PHYSICAL REVIEW D, VOLUME 59, 064005

Rotation and the AdS-CFT correspondence

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(Received 19 November 1998; published 1 February 1999)

In asymptotically flat space a rotating black hole cannot be in thermodynamic equilibrium because the thermal radiation would have to be corotating faster than light far from the black hole. However in asymptotically anti–de Sitter space such equilibrium is possible for certain ranges of the parameters. We examine the relationship between conformal field theory in rotating Einstein universes of dimensions two to four and Kerr–anti–de Sitter black holes in dimensions three to five. The five-dimensional solution is new. We find similar divergences in the partition function of the conformal field theory and the action of the black hole at the critical angular velocity at which the Einstein universe rotates at the speed of light. This should be an interesting limit in which to study large N Yang-Mills theory. [S0556-2821(99)00106-X]

PACS number(s): 04.70.Dy, 04.20.−q, 11.25.Hf

I. INTRODUCTION

In Minkowski space the only Killing vector that is timelike everywhere is the time translation Killing vector $\partial/\partial t$. For instance, in four-dimensional Minkowski space, the Killing vector $\partial/\partial t+\Omega\partial/\partial\phi$ that describes a frame rotating with angular velocity Ω becomes spacelike outside the velocity of the light cylinder $r\sin\theta=1/\Omega$.

This raises problems with the thermodynamic interpretation of the Kerr solution: a Kerr solution with a nonzero rotation parameter a cannot be in equilibrium with thermal radiation in infinite space because the radiation would have to corotate with the black hole and so would have to move faster than light outside the velocity of the light cylinder. The best one can do is consider the rather artificial case of equilibrium with rotating radiation in a box smaller than the velocity of the light radius. This problem is inextricably linked with the fact that the Hartle-Hawking state for a Kerr solution does not exist, as proved in Ref. [1]. The absence of the Hartle-Hawking state has a number of important ramifications, the details of which are discussed in Ref. [1].

On the other hand, even a nonrotating Schwarzschild black hole has to be placed in a finite sized box because otherwise the thermal radiation would have infinite energy and would collapse on itself. There is also the problem that the equilibrium is unstable because the specific heat is negative.

It is now well known [2,3] that the specific heat of large Schwarzschild anti–de Sitter black holes is positive and that the redshift in anti–de Sitter spaces acts as an effective box to remove the infinite energy problem. What was less well known except in the rather special three dimensional case was that anti–de Sitter boundary conditions could also remove the faster than light problem for rotating black holes. That is, in anti–de Sitter space there are Killing vectors that are rotating with respect to the standard time translation Killing vector and yet are timelike everywhere. This means that one can have rotating black holes that are in equilibrium with rotating thermal radiation all the way out to infinity.

One would expect [4,3] the partition function of this black hole to be related to the partition function of a conformal field theory in a rotating Einstein universe on the boundary of the anti–de Sitter space. It is the aim of this paper to examine this relationship and draw some surprising conclusions.

Of particular interest is the behavior in the limiting case in which rotational velocity in the Einstein universe at infinity approaches the speed of light. We find that the actions of the Kerr-AdS solutions in four and five dimensions have similar divergences at the critical angular velocity to the partition functions of conformal field theories in rotating Einstein universes of one dimension lower. This is similar to the behavior of the three-dimensional rotating anti–de Sitter black holes and the corresponding conformal field theory on the two-dimensional Einstein universe or cylinder. There is, however an important difference: in three dimensions one calculates the actions of the Bañados-Teitelboim-Zanelli (BTZ) black holes relative to a reference background that is the $M=0$ BTZ black hole. Had one used three dimensional anti–de Sitter space as the reference background, one would have had an extra term in the action which would have diverged as the critical angular velocity was reached.

On the conformal theory side, this choice of reference background is reflected in a freedom to choose the vacuum energy. However, in higher dimensions there is no analogue of the $M=0$ BTZ black hole to use as a reference background. One therefore has to use anti–de Sitter space itself as the reference background. Similarly, there is not a freedom to choose the vacuum energy in the conformal field theory. Any mismatch between the reference background for anti–de Sitter black holes and the vacuum energy of the conformal field theory will become unimportant in the high-temperature limit for nonrotating black holes or the finite temperature but critical angular velocity case. Thus it might be that the black hole–thermal conformal field theory correspondence is valid only in those limits. In that case, maybe we should not be-

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lieve that the large N Yang-Mills theory in the Einstein universe has a phase transition.

In the $(1+1)$-dimensional boundary of three-dimensional anti–de Sitter space, massless particles move to the left or right at the speed of light. The critical angular velocity corresponds to all the particles moving in the same direction. If the temperature is scaled to zero as the angular velocity approaches its critical value, the energy remains finite and the system approaches a Bogomol'nyi-Prasad-Sommerfield (BPS) state.

In higher-dimensional Einstein universes, however, particles can move in transverse directions as well as in the rotation direction or its opposite. At zero angular velocity, the velocity distribution of thermal particles is isotropic but as the angular velocity is increased the velocity distribution becomes peaked in the rotation direction. When the rotational velocity reaches the speed of light, the particles would have to be moving exclusively in the rotation direction. This is impossible for particles of finite energy. Thus rotating Einstein universes of dimension greater than 2 cannot approach a finite energy BPS state as the angular velocity approaches the critical value for rotation at the speed of light.

Corresponding to this, we shall show that four- and five-dimensional Kerr-AdS solutions do not approach a BPS state as the angular velocity approaches the critical value, unlike the three-dimensional BTZ black hole. Nevertheless critical angular velocity may be of interest because one might expect that in this limit super-Yang-Mills would behave similar to a free theory. We postpone to a further paper the question of whether this removes the apparent discrepancy between the gravitational and Yang-Mills entropies.

We should mention that critical limits on rotation have recently been discussed in the context of black three branes in type IIB supergravity [5]: rotating branes are found to be stable only up to a critical value of the angular momentum density, beyond which the specific heat becomes negative. However, our critical limit is different. It corresponds not to a thermodynamic instability, but rather to a Bose condensation effect in the boundary conformal field theory.

In Sec. II we calculate the partition function for conformal invariant free fields in rotating Einstein universes of dimension two, three, and four in the critical angular velocity limit. In Secs. III, IV, and V, we calculate the entropy and actions for rotating anti–de Sitter black holes in the corresponding dimensions and find agreement with the conformal field in the behavior near the critical angular velocity.

The metric for rotating anti–de Sitter black holes in dimensions higher than four was not previously known. Our solutions have other interesting applications, particularly when regarded as solutions of gauge supergravity in five dimensions, which we will discuss elsewhere [6].

II. CONFORMALLY INVARIANT FIELDS IN ROTATING EINSTEIN UNIVERSES

The Maldacena conjecture [4,3] implies that the thermodynamics of quantum gravity with a negative cosmological constant can be modeled by the large N thermodynamics of quantum field theory. We are interested here in probing the correspondence in the limit that the boundary is rotating at the speed of light; that is, we want to study the large N thermodynamics of conformal field theories in an Einstein universe rotating at the speed of light.

The details of the boundary conformal field theory ultimately depend on the details of the bulk supergravity (or string) theory, but generic features such as the divergence of the entropy in this critical limit should be independent of the precise features of the theory. Thus we are led to making the following simplification: instead of considering, for example, the large N limit of $\mathcal{N}=4$ super Yang-Mills (SYM) theory in four dimensions we can just look at the behavior of conformal scalar fields in a rotating Einstein universe. We find that this does indeed give us generic thermodynamic features at high temperature which agree with those found from the bulk theory.

To go further than this, we would have to embed the rotating black hole solutions within a theory for which we know the corresponding conformal field theory. For instance, we could embed the five-dimensional anti–de Sitter–Kerr black holes into IIB supergravity in ten dimensions; we then know that the corresponding conformal field theory is the large N limit of $\mathcal{N}=4$ SYM theory. However, since we cannot calculate quantities in the large N limit of the latter, to obtain the subleading behavior of the partition function would require some approximations or models such as those used in the discussion of rotating three branes in Ref. [5]. It would be interesting to show that the perturbative SYM calculation gives a discrepancy of 4/3 in the entropy as one expects from the results of Ref. [7].

Of course in two dimensions we can do better than this: the two-dimensional conformal field theory is well understood in the context of an old framework [8], where the correspond between bulk and boundary is effectively provided by the modular invariance of the boundary conformal field theory [9,10]. In recent months, the conformal field theory (CFT) has been discussed in some detail, for example in Ref. [11], and one should be able to obtain the subleading dependences of the partition function on the angular velocity Ω. We leave this issue to future work.

It is interesting to note here that there is no equivalent of the zero mass BTZ black hole in higher dimensions. Since the correspondence between the bulk theory and the boundary conformal field theory is clearest when one takes the background to be the BTZ black hole, the correspondence between the conformal field theory and supergravity in the anti–de Sitter background may only be approximate in higher dimensions, valid for high temperature. This is one reason why it is useful to investigate what happens in the critical angular velocity limit.

Let us start with an analysis of conformal fields in a two-dimensional rotating Einstein universe; the metric on a cylinder is

$$ds^2 = -dT^2 + d\Phi^2, \tag{2.1}$$

where we need to identify $\Phi \sim \Phi + \beta\Omega$, and both the inverse temperature β and the angular velocity Ω are dimensionless. Now consider modes of a conformally coupled scalar field, propagating in this background; for harmonic modes, the frequency ω is equal in magnitude to the angular momentum quantum number L. So we can write the partition function for conformally invariant scalar fields as

$$\ln \mathcal{Z} = -\sum \ln(1-e^{-\beta(\omega - L\Omega)}) - \sum \ln(1-e^{-\beta(\omega+L\Omega)}), \tag{2.2}$$

where the first term counts left moving modes and the second term counts right moving modes. The partition function is manifestly singular as one takes the limit $\Omega \to \pm 1$; in this limit, all the particles rotate in one direction. Provided that β is small we can approximate the summation by an integral so that

$$\ln \mathcal{Z} \approx \frac{\pi^2}{6\beta(1-\Omega^2)}, \tag{2.3}$$

which agrees with the high-temperature result found in the next section (3.12) up to a factor and a scale l. Note that the form of this result could also be derived by requiring conformal invariance in the high-temperature limit.

Let us now consider the conformal field theory in three dimensions; a hypersurface of constant large radius in the four-dimensional anti–de Sitter–Kerr metric has a metric which is proportional to a three-dimensional Einstein universe

$$ds^2 = -dT^2 + d\Theta^2 + \sin^2\Theta\, d\Phi^2, \tag{2.4}$$

where Φ must be identified modulo $\beta\Omega$ with β and Ω dimensionless. Now consider a conformally coupled scalar field propagating in this background: the field equation for a harmonic scalar is

$$\left(\nabla - \frac{R_g}{8}\right) = \left(\nabla - \frac{1}{4}\right)\varphi = 0, \tag{2.5}$$

where ∇ is the d'Alambertian and R_g is the Ricci scalar. Modes of frequency ω satisfy the constraint

$$\omega^2 = L(L+1) + \frac{1}{4} = \left(L + \frac{1}{2}\right)^2, \tag{2.6}$$

where L is the angular momentum quantum number. Then the partition function can be written as

$$\ln \mathcal{Z} = -\sum_{L=0}^{\infty}\sum_{m=-L}^{L} \ln(1-e^{-\beta(\omega - m\Omega)}). \tag{2.7}$$

For small β we can approximate this summation as the integral

$$\ln \mathcal{Z} \approx -\int_0^\infty dx_L \int_{-x_L}^{x_L} dx_M \ln(1-e^{-\beta(x_L - \Omega x_M)})$$

$$= \frac{1}{\beta^2}\int_0^\infty dy \int_{-y}^{y} dx \ln(1-e^{-(y-\Omega x)}). \tag{2.8}$$

We are interested in the divergence of the partition function when $\Omega \to \pm 1$; this divergence arises from the modes for which the frequency is almost equal to $|m|$. Of course the frequency can never be quite equal to $|m|$, but for large m the argument of the logarithm in Eq. (2.7) becomes very small. So picking out the modes for which $y = |x|$ in Eq. (2.8) we find that the leading order divergence in the partition function at small β is

$$\ln \mathcal{Z} \approx \frac{\pi^2}{6\beta^2(1-\Omega^2)}, \tag{2.9}$$

which agrees in functional form with the limit that we will find for the bulk action in Sec. IV. In the critical limit, all the particles are rotating at the speed of light in the equatorial plane.

The metric of the four-dimensional rotating Einstein universe can be written as

$$ds^2 = -dT^2 + d\Theta^2 + \sin^2\Theta\, d\Phi^2 + \cos^2\Theta\, d\Psi^2, \tag{2.10}$$

where Φ and Ψ must be identified modulo $\beta\Omega_1$ and $\beta\Omega_2$. We have only approximated the partition function for conformally coupled scalar fields in lower-dimensional rotating Einstein universes. However in Ref. [12] the thermodynamics of conformally coupled scalars were discussed in detail for a four-dimensional rotating Einstein universe in the limit in which one of the angular velocities vanishes. The general form for the partition function found in Ref. [12] is quite complex, but it takes a simple form when β is small: one finds that

$$\ln \mathcal{Z} \approx \frac{\pi^3}{90\beta^3(1-\Omega^2)}, \tag{2.11}$$

where Ω is the angular velocity, which agrees in form with the bulk result to leading order. In principle we could use the partition function density given in Ref. [12] to probe the correspondence between subleading terms.

Let us now try to approximate the partition function for general angular velocities using the same techniques as before. Consider a conformally invariant scalar field propagating in this background; the field equation is

$$\left(\nabla - \frac{R_g}{6}\right) = (\nabla - 1)\varphi = 0, \tag{2.12}$$

and so modes of the field have frequencies ω which satisfy

$$\omega^2 = L(L+2) + 1 = (L+1)^2, \tag{2.13}$$

where L is the orbital angular momentum number. Then the partition function may be written as

$$\ln \mathcal{Z} = -\sum_{L,m_1,m_2} \ln(1-e^{-\beta(\omega - m_1\Omega_1 - m_2\Omega_2)}), \tag{2.14}$$

where m_1 and m_2 are orbital quantum numbers. Suppose that $\Omega_2 = 0$; then we expect the dominant contribution to the partition function in the critical angular velocity limit to be from the $m_1 = \pm L$ modes. However, there is a constraint on the angular momentum quantum numbers

$$|m_1| + |m_2| \leq L, \tag{2.15}$$

and so we need to set $m_2 = 0$. The dominant contribution to the partition function at high temperature can be expressed as

$$\ln \mathcal{Z} \approx -\frac{1}{\beta^3}\int_0^\infty dx[\ln(1-e^{[1+x(1-\Omega)]}) + \ln(1-e^{[1+x(1+\Omega)]})] = \frac{\pi^2}{6\beta^3(1-\Omega^2)}, \tag{2.16}$$

which agrees with the result (2.11) in functional dependence although not coefficient.

For general angular velocities we find that the factor

$$(L - m_1\Omega_1 - m_2\Omega_2) \tag{2.17}$$

only approaches zero in the limit $\Omega_1, \Omega_2 \to 1$. Thus we expect that there is a divergent contribution to the partition function only when either or both of Ω_1 and Ω_2 tend to 1, as we will find when we look at the black hole metric.

Setting $\Omega_1 = \Omega_2 \equiv \Omega$, the dominant contribution to the partition function will come from modes for which the bound (2.15) is saturated. Then we find that

$$\ln \mathcal{Z} \approx -\frac{1}{\beta^3}\int_0^\infty dx\, x[\ln(1-e^{-x(1-\Omega)}) + \ln(1-e^{-x(1-\Omega)})] = \frac{\zeta(3)}{\beta^3(1-\Omega^2)^2}, \tag{2.18}$$

which has the correct dependence on β and Ω to agree with the bulk result found in Sec. V.

III. ROTATING BLACK HOLES IN THREE DIMENSIONS

A. The BTZ black hole

The Euclidean Einstein action in three dimensions can be written as

$$I_3 = -\frac{1}{16\pi}\int d^3x\sqrt{g}[R_g + 2l^2], \tag{3.1}$$

with the three-dimensional Einstein constant set to 1. The Lorentzian section of the BTZ black hole solution first discussed in Ref. [14] is

$$ds^2 = -N^2 dT^2 + \rho^2(N^\Phi dT + d\Phi)^2 + \left(\frac{y}{\rho}\right)^2 N^{-2} dy^2, \tag{3.2}$$

where the squared lapse N^2, the angular shift N^ϕ, and the angular metric ρ^2 are given by

$$N^2 = \left(\frac{yl}{\rho}\right)^2 (y^2 - y_+^2),$$

$$N^\Phi = -\frac{j}{2\rho^2}, \quad \rho^2 = y^2 + \frac{1}{2}(ml^{-2} - y_+^2), \tag{3.3}$$

with the position of the outer horizon defined by

$$y_+^2 = ml^{-2}\sqrt{1-\left(\frac{jl}{m}\right)^2}. \tag{3.4}$$

Note that in these conventions anti–de Sitter spacetime is the $m = -1, j = 0$ solution. Cosmic censorship requires the existence of an event horizon, which in turn requires either $m = -1$, $j = 0$, or $m \geq |j| l$. This bound in fact coincides with the supersymmetry bound: regarded as a solution of the equations of motion of gauged supergravity with zero gravitini, extreme black holes with $m = |j| l$ have one exact supersymmetry. Both the $m = 0$ and the $m = -1$ black holes have two exact supersymmetries. In higher-dimensional anti–de Sitter Kerr black holes the cosmic censorship bound does not coincide with the supersymmetry bound.

The temperature of the black hole is given by

$$T_H = \frac{\sqrt{2ml}}{2\pi}\left[\frac{1-(jl/m)^2}{1+\sqrt{1-(jl/m)^2}}\right]^{1/2}. \tag{3.5}$$

There has been a great deal of interest recently in the BTZ black hole; the action was first calculated in Ref. [14] and has also been discussed in Ref. [11]. However, the action was calculated with respect to the zero mass black hole background, while in the present context we are interested in the action with respect to anti–de Sitter space itself. The reason for this is that in higher dimensions there is no analogue of the zero mass black hole as a background.[1]

To calculate the action of the rotating black hole one first needs to analytically continue both $t \to i\tau$ and $j \to -i\bar{j}$. Using the Euclidean section one finds the action as a function of m, l, and $\bar{j}$. The physical result is then obtained by analytically continuing the angular momentum parameter. Taking the background to be anti–de Sitter space we then find that the Euclidean action (for $m \geq 0$) is given by

$$I_3 = -\frac{\pi}{8\sqrt{2ml}}\left[\frac{1+\sqrt{f}}{f}\right]^{1/2}[3m\sqrt{f} - (2+m)], \tag{3.6}$$

where $f = 1 - (jl/m)^2$. This action diverges in general as f approaches zero, i.e., as we approach the cosmological and

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[1] The metric for which one replaces the lapse function $(1 + l^2y^2)$ by l^2y^2 certainly plays a distinguished role in all dimensions, since this is the metric that one obtains from branes in the decoupling limit. It is not, however, true that this metric is the natural background for rotating black holes in dimensions higher than three but in the high-temperature limit the distinction between the backgrounds will only affect subleading contributions to the action.

supersymmetry bound. One would expect the action to diverge to positive infinity in this limit; from the gravitational instanton point of view, this implies that there is zero probability for anti–de Sitter spacetime to decay into a supersymmetric BTZ black hole.

It is straightforward to show that the energy $\mathcal{M}$, angular momentum J, angular velocity Ω, and entropy S are given by

$$\mathcal{M}=\frac{1}{8}(m+1), \quad J=\frac{j}{8}, \tag{3.7}$$

$$S=\frac{1}{2}\pi\rho(y_+), \quad \Omega=-\frac{j}{2\rho^2(y_+)}.$$

Note that the zero of energy is defined with respect to the anti–de Sitter space rather than the $m=0$ black hole.

The asymptotic form of the Euclidean section of the BTZ metric is

$$ds^2=y^2l^2\,d\tau^2+y^2\,d\Phi^2+\frac{dy^2}{y^2l^2}. \tag{3.8}$$

Regularity of the solution on the boundary of the Euclidean section at $y=y_+$ requires that we must identify $\tau\sim\tau+\beta$ and $\Phi\sim\Phi+i\beta\Omega$, where β is the inverse temperature. The latter identification is necessary because the boundary is a fixed point set of the Killing vector

$$k=\partial_\tau+i\Omega\partial_\Phi. \tag{3.9}$$

The net result of these identifications is that after one analytically continues back to Lorentzian signature one finds that the boundary at infinity is conformal to an Einstein universe rotating at angular velocity Ω.

In the limit that $\Omega\to\pm l$ the surface is effectively rotating at the speed of light: this gives the critical angular velocity limit. Looking back at the form of the metric for the BTZ black hole we find that this limit implies that

$$\Omega=-\frac{jl^2}{m(1+\sqrt{f})}\to\pm l, \tag{3.10}$$

which in turn requires that $f\to 0$. Hence in three dimensions the cosmological and supersymmetry limits coincide with a critical angular velocity limit. However, the temperature necessarily vanishes while in the conformal field theory we have only probed the high-temperature limit. This suggests that one should be able to find a more general critical angular velocity limit. This is indeed the case: if we rewrite the BTZ metric in Kerr form we will be able to find nonextreme states for which the boundary is rotating at the speed of light.

It is useful to rescale the time coordinate so that $\hat{\beta}$ is both finite and dimensionless in the critical limit

$$\hat{\beta}=\sqrt{f}l\beta\approx\frac{2\pi}{\sqrt{2m}}, \tag{3.11}$$

where the latter equality applies for m large. In this limit of small $\hat{\beta}$ the action for the BTZ black hole diverges as

$$I_3\approx\frac{\pi^2}{8l\hat{\beta}(1-\hat{\Omega})}, \tag{3.12}$$

where $\hat{\Omega}=l^{-1}\Omega$ and is hence dimensionless. We would need to know the CFT partition function at low temperature to compare with the CFT and bulk results.

B. Alternative metric for the BTZ black hole

To elucidate the thermodynamic properties of the black hole as one takes the cosmological and supersymmetric limit it is useful to rewrite the metric in the alternative form

$$ds^2=-\frac{\Delta_r}{r^2}\left(dt-\frac{a}{\Xi}d\phi\right)^2+\frac{r^2\,dr^2}{\Delta_r}+\frac{1}{r^2}\left(a\,dt-\frac{1}{\Xi}(r^2+a^2)d\phi\right)^2, \tag{3.13}$$

where we define

$$\Delta_r=(r^2+a^2)(1+l^2r^2)-2Mr^2. \tag{3.14}$$

The motivation for writing the metric in this form is that it then resembles the higher-dimensional anti–de Sitter–Kerr solutions. We have chosen the normalization of the time and angular coordinates so that the latter has the usual period and the former has norm rl at spatial infinity. Rewriting the BTZ black hole metric in Kerr-Schild and Boyer-Lindquist type coordinates was discussed recently in Ref. [13] in the context of studying the global structure of the black hole. Using the coordinate transformations

$$T=t, \quad \Phi=\phi+al^2t, \tag{3.15}$$

$$R^2=\frac{1}{\Xi}(r^2+a^2), \tag{3.16}$$

with $\Xi=1-a^2/l^2$, followed by a shifting of the radial coordinate, we can bring the metric back into the usual BTZ form. The horizons are defined by the zero points of Δ_r, with the event horizon being at

$$r_+^2=\frac{1}{2l^2}(2M-1-a^2l^2)+\frac{1}{2l^2}\sqrt{(1+a^2l^2-2M)^2-4a^2l^2}. \tag{3.17}$$

Expressed in terms of the variables (M,a) the supersymmetry and cosmic censorship conditions become

$$M\geqslant\frac{1}{2}(1+|a|l)^2, \tag{3.18}$$

where the choice of sign of a determines which Killing spinor is conserved in the BPS limit. In the special case M

$\equiv 0$ both supersymmetries are preserved; this is true for all a and not just for the limiting value $|a|l \to 1$ which saturates Eq. (3.18).

As is the case in higher dimensions, the $M=0$ metric is identified three-dimensional anti–de Sitter space. One can calculate the inverse temperature of the black hole to be

$$\beta_t = 4\pi \frac{r_+^2 + a^2}{\Delta_r'(r_+)}. \tag{3.19}$$

In the calculation of the action, only the volume term contributes; the appropriate background is the $M=0$ solution with the imaginary time coordinate scaled so that the geometry matches on a hypersurface of large radius

$$\tau \to \left(1 - \frac{M}{l^2 R^2}\right)\tau. \tag{3.20}$$

Then the action is given by

$$I_3 = -\frac{\pi(r_+^2 + a^2)}{\Xi \Delta_r'(r_+)}[r_+^2 l^2 + a^2 l^2 - M]. \tag{3.21}$$

In this coordinate system the thermodynamic quantities can be written as

$$\mathcal{M}' = \frac{M}{4\Xi}, \quad J' = \frac{Ma}{2\Xi^2},$$

$$\Omega' = \frac{\Xi a}{(r_+^2 + a^2)}, \quad S = \frac{\pi}{2\Xi r_+}(r_+^2 + a^2). \tag{3.22}$$

We now have to decide how to take the limit of critical angular velocity in this coordinate system. The key point is that this coordinate system is not adapted to the rotating Einstein universe on the boundary. The angular velocity of the black hole in this coordinate system vanishes in the limit $al \to 1$ and is always smaller in magnitude than l.

In both this and following sections, we shall adhere to the notation that primed thermodynamic quantities are expressed with respect to the Kerr coordinate system while unprimed thermodynamic quantities are expressed with respect to the Einstein universe coordinate system. We also assume from here onwards that a is positive.

The angular velocity of the rotating Einstein universe is given by

$$\Omega = \Omega' + al^2; \tag{3.23}$$

that is, we need to define the angular velocity with respect to the coordinates (T,Φ). Now suppose that $\Omega = l(1-\epsilon)$ where ϵ is small. This requires that

$$\epsilon = (1-al)\frac{(r_+^2 - a/l)}{(r_+^2 + a^2)}. \tag{3.24}$$

For the Einstein universe on the boundary to be rotating at the critical angular velocity, either $al=1$ or $r_+^2 = a/l$. Note that not only the action but also the entropy is divergent in the limit $al=1$.

Let us explore the limit $r_+^2 = a/l$ first; it is straightforward to show that this coincides with the supersymmetry limit. This means that in every supersymmetric black hole the boundary is effectively rotating at the speed of light, which is apparent from the limit of Ω given in Eq. (3.7). Cosmic censorship requires that $r_+^2 \geqslant a/l$ and hence the rotating Einstein universe never rotates faster than the speed of light. Put another way, any BTZ black hole can be in equilibrium with thermal radiation in infinite space, no matter what its mass is.

The metric of a supersymmetric BTZ black hole is

$$ds^2 = -l^2 y^2\, dT^2 + \frac{jl}{2}(dT - l^{-1}\, d\Phi)^2 + y^2\, d\phi^2 + \frac{dy^2}{l^2 y^2}. \tag{3.25}$$

Now starting from the black hole metric (3.13) and using the coordinate transformations

$$T = t, \quad \Phi = \phi + al^2 t,$$

$$y^2 = \frac{1}{\Xi}(r^2 - a/l), \tag{3.26}$$

the general supersymmetric metric can also be expressed as

$$ds^2 = -l^2 y^2 dT^2 + y^2\, d\Phi^2 + \frac{dy^2}{l^2 y^2}$$

$$+ \frac{al(1+al)^2}{\Xi^2}(dT - l^{-1}\, d\Phi)^2. \tag{3.27}$$

Correspondence between the two metrics requires that

$$m = jl = \frac{2al}{(1-al)^2}. \tag{3.28}$$

So a supersymmetric black hole has a mass which diverges as we take the limit $al \to 1$. This is apparent if we define the thermodynamic quantities with respect to the coordinates (T,Φ). The energy and inverse temperature are unchanged ($\mathcal{M} \equiv \mathcal{M}'$ and $\beta_t \equiv \beta$) while

$$J = \frac{Ma}{2\Xi(1 + l^2 r_+^2)}. \tag{3.29}$$

So the mass and angular momentum of *any* black hole diverge as we take the limit $al \to 1$. It is useful to consider (very nonextreme) black holes which are at high temperature; this requires that $r_+ l \gg 1$ and so if we define a dimensionless inverse temperature

$$\bar{\beta} = l\beta \approx \frac{2\pi}{lr_+}, \tag{3.30}$$

we find that the other thermodynamic quantities behave for $al \to 1$ as

$$I_3 = -\frac{\pi^2}{l\Xi\bar{\beta}}, \quad S = \frac{\pi^2}{l\Xi\bar{\beta}},$$

$$\mathcal{M} = \frac{\pi^2}{2l\Xi\bar{\beta}^2}, \quad J = \frac{1}{4l^2\Xi}, \tag{3.31}$$

where the latter two quantities are defined with respect to the dimensionless temperature. These thermodynamic quantities are consistent both with the thermodynamic relations, and with the result for the partition function of the corresponding conformal field theory.

IV. ROTATING BLACK HOLES IN FOUR DIMENSIONS

Rotating black holes in four dimensions with asymptotic AdS behavior were first constructed by Carter [15] many years ago. There has been interest in such solutions recently as solitons of $N=2$ gauged supergravity in four dimensions [16] and in the context of topological black holes [17]. The metric is

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left[dt - \frac{a}{\Xi}\sin^2\theta d\phi\right]^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\sin^2\theta\Delta_\theta}{\rho^2}\left[a\,dt - \frac{(r^2+a^2)}{\Xi}d\phi\right]^2, \tag{4.1}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta,$$
$$\Delta_r = (r^2+a^2)(1+l^2r^2) - 2Mr,$$
$$\Delta_\theta = 1 - l^2a^2\cos^2\theta,$$
$$\Xi = 1 - l^2a^2. \tag{4.2}$$

The parameter M is related to the mass, a to the angular momentum, and $l^2 = -\Lambda/3$ where Λ is the (negative) cosmological constant. The solution is valid for $a^2 < l^2$, but becomes singular in the limit $a^2 = l^2$ which is the focus of our attention here. The event horizon is located at $r = r_+$, the largest root of the polynomial Δ_r. One can define a critical mass parameter M_e such that [17]

$$M_e l = \frac{1}{3\sqrt{6}}\left[\sqrt{(1+a^2l^2)^2 + 12a^2l^2} + 2(1+a^2l^2)\right] \times \left[\sqrt{(1+a^2l^2)^2 + 12a^2l^2} - (1+a^2l^2)\right]^{1/2}. \tag{4.3}$$

Cosmic censorship requires that $M \geqslant M_e$ with the limiting case representing an extreme black hole. In the limit of critical angular velocity, the bound becomes

$$Ml \geqslant \frac{8}{3\sqrt{3}}, \tag{4.4}$$

which we will see implies that physical black holes must be at least as large as the cosmological scale. The angular velocity Ω' is

$$\Omega' = \frac{\Xi a}{(r_+^2 + a^2)}, \tag{4.5}$$

while the area of the horizon is

$$\mathcal{A} = 4\pi\frac{r_+^2 + a^2}{\Xi}, \tag{4.6}$$

and the inverse temperature is

$$\beta_t = \frac{4\pi(r_+^2+a^2)}{\Gamma_r'(r_+)} = \frac{4\pi(r_+^2+a^2)}{r_+[3l^2r_+^2 + (1+a^2l^2) - a^2/r_+^2]}. \tag{4.7}$$

We should mention here the issue of the normalization of the Killing vectors and the rescaling of the associated coordinates. One choice of normalization of the Killing vectors ensures that the associated conserved quantities generate the SO(3,2) algebra: this was the natural choice in the context of Ref. [16]. Here we have chosen the metric so that the coordinate ϕ has the usual periodicity while the norm of the imaginary time Killing vector at infinity is lr. Note that we are referring to the issue of the normalization of the Kerr coordinates rather than to the relative shifts between Kerr and Einstein universe coordinates.

If we Wick rotate both the time coordinate and the angular momentum parameter

$$t = -i\tau \quad \text{and} \quad a = i\alpha, \tag{4.8}$$

then we obtain a real Euclidean metric where the radial coordinate is greater than the largest root of Δ_r. The surface $r = r_+$ is a bolt of the corotating Killing vector, $\xi = \partial_\tau + i\Omega\partial_\phi$. However, an identification of imaginary time coordinates must also include a rotation through $i\beta\Omega$ in ϕ; that is, we identify the points

$$(\tau, r, \theta, \phi) \sim (\tau + \beta, r, \theta, \phi + i\beta\Omega). \tag{4.9}$$

We now want to calculate the Euclidean action, defined as

$$I_4 = -\frac{1}{16\pi}\int d^4x\sqrt{g}[R_g + 6l^2], \tag{4.10}$$

where we have set the gravitational constant to 1. The choice of background is made by noting that the $M=0$ Kerr-AdS metric is actually the AdS metric in nonstandard coordinates [18]. If we make the implicit coordinate transformations

$$T = t, \quad \Phi = \phi - al^2t,$$

$$y\cos\Theta = r\cos\theta,$$

$$y^2=\frac{1}{\Xi}[r^2\Delta_\theta+a^2\sin^2\theta], \tag{4.11}$$

this takes the AdS metric

$$d\tilde{s}^2=-(1+l^2y^2)dT^2+\frac{1}{1+l^2y^2}dy^2+y^2(d\Theta^2+\sin^2\Theta\, d\Phi^2), \tag{4.12}$$

to the $M=0$ Kerr-AdS form. To calculate the action we need to match the induced Euclidean metrics on a hypersurface of constant radius R by scaling the background time coordinate as

$$\tau\to\left(1-\frac{M}{l^2R^3}\right)\tau, \tag{4.13}$$

and then we find that

$$I_4=-\frac{\pi(r_+^2+a^2)^2(l^2r_+^2-1)}{\Xi r_+\Delta_r'(r_+)}=-\frac{\pi(r_+^2+a^2)^2(l^2r_+^2-1)}{(1-l^2a^2)[3l^2r_+^4+(1+l^2a^2)r_+^2-a^2]}. \tag{4.14}$$

Features of this result are as follows. The action is positive for $r_+^2\leqslant 1/l^2$ and negative for larger r_+; just as for Schwarzschild anti–de Sitter this indicates that there is a phase transition as one increases the mass. The action is clearly divergent for extreme black holes as one would expect. There is also a divergence when $\Xi\to 0$; for small radius black holes the action diverges to positive infinity, while for large radius black holes the action diverges to negative infinity. In the special case $r_+^2=a/l$ the action is finite and positive in the limit $al\to 1$.

Defining the mass and the angular momentum of the black hole as

$$\mathcal{M}'=\frac{1}{8\pi}\int\nabla_a\delta\mathcal{T}_b\,dS^{ab},\quad J'=\frac{1}{4\pi}\int\nabla_a\delta\mathcal{J}_b\,dS^{ab}, \tag{4.15}$$

where $\mathcal{T}$ and $\mathcal{J}$ are the generators of time translation and rotation, respectively, and one integrates the difference between the generators in the spacetime and the background over a celestial sphere at infinity, then we find that

$$\mathcal{M}'=\frac{M}{\Xi},\quad J'=\frac{aM}{\Xi^2}. \tag{4.16}$$

Allowing for the differences in normalization of the generators, these values agree with those given in Ref. [16]. Using the usual thermodynamic relations we can check that the entropy is

$$S=\pi\frac{r_+^2+a^2}{\Xi}, \tag{4.17}$$

as expected. Note that none of the extreme black holes are supersymmetric: in four dimensions there needs to be a nonvanishing electric charge for such black holes, regarded as solutions of a gauged supergravity theory, to be supersymmetric.

It is well known that small Schwarzschild anti–de Sitter black holes are thermodynamically unstable in the sense that their heat capacity is negative, just as for Schwarzschild black holes in flat space. We find such an instability in dimensions $d\geqslant 4$ for black holes whose radius is less than a critical radius which is dimension dependent but is approximately $1/l$. One can show that small rotating black holes are also unstable in this sense but only for rotation parameters of order $0.1l^{-1}$ or less (again the precise limit is dimension dependent); larger angular velocities stabilize the black holes. In three dimensions no anti–de Sitter black holes have negative specific heat.

To take the limit of critical angular velocity, we need to use the coordinate system adapted to the rotating Einstein universe. As in three dimensions the angular velocity of the Einstein universe is given by

$$\Omega=\Omega'+al^2, \tag{4.18}$$

and is defined with respect to the coordinates (T,Φ). Defining $\Omega=l(1-\epsilon)$ as before we find that

$$\epsilon=(1-al)\frac{(r_+^2-a/l)}{(r_+^2+a^2)}. \tag{4.19}$$

Rotation at the critical angular velocity hence requires that either $al=1$ or $r_+^2=a/l$, as in three dimensions. Generically the thermodynamics of the four-dimensional black hole are similar to those of the BTZ black hole, and in fact to those of higher-dimensional black holes also. The (r_+,a) plane for a single parameter black hole in a general dimension is illustrated in Fig. 1.

There is, however, a novelty compared to the three-dimensional case. The cosmological bound permits solutions with $r_+^2<a/l$; for example, in the limiting case $al=1$, the extreme solution has $r_+^2=a^2/3$. To preserve the Lorentzian signature of the metric we require that $al\leqslant 1$, and so $\Omega'>l$ in the limit $r_+^2<a/l$. That is, only for sufficiently large black holes can one have the rotating black hole in equilibrium with thermal radiation in infinite space. This is reflected in the fact that the action changes sign at $r_+=1/l$. In the limit of zero curvature—by taking l to zero—we find, as expected, that there are no rotating black holes for which there is a Killing vector which is timelike right out to infinity.

One can rewrite the thermodynamic quantities of the black hole with respect to the coordinate system (T,Φ). The temperature is unchanged ($\beta\equiv\beta_t$) while the energy and angular momentum are given by

$$\mathcal{M}=\mathcal{M}',\quad J=\frac{aM}{\Xi(1+l^2r_+^2)}. \tag{4.20}$$

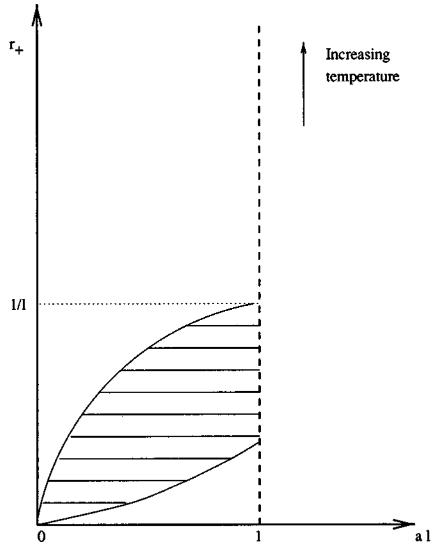

FIG. 1. Plot of black hole radius r_+ against al. For $r_+ < 1/l$, the action is positive, whilst the action blows up along the line $al = 1$. The lower line denotes the radius of the extreme black hole r_c as a function of a. In the hatched region $r_c^2 \leq r_+^2 < a/l$ the Einstein universe on the boundary rotates faster than the speed of light. The action is finite and positive at $r_+^2 = a/l$ but infinite and positive for extreme black holes. In three dimensions the supersymmetric limit coincides with $r_+^2 = a/l$, while in five and higher dimensions the cosmological bound is at $r_c = 0$.

We are particularly interested in the limit of the action as $al \to 1$ at high temperature. Defining a dimensionless quantity

$$\hat{\beta} = l\beta \approx \frac{4\pi}{3lr_+}, \tag{4.21}$$

where the latter relation applies in the high-temperature limit, the action diverges as

$$I_4 = -\frac{8\pi^3}{27l^2\overline{\beta}^2(1-al)}. \tag{4.22}$$

The other thermodynamic quantities behave to leading order as

$$(1-\Omega) = (1-al), \quad \mathcal{M} = \frac{16\pi^3}{27l^2\overline{\beta}^3(1-al)},$$

$$J = \frac{\pi}{3l^3\overline{\beta}(1-al)}, \quad S = \frac{8\pi^3}{9l^2\overline{\beta}^2(1-al)}. \tag{4.23}$$

The entropy diverges at the critical value, as do the energy and the angular momentum. Note that the divergence of the angular momentum is subleading in $\overline{\beta}$. As we stated in the Introduction, there is no sense in which one can take the energy to be finite in the critical limit. If we take $\mathcal{M}$ to be fixed, then M must approach zero in the limit. However according to Eq. (4.4) there is no horizon unless the mass parameter M is of the cosmological scale.

V. ROTATING BLACK HOLES IN FIVE DIMENSIONS

A. Single parameter anti–de Sitter–Kerr black holes

We now consider rotating black holes within a five-dimensional anti–de Sitter background. In five dimensions the rotation group is $SO(4) \cong SU(2)_L \times SU(2)_R$. Black holes may be characterized by two independent projections of the angular momentum vector which may be denoted as the angular momenta J_L and J_R. This is the most natural parametrization when one considers the conformal field theory describing such states but the usual construction of Kerr metrics in higher dimensions will use instead two parameters J_ϕ and J_ψ which we choose such that

$$J_{L,R} = (J_\phi \pm J_\psi), \tag{5.1}$$

where we express the metric on the three sphere in the form

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2. \tag{5.2}$$

The two classes of special cases may be represented by the limits

$$J_R = 0 \Rightarrow J_\phi = J_\psi, \tag{5.3}$$

$$J_L = J_R \Rightarrow J_\psi = 0. \tag{5.4}$$

The former case will be considered in the next subsection. As for the stationary asymptotically flat solutions constructed by Myers and Perry [19], the single parameter Kerr–anti–de Sitter solution in d dimensions follows straightforwardly from the four-dimensional solution. It is convenient to write it in the form

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a}{\Xi}\sin^2\theta\, d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left[a\, dt - \frac{(r^2+a^2)}{\Xi}d\phi\right]^2 + r^2\cos^2\theta\, d\Omega_{d-4}^2, \tag{5.5}$$

where $d\Omega_{d-4}^2$ is the unit round metric on the $(d-4)$ sphere and

$$\Delta_r = (r^2+a^2)(1+l^2r^2) - 2Mr^{5-d},$$

$$\Delta_\theta = 1 - a^2l^2\cos^2\theta,$$

$$\Xi = 1 - a^2l^2,$$

$$\rho^2 = r^2 + a^2\cos^2\theta. \tag{5.6}$$

The angular velocity on the horizon in all dimensions is

$$\Omega' = \frac{\Xi a}{(r_+^2 + a^2)}. \tag{5.7}$$

The thermodynamics of single parameter solutions are generically similar in all dimensions. In five dimensions we can solve explicitly for the horizon position finding that

$$r_+^2=\frac{1}{2l^2}[\sqrt{(1-a^2l^2)^2+8Ml^2}-(1-a^2l^2)]. \tag{5.8}$$

The condition for a horizon to exist is that r_+ must be real, which requires that $2M\geqslant a^2$. The volume of the horizon is

$$V=\frac{2\pi^2}{\Xi}r_+(r_+^2+a^2), \tag{5.9}$$

and the inverse temperature is

$$\beta_t=4\pi\frac{(r_+^2+a^2)}{\Delta_r'(r_+)}=\frac{2\pi(r_+^2+a^2)}{r_+(2l^2r_+^2+1+a^2l^2)}. \tag{5.10}$$

It is useful to note that the $M=0$ Kerr–anti–de Sitter solution reduces to the anti–de Sitter background, with points identified in the angular directions, for all d: this follows from the same coordinate transformation as for the four-dimensional solution. The same coordinate transformation also brings the $M\neq 0$ solution into a manifestly asymptotically anti–de Sitter form.

In calculating the action the appropriate background is the $M=0$ solution, with the imaginary time coordinate rescaled so that the induced metric on a hypersurface of large radius R matches that of the $M\neq 0$ solution. This requires that we scale

$$\tau\rightarrow\left(1-\frac{M}{R^4l^2}\right)\tau. \tag{5.11}$$

The volume term in the action is given by

$$I_5=-\frac{1}{16\pi}\int d^5x(R_g+12l^2) \tag{5.12}$$

(with the gravitational constant equal to one) and the surface term does not contribute. Evaluating the volume term we find that the action is given by

$$I_5=\frac{\pi^2}{4\Xi}\frac{(r_+^2+a^2)^2(1-l^2r_+^2)}{r_+(2l^2r_+^2+1+a^2l^2)}. \tag{5.13}$$

This action has the same generic features as in the lower dimensional cases, namely (i) the sign changes at the critical radius $r_+^2=1/l^2$, (ii) the action diverges as $\Xi\rightarrow 0$ except for black holes of the critical radius $r_+^2=a/l$.

It is straightforward to show that the mass and angular momentum of the rotating black hole with respect to the anti–de Sitter background are given by

$$\mathcal{M}'=\frac{3\pi M}{4\Xi}, \quad J_\phi'=\frac{\pi Ma}{2\Xi^2}. \tag{5.14}$$

Then the usual thermodynamic relations give the entropy of the black hole as

$$S=\beta(\mathcal{M}+\Omega J)-I_5=\pi^2\frac{r_+(r_+^2+a^2)}{2\Xi}, \tag{5.15}$$

which is related to the horizon volume in the expected way.

It is interesting to note that both the temperature and the entropy vanish for black holes with horizons at $r_+=0$, even though the mass and angular momentum can be nonzero. Since a necessary (though nonsufficient) condition for a black hole to be supersymmetric is that the temperature vanishes, only states for which the bound $2M=a^2$ is saturated could be supersymmetric.

Just as the four-dimensional rotating black holes are solutions of $N=2$ gauged supergravity in four dimensions, so we can regard the five-dimensional solutions as solutions of a five-dimensional gauged supergravity theory. However, as in the four-dimensional case, the black holes do not preserve any supersymmetry for nonzero mass unless they are charged.

One can see this as follows. Supersymmetry requires the existence of a supercovariantly constant spinor ϵ satisfying

$$\delta\Psi_m=\hat{D}_m\epsilon=\left(\nabla_m+\frac{1}{2}il\gamma_m\right)\epsilon=0, \tag{5.16}$$

where Ψ is the gravitino, $\hat{D}$ is the supercovariant derivative, ∇ is the covariant derivative, and γ is a five-dimensional gamma matrix. The integrability condition then becomes

$$[\hat{D}_m,\hat{D}_n]\epsilon=0\Rightarrow(R_{mnab}\gamma^{ab}+2l^2\gamma_{mn})\epsilon=0, \tag{5.17}$$

where a,b are tangent space indices. It is straightforward to verify that all components of the bracketed expression vanish for the background while for the rotating black hole the integrability conditions reduce to

$$\frac{M}{\Xi}\gamma_a\epsilon=0. \tag{5.18}$$

Hence only in the zero mass black hole—anti–de Sitter space itself—is any supersymmetry preserved. We expect that supersymmetry can be preserved if we include charges, but leave this as an issue to be explored elsewhere [6]. General static charged solutions of $N=2$ gauged supergravity in five dimensions have been discussed recently in Ref. [20]; one can construct the natural generalizations to general stationary black holes starting from the neutral five-dimensional stationary solutions given here. One can also construct solutions for which the horizon is hyperbolical rather than spherical; such solutions are analogous to those discussed in Ref. [17].

Taking the limit of critical angular velocity requires that we move to the coordinates (T,Φ) which are adapted to the rotating Einstein universe. Then letting $\Omega=l(1-\epsilon)$ with ϵ defined as in Eq. (4.19) we find that in the critical limit either $r_+^2=a/l$ or $al=1$. Since cosmic censorship requires that $r_+\geqslant 0$ with equality in the extreme limit, we can again have solutions for which $\Omega>l$ which in turn implies that the black holes cannot be in equilibrium with radiation right out to infinity. As in four dimensions the action changes sign at the critical value $r_+^2=a/l$.

The thermodynamic quantities relative to the coordinate system (T,Φ) are $\beta\equiv\beta_t$, $\mathcal{M}\equiv\mathcal{M}'$, and

$$J_\Phi = \frac{\pi M a}{2\Xi(1+l^2 r_+^2)}. \tag{5.19}$$

In the limit $al \to 1$ at high temperature such that

$$\bar{\beta} = l\beta \approx \frac{\pi}{l r_+}, \tag{5.20}$$

we can express the thermodynamic quantities as

$$I_5 \approx -\frac{\pi^5}{8l^3 \Xi \bar{\beta}^3}, \quad \mathcal{M} \approx \frac{3\pi^5}{8l^3 \Xi \bar{\beta}^4},$$

$$J_\Phi \approx \frac{\pi^3}{2l^4 \Xi \bar{\beta}^2}, \quad S \approx \frac{\pi^5}{2l^3 \Xi \bar{\beta}^3}, \tag{5.21}$$

where the energy and angular momentum are defined with respect to the dimensionless temperature. Note that the angular momentum is again subleading in $\bar{\beta}$ dependence relative to the mass and the action. The divergence at critical angular velocity is in agreement with that of the conformal field theory.

B. General five-dimensional AdS-Kerr solution

The metric for the two parameter five-dimensional rotating black hole is given by

$$\begin{aligned} ds^2 = &-\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi\right)^2 \\ &+ \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\,dt - \frac{(r^2+a^2)}{\Xi_a}d\phi\right)^2 \\ &+ \frac{\Delta_\theta \cos^2\theta}{\rho^2}\left(b\,dt - \frac{(r^2+b^2)}{\Xi_b}d\psi\right)^2 + \frac{\rho^2}{\Delta}dr^2 \\ &+ \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{(1+r^2l^2)}{r^2\rho^2}\left(ab\,dt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi\right. \\ &\left. - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2, \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} \Delta &= \frac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2l^2) - 2M, \\ \Delta_\theta &= (1 - a^2l^2\cos^2\theta - b^2l^2\sin^2\theta), \\ \rho^2 &= (r^2 + a^2\cos^2\theta + b^2\sin^2\theta), \\ \Xi_a &= (1-a^2l^2), \quad \Xi_b = (1-b^2l^2). \end{aligned} \tag{5.23}$$

It should be straightforward to construct the metric for general rotating black holes in anti–de Sitter backgrounds of higher dimension. As for the single parameter solution, the $M=0$ metric is anti–de Sitter space, with points identified in the angular directions. Using the coordinate transformations

$$\begin{aligned} \Xi_a y^2 \sin^2\Theta &= (r^2+a^2)\sin^2\theta, \\ \Xi_b y^2 \cos^2\Theta &= (r^2+b^2)\cos^2\theta, \\ \Phi &= \phi + a l^2 t, \\ \Psi &= \psi + b l^2 t, \end{aligned} \tag{5.24}$$

the metric can be brought into a form which is manifestly asymptotic to anti–de Sitter spacetime. The parameters a and b are constrained such that $a^2, b^2 \leq l^{-2}$ and the metric is only singular if either or both parameters saturate this limit.

Defining the action as in Eq. (5.12) we find that

$$I_5 = -\frac{\pi\beta l^2}{4\Xi_a\Xi_b}[(r_+^2+a^2)(r_+^2+b^2) - Ml^{-2}], \tag{5.25}$$

where the inverse temperature is given by

$$\beta_t = \frac{4\pi(r_+^2+a^2)(r_+^2+b^2)}{r_+^2\Delta'(r_+)}, \tag{5.26}$$

and r_+ is the location of the horizon defined by

$$(r_+^2+a^2)(r_+^2+b^2)(1+r_+^2l^2) = 2Mr_+^2. \tag{5.27}$$

For real a, b, and l there are two real roots to this equation; when $a=b$ these coincide to give an extreme black hole when

$$\begin{aligned} r_c^2 &= \frac{1}{4l^2}(\sqrt{1+8a^2l^2} - 1), \\ 2M_c l^2 &= \frac{1}{16}(\sqrt{1+8a^2l^2} - 1 + 4a^2l^2) \\ &\quad \times (3\sqrt{1+8a^2l^2} + 5 + 4a^2l^2). \end{aligned} \tag{5.28}$$

The entropy of the general two parameter black hole is given by

$$S = \frac{\pi^2}{2r_+\Xi_a\Xi_b}(r_+^2+a^2)(r_+^2+b^2), \tag{5.29}$$

while the mass and angular momenta are

$$\mathcal{M}' = \frac{3\pi M}{4\Xi_a\Xi_b}, \quad J'_\phi = \frac{\pi M a}{2\Xi_a^2}, \quad J'_\psi = \frac{\pi M b}{2\Xi_b^2}, \tag{5.30}$$

with the angular velocities on the horizon being

$$\Omega'_\phi = \Xi_a \frac{a}{r_+^2+a^2}, \quad \Omega'_\psi = \Xi_b \frac{b}{r_+^2+b^2}. \tag{5.31}$$

Since the black hole is singular only when either or both of Ξ_a and Ξ_b tend to zero, we should look in particular at the latter case for which the two rotation parameters a and b are equal in magnitude. Then we can write the metric in the transformed coordinates as

$$ds^2=-(1+y^2l^2)dT^2+y^2(d\Theta^2+\sin^2\Theta\, d\Phi^2 +\cos^2\Theta\, d\Psi^2)+\frac{2M}{y^2\Xi^2}(dT-a\sin^2\Theta\, d\Phi -a\cos^2\Theta d\Psi)^2 +\frac{y^4\,dy^2}{[y^4(1+y^2l^2)-(2M/\Xi^2)y^2+(2Ma^2/\Xi^3)]}, \quad (5.32)$$

where $\Xi=1-a^2l^2$. The position of the horizon of the extreme solution in these coordinates is

$$y^2=\frac{1}{4\Xi}[4a^2l^2-1+\sqrt{1+8a^2l^2}]. \quad (5.33)$$

In the critical limit, $al\to1$, the size of the black hole becomes infinite in this coordinate system.

One can check to see whether the integrability condition (5.17) is satisfied by the black hole metric (5.32). Preservation of supersymmetry requires that

$$\frac{M}{\Xi^2}\gamma_a\epsilon=0, \quad (5.34)$$

and hence only in the zero mass black hole is any supersymmetry preserved. We have not checked the integrability condition in the general two parameter rotating black hole but do not expect supersymmetry to be preserved. In three dimensions the integrability condition is trivially satisfied since the BTZ black hole is locally anti–de Sitter and supersymmetry preservation relates to global effects. In higher dimensions it does not seem possible to satisfy the integrability conditions without including gauge fields.

The Einstein universe rotates at the speed of light in at least some directions either when one or both of Ξ_a and Ξ_b vanish or when $r_+^2=a/l$ or when $r_+^2=b/l$. The action is singular when either or both of Ξ_a and Ξ_b are zero and when the black hole is extreme. The action is positive for $r_+\leqslant 1/l$; there is a phase transition as the mass of the black hole increases.

If $r_+^2=a/l$ the action will be positive and finite when Ξ_a vanishes and positive and infinite when Ξ_b vanishes. For $lr_+^2<\max[a,b]$ there will be directions in the Einstein universe which are rotating faster than the speed of light. In the limiting case $a=b$ the action diverges for all r_+ as Ξ tends to zero.

In the high-temperature limit, the action for the equal parameter rotating black hole diverges as

$$I_5=-\frac{\pi^5}{8l^3\Xi^2\bar{\beta}^3}, \quad (5.35)$$

with $\bar{\beta}\approx\pi/(r_+l)\ll1$. The other thermodynamic quantities follow easily from this expression, and are in agreement with those derived from the conformal field theory in Sec. II.

We should mention what we expect to happen in higher dimensions. A generic rotating black hole in d dimensions will be classified by $[(d-1)/2]$ rotation parameters a_i, where $[x]$ denotes the integer part of x. Thus we expect both the action and the metric to be singular if any of the a_i vanish. Provided that the black hole horizon is at $r_+>1/l$ the action should diverge to negative infinity in the critical limit, behaving as

$$I_d\sim-\frac{1}{\beta^{d-2}\prod_i\epsilon_i}, \quad (5.36)$$

where $\epsilon_i=1-\Omega_i$ and the product is taken over all i such that $a_il\to1$. The β dependence follows from conformal invariance, whilst one should be able to derive the ϵ_i dependence by looking at the behavior of the spherical harmonics.

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