Mapping Hawking into Unruh thermal properties

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By globally embedding curved spaces into higher dimensional flat ones, we show that Hawking thermal properties map into their Unruh equivalents: The relevant curved space detectors become Rindler ones, whose temperature and entropy reproduce the originals. Specific illustrations include Schwarzschild, Schwarzschild–(anti-)de Sitter, Reissner-Nordström, and Bañados-Teitelboim-Zanelli spaces. [S0556-2821(99)07004-6]

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I. INTRODUCTION

It is well understood that, for both Hawking and Unruh effects, temperature emerges from information loss associated with real and accelerated-observer horizons, respectively. Given that any D-dimensional geometry has a higher-dimensional global embedding Minkowskian (possibly with more than one timelike coordinate) spacetime (GEMS) [1], it is natural to ask whether these mappings can unify the two effects, by associating the relevant detectors of the curved spaces and their horizons with (constant acceleration) Rindler detectors and their horizons. Confirmation of these ideas was recently given in an analysis of de Sitter (dS) [2] and anti–de Sitter (AdS) [3] geometries and their GEMS. There, constantly accelerated observers were mapped into similar ones in the GEMS. The resulting Unruh temperatures associated with these Rindler motions agreed with those in the original dS and AdS spaces. (Actually, AdS has no real horizon, but temperature is well defined for sufficiently large accelerations and the two methods agree both as to the range where T exists and to its magnitude.) In the present paper,[1] we will show that the GEMS approach indeed provides a unified derivation of temperature for a wide variety of curved spaces, including general rotating Bañados-Teitelboim-Zanelli (BTZ), Schwarzschild together with its dS and AdS extensions, and Reissner-Nordström. In each case the usual black-hole (BH) detectors are mapped into Rindler observers with the correct temperature as determined from their (constant) accelerations. Conversely, we will also connect surface gravity and Unruh temperatures, for both Rindler observers in flat space and various accelerated observers in de Sitter (dS) and anti–de Sitter (AdS) spaces, thereby establishing the equivalence principle between constant acceleration and ''true'' gravity effects. We will also consider the associated extensive quantity, the entropy, and again show the mapping correctly matches the area of the GEMS Rindler motion and ''true'' horizons, thereby confirming the equivalence for entropy as well.

We will first review how temperature measured by an accelerated detector in dS and AdS geometries, say in $D=4$, is just its Unruh temperature (i.e., Rindler acceleration divided by 2π) in the $D=5$ GEMS, by relating the corresponding 4- and 5-accelerations. In this connection we will also explicitly relate surface gravity to the associated temperatures. Next we shall treat rotating and non-rotating $D=3$ BTZ spaces [5,6]. Since BTZ is obtained from AdS through geodesic identification, we will show that we can use the treatment of Unruh observers in AdS to calculate the BH temperature here as well, in agreement with earlier results. Our final applications will be to Schwarzschild, Schwarzschild–dS, Schwarzschild–AdS and Reissner-Nordström spacetimes, where the same connections are made, this time the required GEMS extensions having $D\geqslant 6$. More generally, it will be seen that for any geometry admitting a group of constantly accelerated observers which encounter a horizon as they follow a ''bifurcate'' timelike Killing vector field, the temperature measured by each observer is simply $2\pi T=a_G$ when a_G is their acceleration as mapped into the GEMS. Finally, we will establish equivalence of entropies using the Unruh definition in terms of the ''transverse'' Rindler area [7], together with the fact that horizons map into horizons.

II. SURFACE GRAVITY-UNRUH EFFECT CONNECTION IN dS AND AdS

We begin with a brief summary of the GEMS approach to temperature given in [3], for dS/AdS spaces of cosmological constant $\Lambda\equiv\pm 3R^{-2}$; these are hyperboloids in the $D=5$ GEMS $ds^2=\eta_{AB}(dz^A)^2(dz^B)^2$,

$$\eta_{AB}(z^A)^2(z^B)^2=\mp R^2. \tag{1}$$

Here $A,B=0\dots 4$, $\eta_{AB}=\mathrm{diag}(1,-1,-1,-1,\mp 1)$; throughout, upper/lower signs refer to dS/AdS, respectively. We specifically consider $z^1=z^2=0$ and $z^4=Z=$const trajectories, obeying $(z^1)^2-(z^0)^2=\pm R^2\mp Z^2\equiv a_5^{-2}$. Now the Unruh effect states that flat space detectors with constant acceleration a along the x direction, whose motions are thus on $x^2-t^2=a^{-2}$, measure temperature $2\pi T=a$. Since our embedding space detectors follow precisely such trajectories i.e., have a Rindler-like motion with constant acceleration a_5, they measure

$$2\pi T=a_5=(\pm R^2\mp Z^2)^{-1/2}\equiv(\pm R^{-2}+a^2)^{1/2}. \tag{2}$$

The last equality expresses the temperature in term of the $D=4$ quantities, using $a_5^2=\pm R^{-2}+a^2$.

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[1]A brief summary of part of this work was given in [4].

The relation between the Hawking-Bekenstein horizon surface gravity k_H and the BH temperature (originally found for Schwarzschild BH) [8,9]

$$T=\frac{1}{2\pi}\frac{k_H}{\sqrt{g_{00}}}, \tag{3}$$

where x^0 is the time-like Killing vector of a detector in its rest frame, holds also for Schwarzschild–AdS and BTZ spacetimes [10]. For these latter two, the local temperature vanishes at infinity, and no Hawking particles are present far from the BH: created at the horizon, they do not have enough energy to escape to infinity (where the ''effective potential'' becomes infinite). The connection (3) between temperature and surface gravity also holds [11] for Rindler motions, reinforcing the connection between the Hawking and Unruh effects as being based on the existence of horizons, whether ''real'' or just seen by accelerated observers. In both cases, inserting the horizon surface gravity in Eq. (3) will give the temperature. To calculate T, it is convenient to use the detector rest frame.[2] The simplest example is the flat space Rindler observer, best described by Rindler coordinates (τ,ζ)

$$ds^2=L^2\exp(2\zeta)(d\tau^2-d\zeta^2)-(dy^2+dz^2). \tag{4}$$

A $\zeta=$const detector (following the timelike Killing vector $\xi=\partial_\tau$) has a constant acceleration $a=L^{-1}\exp(-\zeta)$. This group of accelerated observers sees an event horizon at $\zeta=-\infty$. Since ξ is perpendicular to the horizon (and therefore null) we can calculate the surface gravity using its definition [11]

$$k_H^2=-\frac{1}{2}(\nabla^\mu\xi^\nu)(\nabla_\mu\xi_\nu), \tag{5}$$

where the right side is to be evaluated at the horizon. For us

$$k_H^2=k^2(\zeta=-\infty)=1. \tag{6}$$

Inserting k_H in Eq. (3) gives the desired result

$$2\pi T=L^{-1}\exp(-\zeta)=a. \tag{7}$$

Let us show that use of surface gravity to calculate temperature also works for dS and AdS. Consider first dS with its real horizon, expressed in the static coordinates (t,r,θ,ϕ) related to the z^A according to[3]

$$z^0=\sqrt{R^2-r^2}\sinh(t/R)\quad z^1=\sqrt{R^2-r^2}\cosh(t/R),$$
$$z^2=r\sin\theta\cos\phi\quad z^3=r\sin\theta\sin\phi\quad z^4=r\cos\theta. \tag{8}$$

The metric

$$ds^2=\left[1-\frac{r^2}{R^2}\right]dt^2-\left[1-\frac{r^2}{R^2}\right]^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{9}$$

has an intrinsic horizon at $r=R$. It is seen by ''static'' detectors (r,θ,ϕ const), or equivalently (choosing $\theta=0$, as is allowed by symmetry) $z^1=z^2=0$ and $z^4=r=Z=$const. They follow the time-like Killing vector ∂_t and have constant acceleration $a=r/(R\sqrt{R^2-r^2})$. Hence, using Eq. (5), we have

$$k_H=1/R \tag{10}$$

and the temperature measured by these detectors agrees with the known results of [2],

$$T=\frac{1}{2\pi}\frac{1}{\sqrt{R^2-r^2}}=\frac{1}{2\pi}\sqrt{\frac{1}{R^2}+a^2}. \tag{11}$$

In AdS,

$$ds^2=\left[1+\frac{r^2}{R^2}\right]dt^2-\left[1+\frac{r^2}{R^2}\right]^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{12}$$

there is no intrinsic horizon. So although $r=$const detectors have constant acceleration $a=r/(R\sqrt{R^2+r^2})<R^{-1}$, they will not measure any temperature. The intrinsic horizon of dS causes even inertial detectors to measure temperature, while in AdS the absence of a real horizon causes sufficiently slowly ($a<R^{-1}$) accelerated detectors not to measure one. There is no contradiction with the Unruh picture: as we will see, the GEMS acceleration a_5^2 becomes negative for them.[4,5] Indeed the ''GEMS temperature'' was obtained only for $(z^4)^2=\text{const}^2>R^2$ ($a>R^{-1}$) trajectories there [3]. Using the formula for time-like trajectories with $a<R^{-1}$ [not $(z^4)^2>R^2$ trajectories, but for example the $z^1=$const, or the $r=$const case we discussed above] would lead to imaginary T: the detector will not measure any temperature because it sees no event horizon, hence no loss of information. To calculate the temperature using Eq. (3) when $a>R^{-1}$, it is con-

[2]The vacuum states in these timelike Killing coordinate systems are Schwarzschild-like. Therefore, determining the temperature by the (lowest order) transition rate obtained from the Wightman function for these vacua gives zero temperature, while the same calculation for Hawking-Hartle and Kruskal-like vacua gives the temperature (3).

[3]Although this coordinate transformation covers only part of the space, it is easy to extend it continuously to the whole dS, resulting in a global embedding.

[4]If we take the imaginary point $r_H=\pm iR$ to define the AdS ''horizon'' and calculate the surface gravity at that point, Eq. (3) will give, as expected, an imaginary temperature $2\pi T=\pm i(R^2+r^2)^{-1/2}=\sqrt{-R^{-2}+a^2}$, but (by the last equality) the correct temperature formula for AdS [3].

[5]It is also possible to get the AdS result from that of Schwarzschild–AdS [10], not by taking the limit $m\to0$ but only by setting $m=0$ initially. This is exactly like the impossibility of reaching flat space by taking the $m\to0$ limit of the Hawking temperature formula for Schwarzschild space.

venient to use a new coordinate system (the one in [3] is not suitable here since its x^0 is not the time-like Killing vector followed by the observers). Instead we introduce an ''accelerated'' coordinate system obtained by the GEMS coordinates defined from the $D=4$ covering of AdS,

$$ds^2=\frac{-R^2+\rho^2}{R^2}d\eta^2-\frac{R^2}{-R^2+\rho^2}d\rho^2 -\rho^2(d\psi^2+\sinh^2\psi d\theta^2), \tag{13}$$

as follows:

$$z^0=\sqrt{-R^2+\rho^2}\sinh(\eta/R),\quad z^1=\sqrt{-R^2+\rho^2}\cosh(\eta/R),$$
$$z^2=\rho\sinh\psi\cos\theta,\quad z^3=\rho\sinh\psi\sin\theta,\quad z^4=\rho\cosh\psi. \tag{14}$$

Here $-\infty<\eta,\ \psi<\infty,\ -\pi<\theta<\pi$; while this coordinate patch only covers the region $\rho>R$, it can be extended to the entire space. Since we are interested in $z^1=z^2=0$, $z^4=\text{const}$ trajectories, ψ is set to zero, and ρ to a constant Z; their accelerations are $a^2=Z^2R^{-2}(Z^2-R^2)^{-1}>R^{-2}$. For AdS, the horizon appears in this ''accelerated'' frame exactly as it did upon transforming from Minkowski to Rindler coordinates in flat space. These trajectories follow the time-like Killing vector field ∂_η which is null at the event horizon $\rho=R$, so Eq. (5) gives

$$k_H=R^{-1}. \tag{15}$$

The corresponding temperature, from Eq. (3), is

$$2\pi T=(-R^2+Z^2)^{-1/2}=(-R^{-2}+a^2)^{1/2}, \tag{16}$$

which is exactly the result obtained using the kinematical behavior of these trajectories in the GEMS, as well as by calculating the transition rate in the ''nonaccelerated'' coordinate system.

III. BTZ SPACES

In the previous section, we demonstrated the feasibility of using surface gravity (or equivalently the Hawking-Bekenstein temperature) to calculate the temperature measured in dS and AdS, in agreement with that obtained by purely kinematical Unruh considerations. This immediately raises the converse question: calculate Hawking temperature entirely from GEMS kinematics when ''real,'' mass-related, horizons are present. The simplest candidate for this would seem to be the BTZ black hole solution, due to its relation to AdS; we now use our method to calculate BTZ temperature, at least for some observers, and compare with previous calculations using surface gravity [5,12].

The general rotating BTZ black hole is described by the 3-metric

$$ds^2=N^2dt^2-N^{-2}dr^2-r^2(d\phi+N^\phi dt)^2,$$
$$N^2\equiv(r^2-r_+^2)(r^2-r_-^2)/(r^2R^2),\quad N^\phi\equiv-r_+r_-/(r^2R). \tag{17}$$

It arises from AdS upon making the geodesic identification $\phi=\phi+2\pi$. The coordinate transformations to the $(2+2)$ AdS GEMS $ds^2=(dz^0)^2-(dz^1)^2-(dz^2)^2+(dz^3)^2$ are, for $r\geqslant r_+$ (the extension to $r<r_+$ is given in [6]),

$$z^o=R\sqrt{\frac{r^2-r_+^2}{r_+^2-r_-^2}}\sinh\left(\frac{r_+}{R^2}t-\frac{r_-}{R}\phi\right),$$
$$z^1=R\sqrt{\frac{r^2-r_+^2}{r_+^2-r_-^2}}\cosh\left(\frac{r_+}{R^2}t-\frac{r_-}{R}\phi\right),$$
$$z^2=R\sqrt{\frac{r^2-r_-^2}{r_+^2-r_-^2}}\sinh\left(\frac{r_+}{R}\phi-\frac{r_-}{R^2}t\right),$$
$$z^3=R\sqrt{\frac{r^2-r_-^2}{r_+^2-r_-^2}}\cosh\left(\frac{r_+}{R}\phi-\frac{r_-}{R^2}t\right), \tag{18}$$

where the constants (r_+,r_-) are related to the mass and angular momentum. This AdS GEMS can serve as the BTZ embedding space for our purpose. In spite of the fact that there is no longer a one to one mapping between it and the BTZ space due to the ϕ identification, following a detector motion with certain initial condition such as $\phi(t=0)=0$ still gives a unique trajectory in the embedding space which is the basic requirement of our approach based on the observer's kinematical behavior in the GEMS: If the detector trajectory maps (without ambiguity) into an Unruh one in the GEMS, then we can use it for temperature calculation.

Consider first non-rotating BTZ $(r_-=0)$ and focus on ''static'' detectors $(\phi,r=\text{const})$. These detectors have constant 3-acceleration $a=rR^{-1}(r^2-r_+^2)^{-1/2}$, and are described by a (fixed) point in the (z^2,z^3) plane (for example $\phi=0$ gives $z^2=0$ $z^3=\text{const}$), and constant accelerated motion in (z^0,z^1) with $a_4=r_+R^{-1}(r^2-r_+^2)^{-1/2}$. So in the GEMS we have a constant Rindler-like accelerated motion and the temperature measured by the detector is

$$2\pi T=a_4=r_+R^{-1}(r^2-r_+^2)^{-1/2}=(-R^{-2}+a^2)^{1/2}, \tag{19}$$

which is that obtained using Eq. (3), and agrees with the temperature given by the response function of particle detectors [13]. In the asymptotic limit $r\to\infty$, BTZ tends to AdS, the acceleration $a\to R^{-1}$, which is of course the acceleration of a ''static'' detector at infinity in AdS; both detectors measure zero temperature[6] (no Hawking particle at infinity). The rotating case is more complicated. The Hawking temperature $2\pi T=(Rr)(r^2-r_+^2)^{-1/2}(r^2-r_-^2)^{-1/2}k_H$, $k_H=(r_+^2-r_-^2)(r_+R^2)^{-1}$, was calculated [12,10] for trajectories that follow the time-like Killing vector

[6]BTZ formally becomes AdS in our coordinates by setting $r_-=0$ and $r_+=\pm iR$; Eq. (17) and the $D=3$ version of Eq. (12) are the same. This shows again that AdS has a hidden imaginary horizon which causes the threshold in the temperature (acceleration smaller than R^{-1} measures no temperature).

$\xi=\partial_t-N^\phi\partial_\phi$, i.e., observers that obey $\phi=-N^\phi t$, $r=$const (and hence are ''static'' at infinity). Although they have a constant $D=3$ acceleration,

$$a=(r^4-r_-^2r_+^2)/[r^2R\sqrt{(r^2-r_-^2)(r^2-r_+^2)}],$$

these trajectories do not describe pure Rindler motion in the GEMS, combining accelerated motion in the (z^0,z^1) plane with a space-like motion in (z^3,z^2). Therefore, we cannot use their kinematical behavior in these GEMS to calculate the temperature they measure. Exactly the same problem would arise for any AdS detector with $\psi\neq$const in Eq. (14). This particular case resembles AdS motion with $\psi=\alpha(r)t$, $\theta=0$. Our method can be used only for a group of detectors that maps into a group of pure Unruh observers in the GEMS. Hence, it is only possible to use it for those observers for whom the map of the detector trajectory into the ''transverse'' embedding space (for BTZ the z^2,z^3 plane) is time-independent, i.e., the detector motion at any time is described by a fixed point in that plane. There is one group of time-like observers obeying

$$\phi=\frac{r_-}{r_+R}t, \quad r=\text{const}, \tag{20}$$

which does allow us to use the above GEMS and hence to compare the two calculation of T. These detectors have a constant acceleration $a=(r^2-r_-^2)^{1/2}(r^2-r_+^2)^{-1/2}R^{-1}$ in BTZ and a Rindler-like motion in the GEMS with acceleration $a_4=R^{-1}(r_+^2-r_-^2)^{1/2}(r^2-r_+^2)^{-1/2}$ and, therefore, measure

$$2\pi T=a_4=\frac{1}{R}\sqrt{\frac{r_+^2-r_-^2}{r^2-r_+^2}}=\sqrt{-R^{-2}+a^2}. \tag{21}$$

On the other hand, inserting $\phi=(r_-/r_+l)t$ into Eq. (17) gives

$$ds^2=\frac{(r^2-r_+^2)(r_+^2-r_-^2)}{r_+^2R^2}dt^2-\frac{r^2R^2}{(r^2-r_+^2)(r^2-r_-^2)}dr^2, \tag{22}$$

which show us that they follow the time-like Killing vector field $\xi=\partial_t$ for this metric [or $\xi=\partial_t+r_-(r_+R)^{-1}\partial_\phi$, if we use Eq. (17)] and see an event horizon at the metric's own ''real'' event horizon $r=r_+$. The surface gravity is

$$k_H=(r_+^2-r_-^2)/(r_+R^2), \tag{23}$$

which is the same as that calculated for the other group by using the other Killing vector. This equivalence exists since both have the same horizon $r=r_+$ and the Killing vectors they follow are the same there. Any scaling problems are avoided since we used a common coordinate system. While surface gravity can be obtained from either of the metrics (22) or (17), the appropriate g_{00} must taken from Eq. (22) because only there is x^0 the time-like Killing vector followed by the observer. This gives

$$2\pi T=\frac{1}{R}\sqrt{\frac{r_+^2-r_-^2}{r^2-r_+^2}}, \tag{24}$$

exactly the result obtained by using the GEMS. Finally, we note that a common alternative definition of BH temperature is to scale T by $\sqrt{g_{00}}$: $T_0=\sqrt{g_{00}}T=k_H/2\pi$; as distinct from the local temperature T, it is T_0 that enters into the BH thermodynamics relations. Since there is one observer (the $r=r_+$ one) that belongs to both of the different observer groups [$\phi=-N^\phi t$ and $\phi=r_-t/(r_+R)$], and since T_0 is a global feature of all the members in the group, it is obvious that both groups should give the same temperature (this of course could be seen immediately from their surface gravity equivalence). On the other hand, it should be no surprise that detectors in the two different observer groups measure different temperatures even though their absolute accelerations are the same (the Rindler relation $2\pi T=a_4$ does not apply to the $\phi=-N^\phi t$ group) because the temperature T is observer-dependent in general. Since BTZ is asymptotically AdS, both detectors will again measure zero temperature at $r\to\infty$, where $a\to R^{-1}$.

IV. SCHWARZSCHILD AND RELATED GEOMETRIES

We now come to spaces with ''more manifest'' real horizons. Once a GEMS has been found (they always exist [1]) for the desired physical space, it is a mechanical procedure, using the familiar embedding Gauss-Codazzi-Ricci equations to relate constant acceleration a_G in GEMS to the embedded space physics; this is also possible when (as for Schwarzschild) the GEMS is more than one dimension higher. The acceleration of detectors that follow a time-like Killing vector ξ in the physical space is [2] $a=\nabla_\xi\xi/|\xi|^2$ where $|\xi|$ is the norm of ξ. It is related to a_G in the GEMS according to

$$a_G^2=a^2+\alpha^2|\xi|^{-4}, \tag{25}$$

where α is the second fundamental form [1]. Thus the temperature should simply be $2\pi T=a_G=[a^2+\alpha^2|\xi|^{-4}]^{1/2}$. One should not, however, assume from this formula that there is always a temperature, since in fact α^2 need not always be positive (it is $\alpha^2|\xi|^{-4}=-R^{-2}$ in AdS). After all, it is only when a_G^2 is non-negative that the Unruh description itself is meaningful in a flat space.

We apply these ideas first to the three types of Schwarzschild (vacuum) spaces, beginning with the usual case without cosmological constant; it can be globally embedded in flat $D=6$,

$$ds^2=(dz^0)^2-(dz^1)^2-(dz^2)^2-(dz^3)^2-(dz^4)^2-(dz^5)^2, \tag{26}$$

using the coordinate transformation [14]

$$\begin{aligned} z^0&=4m\sqrt{1-2m/r}\sinh(t/4m),\\ z^1&=4m\sqrt{1-2m/r}\cosh(t/4m),\\ z^2&=\int dr\sqrt{(2mr^2+4m^2r+8m^3)/r^3},\\ z^3&=r\sin\theta\sin\phi \quad z^4=r\sin\theta\cos\phi \quad z^5=r\cos\phi. \end{aligned} \tag{27}$$

This transformation can be extended to cover the $r<2m$ interior thanks to the analyticity of $z^2(r)$ in $r>0$. Indeed, the extension is just the maximal Kruskal one [15]. The original Hawking detectors (moving according to constant r, θ, ϕ), are here Unruh detectors; their six-space motions are the now familiar hyperbolic trajectories

$$(z^1)^2-(z^0)^2=16m^2(1-2m/r)=a_6^{-2}. \tag{28}$$

Hence, we immediately infer the local Hawking and BH temperatures

$$T=a_6/2\pi=(8\pi m\sqrt{1-2m/r})^{-1},\quad T_0=\sqrt{g_{00}}T=(8\pi m)^{-1}. \tag{29}$$

It should be cautioned that use of incomplete embedding spaces, that cover only $r>2m$ (as, for example, in [16]), will lead to observers there for whom there is no event horizon, no loss of information, and no temperature.

The above calculation is easily generalized to Schwarzschild–AdS spaces (where $1-2m/r$ is replaced by $1-2m/r+r^2/R^2$) using a $D=7$ GEMS with an additional timelike dimension z^6,

$$z^0=k_H^{-1}\sqrt{1-2m/r+r^2/R^2}\,\sinh(k_Ht),$$

$$z^1=k_H^{-1}\sqrt{1-2m/r+r^2/R^2}\,\cosh(k_Ht),$$

$$z^2=\int \frac{R^3+Rr_H^2}{R^2+3r_H^2}\sqrt{\frac{r^2r_H+rr_H^2+r_H^3}{r^3(r^2+rr_H+r_H^2+R^2)}}\,dr, \tag{30}$$

$$z^6=\int \sqrt{\frac{(R^4+10R^2r_H^2+9r_H^4)(r^2+rr_H+r_H^2)}{r^2+rr_H+r_H^2+R^2}}\frac{dr}{R^2+3r_H^2},$$

and (z^3,z^4,z^5) as in Eq. (27); $k_H=(R^2+3r_H^2)/2r_HR^2$ is the surface gravity at the root r_H of $(1-2m/r+r^2/R^2)=0$. Using this GEMS,[7] we obtain

$$2\pi T=k_H(1-2m/r+r^2/R^2)^{-1/2},$$

equal to that calculated in [10]. (It may seem that we have the freedom to choose an arbitrary constant rather than k_H in z^0 and z^1 and thereby get a different temperature. But for any other choice, z^2 and z^6 cannot be chosen so that both their integrands are finite at the horizon. Hence, such embedding spaces are not global, cover only the area outside the horizon and cannot be extended; they are therefore excluded.)

For Schwarzschild–dS, which differs formally from Schwarzschild–AdS by $R^2\to-R^2$, there are two real horizons (r_+,r_-) in general, both of which could be seen by physical detectors (such as constant r, with $r_-<r<r_+$). This requires the use of a GEMS that captures both horizons. Although we have not tried to define this bigger GEMS, we do reproduce the known results [17] for the temperature of each separate horizon, by using Eq. (30), with $R^2\to-R^2$ and the respective $k_H(r_+)$, $k_H(r_-)$. [Our method becomes meaningless for the extremal $(r_+=r_-)$ case since the whole Rindler wedge vanishes there.]

We turn now to an example with matter, the Reissner-Nordström solution with

$$ds^2=\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)dt^2-\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)^{-1}dr^2 -r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{31}$$

Although there are two horizons $(r_\pm=m\pm\sqrt{m^2-e^2})$ in the nonextremal case $(m>e)$, it is still simple to calculate the temperature via the embedding space. As explained earlier, a reliable GEMS has to cover (or be extendable to cover) both sides of the horizon, or else there is no loss of information for a detector in that space. But physical $(r>r_+)$ $r=$const Reissner-Nordström detectors are aware only of the existence of one horizon r_+, unlike the physical Schwarzschild–dS $r=$const detectors $(r_+>r>r_-)$ that see two horizons. Therefore, it is enough to use as the embedding space, again with an added timelike z^6 dimension,

$$z^0=k_H^{-1}\sqrt{1-2m/r+e^2/r^2}\,\sinh(k_Ht),$$

$$z^1=k_H^{-1}\sqrt{1-2m/r+e^2/r^2}\,\cosh(k_Ht),$$

$$z^2=\int\left(\frac{r^2(r_++r_-)+r_+^2(r+r_+)}{r^2(r-r_-)}\right)^{1/2}dr, \tag{32}$$

$$z^6=\int\left(\frac{4r_+^5r_-}{r^4(r_+-r_-)^2}\right)^{1/2}dr$$

with (z^3,z^4,z^5) as in Eq. (27), and $k_H=k(r_+)$ $=(r_+-r_-)/2r_+^2$. [In the neutral, $e=0$, limit, z^6 vanishes and this GEMS becomes the $(D=6)$ Schwarzschild one.] Even though it does not reach down to $r\leq r_-$, this embedding suffices, because it covers r_+, for the purpose of calculating the Reissner-Nordström temperature in the nonextremal case.[8] It is clear from Eq. (32) that the relevant $D=7$ acceleration

$$a_7=[(z^1)^2-(z^0)^2]^{-1/2}$$
$$=(r_+-r_-)/(2r_+^2\sqrt{1-2m/r+e^2/r^2})$$

[7] It is easy to see that when $R\to\infty$ (the Schwarzschild limit), $r_H\to 2m$, $k_H\to(2r_H)^{-1}$ and z^2 becomes identical to the Schwarzschild one while z^6 vanishes, so that we indeed get back the Schwarzschild GEMS. When $m=0$ (the AdS limit), we have $r_H^2=k_H^{-2}=-R^2$ and both z^2 and z^6 vanish, leaving the AdS GEMS of Eq. (12).

[8] To be sure, our mapping approach has limitations: since the Rindler horizon of the GEMS is Killing bifurcate, one can only map from spaces whose horizons also are; this excludes the strictly extremal $(m=e)$; Reissner-Nordström space, which is also exceptional from the $D=4$ point of view [18].

gives the correct Hawking temperature $T=(r_+-r_-)/(4\pi r_+^2\sqrt{1-2m/r+e^2/r^2})$.

V. ENTROPY

We turn now to the ''extensive'' companion of temperature, the entropy. For those of our curved spaces with intrinsic horizons, and at our semiclassical level, entropy is just one quarter of the horizon area. Entropy can also be defined for a Rindler wedge [7], using arguments similar to those used originally [19] for Schwarzschild and dS. Here the relevant area is that of the null surface $x^2-t^2=0$. This ''transverse'' area is in general infinite for otherwise unrestricted Rindler motion, being just the Cartesian $\int dy dz$ for $D=4$, say. For our purposes, however, we must evaluate this area subject to the embedding constraints, and we shall see, the resulting integral becomes finite and agrees with that of the original horizon. [This is not a tautology: we are not initially writing the original horizon area in embedding coordinates, although the result is indeed that real and embedding horizon areas agree. Nor is it a surprise: we have insured that (when present) horizons map to horizons.]

Let us begin with the dS case, where the Rindler horizon condition is $(z^1)^2-(z^0)^2=0$ which was $Z=R$, and of course $(z^2)^2+(z^3)^2+(z^4)^2=R^2$. Thus, the integration over $dz^2dz^3dz^4$ is restricted to the surface of the sphere of radius R, precisely that of the true horizon. The AdS case differs, (as expected from lack of an intrinsic horizon) and the corresponding restrictions are $(z^1)^2-(z^0)^2=0$ which again implies $Z=R$, but now $(z^2)^2+(z^3)^2-(z^4)^2=-R^2$, and the area of this hyperboloidal surface diverges, having no further restrictions. For comparison with the BTZ case below, the cause of the infinity can be traced to the fact that the limits on the z^4 integral are $\pm R\sinh\psi$, with $-\infty<\psi<\infty$.

We now see how the BTZ solution leads to a finite Unruh area due to the periodic identification of ϕ mod 2π. The $(z^1)^2-(z^0)^2=0$ Rindler horizon condition implies $r=r_+$, while $(z^3)^2-(z^2)^2=R^2(r^2-r_-^2)/(r_+^2-r_-^2)=R^2$ still looks hyperbolic. However, the relevant bounds on z^3 due to the periodicity are $R\sinh(r_+\pi/R^2)$ and $-R\sinh(r_+\pi/R^2)$ for the nonrotating case, so that one has the integral

$$\int_{-R\sinh(r_+\pi/R)}^{R\sinh(r_+\pi/R)}\int_0^{\sqrt{R^2+(z^2)^2}}\delta\left(\sqrt{(z^3)^2-(z^2)^2}-R\right)dz^3dz^2$$

$$=\int_{-R\sinh(r_+\pi/R)}^{R\sinh(r_+\pi/R)}\frac{R}{\sqrt{R^2+(z^2)^2}}dz^2$$

[and limits $R\sinh(r_+\pi/R-r_-t/R^2)$ and $R\sinh(r_+\pi/R+r_-t/R^2)$ in the rotating case] and it yields the desired area integral $2\pi r_+$. It is clear that the limits differ from the AdS ones precisely in having the ''angle's'' bounds be finite here.

The Schwarzschild case, where there are two additional dimensions in the transverse area, $\int dz^2\ldots dz^5$, is correspondingly subject to three constraints: $(z^1)^2-(z^0)^2=0$ leads to $r=2m$, (horizon to horizon mapping) $z^2=f(r)$ and $(z^3)^2+(z^4)^2+(z^5)^2=r^2$. Thus the z^2 integral, $\int dz^2\,\delta(z^2-f(r))$, is unity, while the remaining integrals of course reproduce the area of the $r=2m$ sphere in $D=3$. The Reissner-Nordström and the Schwarzschild–AdS calculations are essentially the same,[9] except that in these cases there are three (rather than two) additional dimensions, and four constraints: $(z^1)^2-(z^0)^2=0$ leads to $r=r_+$, $z^2=f_1(r)$, $z^6=f_2(r)$ and $(z^3)^2+(z^4)^2+(z^5)^2=r^2$. Thus the z^2,z^6 integrals, $\int dz^2dz^6\,\delta(z^2-f_1(r))\,\delta(z^6-f_2(r))$, are unity, and the z^3,z^4,z^5 integrals gives the desired area, that of the $r=r_+$ sphere. Having two separate horizons, the Schwarzschild–dS system is more delicate to handle, but just as for temperature, we can calculate entropy for each horizon separately, to obtain the corresponding $D=4$ results [17].

VI. SUMMARY

We have formulated a uniform mechanism for reducing curved space BH horizon temperatures and entropies to those of the kinematical Unruh effect due to Rindler motion in their GEMS. The latter must, of course, first be found and cover enough of the underlying space to include the horizon in question. This method has been applied to a variety of ''true'' BH spacetimes, both vacuum ones such as BTZ, Schwarzschild, and its dS and AdS extensions, as well as Reissner-Nordström spacetime. It would be interesting to consider other possible applications of GEMS, for example to superradiance in rotating geometries.

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[9]We may still use the Reissner-Nordström partial GEMS for the area calculation, since it covers the original $r=r_+$ horizon.

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