2\pi}\sin\theta\, d\theta\, d\varphi, \tag{2.20}$$

so that the total volume difference is given as the limit, as $R\to\infty$, of $\mathrm{Vol}_b(R)-\mathrm{Vol}_n(R)$. Recalling that we must ensure that the induced metrics of TN-AdS and TB-AdS space-times match on the hypersurface $(r=R)$, we see that we must rescale the nut parameter n_n of TN-AdS space-time to $\lambda(r)n_b$ (where n_b is the nut parameter of TB-AdS space-time), in order that their Euclidean times have the same period to sufficiently high order. [The function $\lambda(r)^2$ is obtained by expanding the ratio of the metric functions $V(r,n,m)$ obtained in each case.]

In this way we find

$$n_b=n_n\left(1+\frac{l^2(m_b-m_n)}{R^3}+O(R^{-4})\right). \tag{2.21}$$

Putting all of this together one therefore obtains the final result for the action of TB-AdS space-time after considerable algebra:

$$I_b=-\frac{2\pi n}{Gl^2}\left(\frac{(r_b-n)^2(r_b^2-6nr_b-n^2)}{r_b-2n}\right). \tag{2.22}$$

We can now analyze for which values of the nut parameter n the action of TB-AdS space-time is larger or smaller than that of TN-AdS space-time, i.e., where I_b is positive or negative. A short inspection shows that I_b is positive only in the range $2n<r_b<(3+\sqrt{10})n$ (of course, we are always considering $r_b\geqslant n$). Figure 1 is a plot of $r=r_b$ as a function of n, in the allowed range of variables $r_b<n_{\max}$. We also include the lines $r=2n$ (dotted line) and $r=(3+\sqrt{10})n$ (dashed line).

We can see that we always have $r_b>2n$ from Eq. (2.14). Note that $r_{b+}\to 2n$ as $n\to 0$. The lower branch r_{b-} lies entirely between $r=2n$ and $r=(3+\sqrt{10})n$, and so the action is always positive for these solutions. On the upper branch

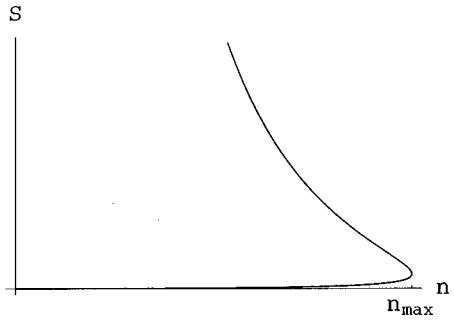

FIG. 2. Entropy S as a function of n.

r_{b+}, the action is positive for the smallest values of r_{b+} (the largest values of n), but as r_{b+} grows (n becomes smaller), the action becomes negative. The crossover point, i.e., $I_b=0$, lies at $n=n_{\rm crit}=l(7-2\sqrt{10})^{1/2}/6$.

III. SOME THERMODYNAMICS

We have performed a covariant computation of the action, as distinct from a Hamiltonian calculation, which would have required a specific time slicing. Such a calculation would have identified a periodic time in an Arnowitt-Deser-Misner (ADM) manner [21], using the temperature $T=1/(8\pi n)$. We expect that such a calculation would have shown that the action decomposes into contributions from the Hamiltonian at infinity and the Misner strings, in addition to the usual terms corresponding to the area of the bolt [22].

We will not carry out a Hamiltonian calculation here, instead moving on to compute various state functions and hence study the physics of the present situation.

We have for the entropy the formula $S=(\beta\partial_\beta-1)I$. Lengthy algebraic manipulations finally yield the entropy in a simple form

$$S=\frac{\pi}{G}(r_b-n)^2\left(1+12\frac{n^2}{l^2}\right). \tag{3.1}$$

This is manifestly positive. It should be noted that this expression differs from $A_{\rm bolt}/4G$: there are contributions to the entropy from the nut charge and nut potential at the bolt [22].

We plot the entropy as a function of n in Fig. 2, including that of the lower branch solutions.

We can compute the thermodynamic energy E:

$$E=\partial_\beta I=\frac{1}{2Gl^2}\frac{(r_b-n)^3(r_b+7n)}{r_b-2n}=\frac{m_b-m_n}{G}, \tag{3.2}$$

where $m_{b,n}$ are the mass parameters for TB-AdS and TN-AdS space-times as given in Eqs. (2.4) and (2.12) above. Since $r_b>2n$, the energy is strictly positive.

We are particularly interested in the very high temperature regime $n\to 0$. In this limit we have

$$r_{b+}=\frac{l^2}{6n}-2n+O(n^3). \tag{3.3}$$

For the upper branch solutions, the action and entropy become

$$I=-\frac{\pi l^4}{108Gn^2}+O(n^0),\quad S=\frac{\pi l^4}{36Gn^2}+O(n^0). \tag{3.4}$$

The entropy coincides in this limit with the limiting value of $A_{\rm bolt}/4G$, showing that in the high temperature regime the effect of the nontrivial topological fibering of the manifold (the contribution from the Misner string [16]) becomes invisible, as could be expected.

Note that the lower branch solutions (which have higher action and lower entropy) have the following behavior at high temperature in the limit $\Lambda\to 0$:

$$r_b=2n+O(n^3),\quad I_b=\frac{\pi n^2}{G}+O(n^4),$$

$$S=\frac{\pi n^2}{G}+O(n^4). \tag{3.5}$$

These are the values obtained in the $\Lambda=0$ Taub-NUT or bolt action calculations of Ref. [13]. This is entirely consistent with the observation, made in Sec. II B, that the lower branch bolt solutions tend to the $\Lambda=0$ solutions in this limit.

Focusing on the upper branch solutions (which will always be more stable: see later), we immediately see[5] that the free energy $F\sim V_2T^3N^{3/2}$ and entropy $S\sim V_2T^2N^{3/2}$ (V_2 is the spatial volume of the field theory), which corresponds to the expected high temperature behavior of a field theory in three space-time dimensions. It is important to note that the growth with N is slower than N^2, confirming that the N is *not* associated with the gauge theory of N $D2$-branes in 10 spacetime dimensions, but rather the more exotic field theory associated with N $M2$-branes in 11 dimensions. (The former flows to the latter in the infrared [23].) The power $N^{3/2}$ counts the number of degrees of freedom of the theory, showing that we are in, roughly speaking, a deconfined phase of the theory. The $N^{3/2}$ factor was first noted in Ref. [24] as associated with the entropy of N coincident $M2$-branes. We consider our present calculations, with their holographic interpretation, as independent support for the conclusion of Ref. [24] that the $(2+1)$-dimensional CFT has $O(N^{3/2})$ degrees of freedom. (This also follows from the results of Refs. [5,7] for the AdS_4–AdS_4-Schwarzschild case, once the appropriate conversions have been made.)

Recall that Taub-bolt-AdS solutions only existed for $n<n_{\rm max}$; the radius of the bolt becomes unphysical. This means that below a certain temperature $T_{\rm min}=1/(8\pi n_{\rm max})$, the solution does not exist, and the TN-AdS solution is the allowed one. Above that temperature, there is apparently a transition to the TB-AdS solution as is evident from the displayed plots in Figs. 2 and 3. However, the transition at $T_{\rm min}$ is merely an artifact of the fact that we not truly in the ther-

[5]Crucially, use the fact that this is an 11-dimensional supergravity compactification; so $G\sim l^{-7}$ (in units where the 11-dimensional Planck length is unity) and $l\sim N^{1/6}$.

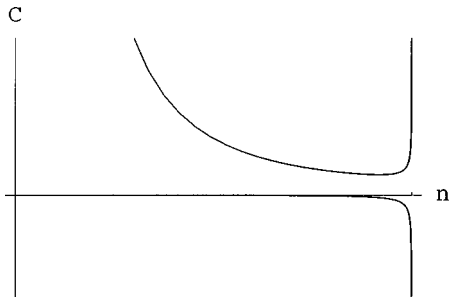

FIG. 3. Specific heat C, at constant volume, as a function of n. It diverges sharply at $n=n_{\max}$.

modynamic limit. More careful consideration below will reveal the transition to be at a higher temperature $T_{\rm crit}$.

In order to study the thermal stability of the system, it is convenient to examine the specific heat $C=-\beta\partial_\beta S=-n\partial_n S$. The analytical expression, however, is not very illuminating. Instead, we provide in Fig. 3 a plot of C as a function of n, which remains positive for the upper branch solutions, negative for the lower branch solutions, and begins to grow rapidly near $T_{\rm min}$ for both branches.

Notice, however, that the action (2.22) is *positive* for $T_{\rm min}<T<T_{\rm crit}=1/(8\pi n_{\rm crit})$. Above $T_{\rm crit}$, it goes negative. This means that the TN-AdS solution is still relevant above $T_{\rm min}$. As the specific heat of the TB-AdS solution is positive, however, we can have a stable bolt as well, and therefore we can nucleate long-lived bolt solutions.

This initial conclusion that there is a phase transition to a (nearly) coexistence phase is modified by the fact that we have not taken the limit $N\to\infty$. To see how the modification comes about, first recall that (after converting $G\sim l^{-7}$ and $l\sim N^{1/6}$) there is a positive power of N multiplying the action I_b, and remember that I_b is the *difference* between that of AdS-TB and AdS-TN space-times.

Therefore, in the thermodynamic limit where we take the number of degrees of freedom (measured by N) to infinity, the action difference is *infinite*. We must conclude therefore that the true phase transition takes place where the *sign* of I_b changes, which is at $T_{\rm crit}$. The free energy is continuous there, but the energy E is discontinuous, and so we conclude that it is a *first order* phase transition: The degrees of freedom are distinct in each case, as shown by the fact that the amount of entropy associated with thermal radiation in the TN-AdS solution is vastly exceeded by the amount which can be stored in the area of the bolt (and Misner strings) in the TB-AdS solution. (Notice that this analysis and discussion also applies to the AdS_4–AdS_4-Schwarzschild case studied in Refs. [5,7], although the more complex thermodynamic conclusions made about the same bulk physics in Ref. [10] are more general, as they are not restricted to the large N limit of this context.)

We remark again that although this represents the physics of transitions between very different gravitational solutions, the complete physics is very plausibly described by the unitary conformal field theory living on the twisted three-sphere at the boundary.

IV. TOPOLOGICALLY TRIVIAL NUTS AND BOLTS

The Taub-NUT-AdS family of metrics contains solutions where the angular spheres (θ,φ) are replaced by planes or hyperboloids. For vanishing nut charge, the solutions correspond to topological black holes [25], studied in Ref. [26] in their M-theory context.

A. Planar nuts and bolts

Let us focus first on the planar (or toroidal) solutions

$$ds^2=V(r)\left(d\tau+\frac{n}{l^2}(xdy-ydx)\right)^2+\frac{dr^2}{V(r)}+\frac{r^2-n^2}{l^2}(dx^2+dy^2),\tag{4.1}$$

where, now,

$$V=\frac{-2mr+l^{-2}(r^4-6n^2r^2-3n^4)}{r^2-n^2}.\tag{4.2}$$

The coordinates x,y here have dimensions of length. Notice that the fibration is now trivial: there are no Misner strings. The topology of the boundary at $r\to\infty$ is therefore $\mathbb{R}^3$. However, although the boundary is topologically a direct product of the Euclidean time line and the spatial plane (x,y), the product is ''twisted'' or warped and the boundary is not flat.

An immediate consequence of the trivial topology is that the Euclidean time period β will not be fixed, as it was in the spherical case, by the value of the nut parameter[6] n. Therefore, in the present case, we can vary the temperature of the system while leaving n fixed. In other words, n labels different sectors of the theory, characterizing the ''warpage'' of the product $\mathbb{R}\times\mathbb{R}^2$. For each sector, we can consider the phase structure as a function of temperature separately.

In the absence of Misner strings, we expect the entropy of the solutions to receive contributions solely from the area of bolts. This expectation will be confirmed below.

Let us now proceed to examine the fixed-point sets of the isometry generated by ∂_τ—the planar nuts and bolts. Nuts will appear as fixed-point sets at $r=n$. One finds that the mass parameter must take the value

$$m_n=-\frac{4n^3}{l^2}\tag{4.3}$$

so that

$$V_n(r)=\frac{(r-n)^2(r+3n)}{l^2(r+n)}.\tag{4.4}$$

Notice that $V_n(r)$ has a double zero at $r=n$. This is, the solution must be regarded as an extremal, zero-temperature

[6]Note that if τ, x, and y are all compactified on a (warped) torus T^3, consistency will demand that the period β be fixed in terms of n. We will not do such a compactification here.

background, since the Euclidean time τ can be identified with arbitrary period. In fact, when $n=0$ we simply recover the AdS_4 metric in horospheric coordinates ($r=1/z$). It is also interesting to note that the mass parameter is negative for all other cases. This might be an indication that the CFT defined on these boundary geometries might be unstable. Although we have not checked this point, these backgrounds are presumably nonsupersymmetric for $n\neq 0$.

Now let us find Taub-bolt-AdS solutions, where ∂_τ has a two-dimensional fixed-point set at some radius $r=r_b>n$. In this case we find that the mass parameter has to be

$$m_b=\frac{1}{2l^2}\left(r_b^3-6n^2r_b-\frac{3n^4}{r_b}\right). \tag{4.5}$$

This time, Euclidean regularity at the bolt requires the period of τ to be

$$\beta=\frac{4\pi}{V'(r_b)}=\frac{4\pi l^2}{3}\frac{r_b}{r_b^2-n^2}. \tag{4.6}$$

As r_b varies from n to infinity, we cover the whole temperature range from 0 to ∞. Notice that m_b can be either negative, zero, or positive. When $n=0$ we recover the standard results for Schwarzschild-AdS_4 spacetime.

As we said above, we can thermally excite each of the sectors labeled by n, keeping n fixed. This requires us to study the thermodynamics of TB-AdS solutions above a TN-AdS background with the same nut charge. As usual, in order to match the geometries at large radius R, we must set

$$\beta_n\sqrt{V_n(R)}=\beta_b\sqrt{V_b(R)}. \tag{4.7}$$

We must also match the values of the nut charges, but this turns out to yield a contribution to the action that vanishes as $R\to\infty$ and, therefore, will be neglected. The computation of the action, which is reduced to a difference of volume terms, is straightforward and yields

$$I_b=-\frac{L^2}{12Gl^2}\left(\frac{r_b-n}{r_b+n}(r_b^2+2nr_b+3n^2)\right), \tag{4.8}$$

where L^2 accounts for the area of the (x,y) plane, $-L/2\leqslant x,y\leqslant L/2$.

Now we find

$$E=\frac{L^2}{8\pi Gl^4}(r_b-n)^2(r_b+2n). \tag{4.9}$$

Notice that, for $n\neq 0$, this is different from the value

$$\frac{L^2}{4\pi Gl^2}(m_b-m_n)=\frac{L^2}{8\pi Gl^4}\frac{(r_b-n)^3(r_b+3n)}{r_b}, \tag{4.10}$$

which could, perhaps, have been expected. This means that in this case one should not think of m as a parameter directly related to the mass.

The action is, for $r_b>n$, always negative. Therefore, as in the $n=0$ case, there are no phase transitions as a function of the temperature and the system stays always in the ''deconfined'' phase.

Finally, the entropy

$$S=(\beta\partial_\beta-1)I=\frac{L^2(r_b^2-n^2)}{4Gl^2}=\frac{A_{\rm bolt}}{4G} \tag{4.11}$$

reproduces the Bekenstein-Hawking [27] result, as it should in the absence of Misner strings.

At high temperatures the entropy behaves in the conformally invariant way, $S\sim\beta^{-2}$. In this regime, the nontrivial warpage for $n\neq 0$ is invisible. However, at lower temperatures the entropy departs from the CFT behavior. This is as expected, since the warpage breaks conformal invariance by introducing a nonvanishing scale, namely, the mass parameter m.

B. No hyperbolic nuts

There is also the possibility of having hyperbolic, instead of spheric or planar, fixed-point sets of ∂_τ. The metric to be used is, in this case,

$$ds^2=V(r)[d\tau+2n(\cosh\theta-1)d\varphi]^2+\frac{dr^2}{V(r)}+(r^2-n^2)(d\theta^2+\sinh^2\theta\,d\varphi^2), \tag{4.12}$$

with

$$V=\frac{-(r^2+n^2)-2mr+l^{-2}(r^4-6n^2r^2-3n^4)}{r^2-n^2}. \tag{4.13}$$

The coordinates (θ,φ) parametrize a hyperboloid and, upon appropriate quotients, surfaces of any genus higher than 1. The fibration is trivial, and, again, there are no Misner strings.

However, if we try to make $r=n$ into a fixed point of ∂_τ, we find that $V(r)$ becomes negative for r close enough to (and bigger than) n. That is, V vanishes at some $r>n$, and instead of a nut, we find a bolt. Thus there are no hyperbolic nuts.

One could study the thermodynamics of these solutions by taking as a background a singular, extremal bolt. However, the holographic significance of these solutions is obscure, as it is for $n=0$, where it has been argued that these systems are likely to be unstable [26].

V. CONCLUSIONS

Having proposed that it should be instructive to revisit the program of studying various quantum gravity processes in the light of the holographic principle (as embodied by the use of AdS space-time), we have enlarged the arena somewhat by studying some examples which are only *locally* asymptotically AdS space-time.

The boundary conformal field theory is the Euclideanized

$(2+1)$-dimensional superconformal field theory compactified on a squashed three-sphere, in one case, and a twisted plane in another, mapping its phase structure to that of Taub-NUT-AdS and Taub-bolt-AdS systems in the bulk.

We find that at high and (to a lesser extent) low temperatures, the thermodynamic properties of the theory are those we expect from general considerations and are consistent with the properties of the dual conformal field theory, including an unambiguous phase transition at $T_{\rm crit}$. It would be interesting to study further the properties of the field theory at intermediate and low temperatures.

In Sec. II C we suggested that AdS_4 space-time could not be used as a background solution in the action calculations. This is because we were unable to embed the asymptotic squashed S^3 into AdS_4 space-time. If this were possible, the phase structure could be even more complicated by the introduction of an AdS_4 phase. However, the fact that our results are consistent with the field theory equivalence suggests that our calculations are correct without such a contribution (at least at high temperatures).

We now have three concrete families of holographic examples of the map between the large N, finite temperature properties of $(2+1)$-dimensional field theory and gravity: that of Refs. [5,7] and those presented here.

As we stressed in the Introduction, this program has sharpened the debate about the nature of various instanton calculations in quantum gravity and may provide a practical answer to the question of the unitarity of processes which include spacelike topology change. We intend to report on further examples in the near future.

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