Operator ordering and consistency of the wave function of the Universe

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We demonstrate in the context of the minisuperspace model consisting of a closed Friedmann-Robertson-Walker universe coupled to a scalar field that Vilenkin's tunneling wave function can only be consistently defined for particular choices of operator ordering in the Wheeler-DeWitt equation. The requirement of regularity of the wave function has the particular consequence that the probability amplitude, which has been used previously in the literature in discussions of issues such as the prediction of inflation, is likewise ill defined for certain choices of operator ordering with Vilenkin's boundary condition. By contrast, the Hartle-Hawking no-boundary wave function can be consistently defined within these models, independently of operator ordering. The significance of this result is discussed within the context of the debate about the predictions of semiclassical quantum cosmology. In particular, it is argued that inflation cannot be confidently regarded as a ''prediction'' of the tunneling wave function, for reasons similar to those previously invoked in the case of the no-boundary wave function. A synthesis of the no-boundary and tunneling approaches is argued for. [S0556-2821(99)04306-4]

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I. INTRODUCTION

Recent developments concerning the possibility of open inflation in quantum cosmology [1,2] have revived an old debate about the foundational issues of that subject [2–4]. It is therefore timely to raise an issue which has been largely overlooked previously, but which in our opinion has a direct bearing on these foundational issues.

Much of the current debate originates from differing probability amplitudes calculated in an approximation which assumes that the transition amplitude for nucleation of a universe from ''nothing'' is dominated by a Euclidean instanton. Different probability amplitudes are assumed to correspond to the different boundary conditions implied by the Hartle-Hawking ''no-boundary'' proposal [5] and the ''tunneling'' proposals of Vilenkin [6,7] and Linde [2,8], which are themselves distinct. In particular, the nucleation probability for instanton-dominated transitions is assumed to be

$$\mathcal{P}\propto|\Psi|^2\propto\begin{cases}\mathrm{e}^{-2I_{\mathrm{cl}}}, & \Psi_{\mathrm{NB}},\\ \mathrm{e}^{+2I_{\mathrm{cl}}}, & \Psi_{\mathrm{TL}},\Psi_{\mathrm{TV}},\end{cases}\tag{1}$$

where the subscripts (NB), (TL) and (TV) refer to the no-boundary wave function and the tunneling wave functions of Linde and Vilenkin respectively. For the solutions in question, which correspond to a model in which gravity is coupled to a scalar field, ϕ, with potential, $V(\phi)$, in dimensionless units [see Eq (3) below for our conventions], the Euclidean action of the instanton is

$$I_{\mathrm{cl}}=\frac{-1}{3V(\phi_0)},\tag{2}$$

ϕ_0 being the value of the scalar field at nucleation.

According to the standard folklore the above nucleation probabilities can be determined by a somewhat more careful analysis of the appropriate minisuperspace models [9–12]. It is this folklore which we wish to challenge here. We will explicitly demonstrate that the prefactor in the wave function cannot be ignored when calculating the appropriate probability amplitudes if one is to implement the boundary conditions carefully in minisuperspace models. In particular, the identification of Eqs. (1), (2) as representing the relevant probabilities from which a comparison of the consequences of the competing boundary condition proposals is to be made depends crucially on Planck scale physics on account of such ambiguities. While the ''no-boundary'' proposal turns out to yield a well-defined probability amplitude independently of such ambiguities, Vilenkin's boundary condition does not. Vilenkin has previously noted [6] that Ψ_{TV} cannot be normalized for one particular operator-ordering (the ''D'Alembertian ordering''). However, in our view he appears to have overlooked the full consequences of this issue, which turns out to be quite a generic problem, as we will show. In particular, it has often been stated [4,7,12] that operator ordering is unimportant to the discussion, especially with regard to probability measures [7]. Our findings contradict such a viewpoint when it comes to the actual calculations [7,13–17] which attempt to discriminate between consequences of the wave function proposals.

Nevertheless, we do believe that some of the viewpoints expressed by each of the parties to the ''wave function debate'' do have some merits. In the last section of this paper, we will discuss these relative merits in detail, in light of the mathematical results we will present here. We shall confine our discussion to minisuperspace, not because we believe that that is the ultimate arena in which the issue should be decided, but because particular results which we wish to criticize are derived in this setting and because even in more general discussions it is usually semiclassical probability measures which are nonetheless actually used. Furthermore, while the dangers of too readily associating Ψ_{TV} with the semiclassical probability (1) have been commented on before

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[18], it does not seem to have been appreciated that this can be a problem in even the most well-studied minisuperspace model and at the level of affecting commonly claimed ''predictions'' of quantum cosmology such as the prediction of inflation.

II. MINISUPERSPACE MODEL

To be more specific, let us consider the 2-dimensional minisuperspace corresponding to the classical action for gravity coupled to a scalar field,

$$S=\frac{1}{4\kappa^2}\left[\int_{\mathcal{M}}\mathrm{d}^4x\sqrt{-g}\mathcal{R}+2\int_{\partial\mathcal{M}}\mathrm{d}^3x\sqrt{h}\mathcal{K}\right]+\frac{3}{\kappa^2}\int_{\mathcal{M}}\mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{V(\phi)}{2\sigma^2}\right), \quad (3)$$

where $\kappa^2=4\pi G=4\pi m_{\mathrm{Planck}}^{-2}$, $\mathcal{K}$ is the trace of the extrinsic curvature, and the metric is assumed to take the closed Friedmann-Robertson-Walker form

$$\mathrm{d}s^2=\sigma^2\{-\mathcal{N}^2\mathrm{d}t^2+a^2(t)\mathrm{d}\Omega_3{}^2\}, \quad (4)$$

where $\mathrm{d}\Omega_3{}^2$ is a round metric on the 3-sphere, and $\sigma^2=\kappa^2/(6\pi^2)$.

The Hamiltonian constraint obtained from the $(3+1)$-decomposition of the field equations may be quantized to yield the Wheeler-DeWitt equation

$$\left[\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a}-\frac{1}{a^2}\frac{\partial^2}{\partial\phi^2}-a^2\mathcal{U}(a,\phi)\right]\Psi=0, \quad (5)$$

where

$$\mathcal{U}(a,\phi)\equiv 1-a^2V(\phi), \quad (6)$$

and we have allowed for possible operator-ordering ambiguities through the integer power, p, in the first term. The approximation that has been adopted in previous treatments [6,7] is to confine the discussion to regions in which the potential $V(\phi)$ can be approximated by a cosmological constant, so that the ϕ dependence in Eq. (5) can be effectively ignored. The resulting equation is then amenable to a standard 1-dimensional WKB analysis. In this ''de Sitter minisuperspace'' approximation the WKB solutions are readily found to be

$$\Psi(a,\phi)\simeq\frac{\mathcal{A}_\pm(\phi)\mathrm{e}^{\mp i\pi/4}}{a^{(p+1)/2}[\mathcal{U}(a,\phi)]^{1/4}}\exp\left\{\frac{\pm i[-\mathcal{U}(a,\phi)]^{3/2}}{3V(\phi)}\right\} \quad (7)$$

if $a^2V>1$ and

$$\Psi(a,\phi)\simeq\frac{\mathcal{B}_\pm(\phi)}{a^{(p+1)/2}[\mathcal{U}(a,\phi)]^{1/4}}\exp\left\{\frac{\pm[\mathcal{U}(a,\phi)]^{3/2}}{3V(\phi)}\right\} \quad (8)$$

if $a^2V<1$.

Different boundary conditions will then lead to a solution, Ψ, corresponding to different linear combinations of these WKB components in the ''oscillatory'' and ''tunneling'' regions of the minisuperspace, which correspond to the oscillatory [Eq. (7)] and exponentially [Eq. (8)] dominated solutions respectively.

Both the Hartle-Hawking [5] and Vilenkin [7] boundary conditions on the wave function require regularity of Ψ as $a\to 0$. From Eq. (5) one can see that a potential divergence in the $a^{-2}\Psi_{,\phi\phi}$ term can be avoided by requiring Ψ to be independent of ϕ as $a\to 0$. Since $\mathcal{U}(a,\phi)\to 1$ in this limit, one would thus naively expect that the prefactor $\mathcal{B}_\pm(\phi)$ of the tunneling WKB modes of Eq. (8) should take the form

$$\mathcal{B}_\pm(\phi)\propto\exp\left(\frac{\mp 1}{3V(\phi)}\right) \quad (9)$$

for the respective modes. There are two problems with this, however. First, for operator orderings other than the $p\leqslant -1$, the factor $a^{-(p+1)/2}$ in the prefactor of Eq. (8) will alter any considerations based on the regularity of Ψ. Second, the WKB approximation does not hold all the way down to $a\to 0$ in any case, and a more careful analysis of the solutions of Eq. (5) is required in this limit. Such an analysis has been given by Hawking and Page [19] in the case of Ψ_{NB} with the ''D'Alembertian operator ordering'' $p=1$ and by Vilenkin [7] for the case of Ψ_{TV} and Ψ_{NB} with operator ordering $p=-1$.

We will now extend the analysis of Refs. [7,19] to both wave functions for arbitrary operator ordering, p. In following [7,19] we shall assume that the ϕ dependence in Eq. (5) can be ignored. Such an approximation can be justified if we assume that we are close to semiclassical solutions (7), (8) for which ϕ varies slowly. This means that the potential $V(\phi)$ should be suitably flat, which physically is one example of a model leading to ''slow-roll'' inflationary cosmologies. With such a simplification, the $a^{-2}\Psi_{,\phi\phi}$ term in Eq. (5) is dropped and $V(\phi)$ is approximated by a constant. It is still not possible to solve Eq. (5) exactly for arbitrary p in terms of known elementary functions with these approximations [20]. However, it can be solved in a direct fashion in two separate regimes.

First, if $a^2V\ll 1$, which for constant finite V will pertain to the $a\to 0$ limit, the curvature term dominates the ''potential'' (6), and with the redefinition

$$\Psi\equiv z^{-(p-1)/4}y(z), \quad (10)$$

where $z\equiv\frac{1}{2}a^2$, we find that Eq. (5) reduces to a modified Bessel equation

$$z^2\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+z\frac{\mathrm{d}y}{\mathrm{d}z}-(z^2+\nu^2)y=0, \quad (11)$$

where $\nu=\pm\frac{1}{4}(p-1)$. The general solution for $y(z)$ is thus a linear combination of modified Bessel functions, $I_{(p-1)/4}(z)$ and $K_{(p-1)/4}(z)$.

Second, if $a^2V\gg 1$, which for constant finite V will pertain to the large a limit, the $a^2V(\phi)$ term dominates the ''potential'' (6), and with the redefinition

$$\Psi\equiv x^{-(p-1)/6}w(x), \quad (12)$$

where $x \equiv \frac{1}{3}a^3\sqrt{V}$, we find that Eq. (5) reduces to a Bessel equation

$$x^2\frac{d^2w}{dx^2}+x\frac{dw}{dx}+(x^2-n^2)w=0, \tag{13}$$

where $n=\pm\frac{1}{6}(p-1)$. The general solution for $w(x)$ is thus a linear combination of ordinary Bessel functions, $J_{(p-1)/6}(x)$ and $Y_{(p-1)/6}(x)$.

The two sets of Bessel function solutions must agree with the respective WKB solutions (7), (8) in the limits in which all relevant approximations mutually hold. Using a combination of an analysis of these limits and the WKB matching procedure we can constrain the particular linear combinations of solutions which correspond to the boundary conditions of $\Psi_{\rm NB}$, $\Psi_{\rm TL}$ and $\Psi_{\rm TV}$.

III. "NO BOUNDARY" WAVE FUNCTION

Since the Hartle-Hawking boundary condition [5] is stated in terms of the path integral, some further arguments are required to translate this into boundary conditions on Eq. (5) in minisuperspace. However, the statement of the relevant boundary conditions on Eq. (5) is uncontroversial and we will thus follow Hawking and Page [19,22] in demanding the following:

(i) In the tunneling region the relevant WKB mode is the $(-)$ solution of Eq. (8) only, viz.

$$\Psi_{\rm NB}\simeq\frac{\mathcal{B}_-}{a^{(p+1)/2}[1-a^2V]^{1/4}}\exp\left(\frac{-[1-a^2V]^{3/2}}{3V}\right), \tag{14}$$

as is appropriate to the standard Wick rotation $t\rightarrow -i\tau$ in the definition of the Euclidean path integral.

(ii) The wave function must be bounded as $a\rightarrow 0$ for all finite values of ϕ and on the past null boundaries of minisuperspace. Thus in a suitable measure we can take

$$\Psi_{\rm NB}(a=0,\phi)=1. \tag{15}$$

First consider $p\geqslant 1$. The only modified Bessel function solution of Eq. (11) which leads to a regular wave function (10) as $a\rightarrow 0$ for values of $p\geqslant 1$ is $I_{(p-1)/4}(z)$, yielding a wave function

$$\Psi_{\rm NB}=\frac{C_1}{a^{(p-1)/2}}I_{(p-1)/4}(\tfrac{1}{2}a^2) \tag{16}$$

in the $a^2V\ll 1$ limit. The constant C_1 may be fixed by the normalization condition (15) and the small value limit [23] of the Bessel function, giving

$$C_1=2^{(p-1)/2}\Big/\,\Gamma\left(\frac{p+3}{4}\right).$$

We can now check that Eq. (16) does agree with Eq. (14) by taking the limit of both expression for a finite large a for which $a^2V\ll 1$ nonetheless, which is the limit in which they should match. In practice, this requires very small values of $V(\phi)\ll 1$; i.e., the potential must be much less than the Planck scale, which is physically reasonable. One finds that since, for finite large a [23],

$$I_{(p-1)/4}(\tfrac{1}{2}a^2)=\frac{1}{\sqrt{\pi}a}\exp(\tfrac{1}{2}a^2)[1+{\rm O}(a^{-2})], \tag{17}$$

the leading term in the appropriate limit of Eq. (16) does agree with that from Eq. (14) if

$$\mathcal{B}_-=\frac{C_1}{\sqrt{\pi}}\exp\left(\frac{1}{3V(\phi)}\right). \tag{18}$$

Using the WKB connection formulas [24] we find

$$\Psi_{\rm NB}=\frac{2C_1\exp\left(\frac{1}{3V}\right)\cos\left[\frac{1}{3V}(a^2V-1)^{3/2}-\frac{\pi}{4}\right]}{\sqrt{\pi}a^{(p+1)/2}(a^2V-1)^{1/4}} \tag{19}$$

for the WKB solution in the oscillatory region, which is the linear superposition of the modes (7) with $\mathcal{A}_+=\mathcal{B}_-$. This can be checked against linear combinations of the Bessel function solutions in the limit $a^2V\gg 1$. We find that the solution does indeed match the linear combination of solutions to Eqs. (12), (13) given by

$$\Psi_{\rm NB}=\frac{\mathcal{C}}{a^{(\varphi-1)/2}}\{J_{(p-1)/6}(x)+J_{(1-p)/6}(x)\}, \tag{20}$$

where $x\equiv\frac{1}{3}a^3\sqrt{V}$ as before, and $\mathcal{C}(\phi)\propto\exp[1/3V(\phi)]$.

In the case that $p<1$, any arbitrary linear combination of the independent modified Bessel function solutions $I_{(1-p)/4}(z)$ and $K_{(1-p)/4}(z)$ yields a convergent wave function as $a\rightarrow 0$. Therefore, the Hartle-Hawking condition does not restrict the wave function except by appealing to the semiclassical behavior (14). Since similarly to Eq. (17) $K_{-\nu}(z)\equiv K_\nu(z)$ is given in the limit of finite large a by [23]

$$K_{(p-1)/4}(\tfrac{1}{2}a^2)=\frac{\sqrt{\pi}}{a}\exp(-\tfrac{1}{2}a^2)[1+{\rm O}(a^{-2})], \tag{21}$$

we see that the semiclassical condition only makes the restriction that the coefficient of $I_{(1-p)/4}(z)$ must be non-zero so as to dominate over $K_{(1-p)/4}(z)$ in the appropriate limit. If we take the particular choice

$$y(z)=I_{(1-p)/4}(z)+\frac{2}{\pi}\sin\left((1-p)\frac{\pi}{4}\right)K_{(1-p)/4} \tag{22}$$

for $p\notin\{-3,-7,-11,\ldots\}$, then the wave function is once again given by Eq. (16) as $a\rightarrow 0$ and the previous analysis applies exactly. For $p\in\{-3,-7,-11,\ldots\}$ the linear combination (22) must be replaced by one for which the coefficient of $K_{(1-p)/4}$ is nonzero, since otherwise we would have $\Psi\rightarrow 0$ as $a\rightarrow 0$, in violation of Eq. (15). However, provided a linear combination consistent with the semiclassical behavior (14) is chosen, then the above analysis is not changed in any substantial way. (The exact solutions will be discussed elsewhere [21].)

We have thus shown that $\Psi_{\rm NB}$ can be consistently defined for arbitrary p in accordance with the approximations usually assumed for specific operator orderings.

IV. ''TUNNELING'' WAVE FUNCTIONS

Vilenkin's tunneling wave function is defined in reverse by placing ''boundary'' conditions in the oscillatory region of the minisuperspace. In accordance with [7] we require the following:

(i) In the oscillatory region the relevant WKB mode is the $(-)$ solution of Eq. (7) only, viz.

$$\Psi_{\rm TV} \simeq \frac{\mathcal{A}_- \, {\rm e}^{i\pi/4}}{a^{(p+1)/2}[a^2V-1]^{1/4}} \exp\left(\frac{-i[a^2V-1]^{3/2}}{3V}\right), \quad (23)$$

so that $(i/\Psi_{\rm TV})\partial\Psi_{\rm TV}/\partial a > 0$ there, as required.

(ii) The wave function must be everywhere bounded:

$$|\Psi_{\rm TV}| < \infty. \quad (24)$$

Beginning with the WKB mode (23) in the oscillatory region we can use the WKB matching procedure [24] to obtain the appropriate linear combination of the modes (7) in the tunneling region, $a^2V<1$, viz. [25]

$$\Psi_{\rm TV} = \tfrac{1}{2}\Psi_- + i\Psi_+, \quad (25)$$

where

$$\Psi_\pm \equiv \frac{\mathcal{A}_-}{a^{(p+1)/2}(1-a^2V)^{1/4}} \exp\left[\frac{\pm 1}{3V}(1-a^2V)^{3/2}\right]. \quad (26)$$

We can separately match the real and imaginary parts of Eq. (25) with appropriate linear combinations of modified Bessel function solutions to Eq. (10), (11) in the limit that $a^2V \ll 1$ with finite large a using their asymptotic limits (17) and (21) similarly to the case of $\Psi_{\rm NB}$. In this manner, we find that the appropriate solution in the $a^2V \ll 1$ region which corresponds to Vilenkin's boundary condition is

$$\Psi_{\rm TV} = \frac{\mathcal{A}_-}{a^{(p-1)/2}}\left\{\frac{\sqrt{\pi}}{4}\,{\rm e}^{-1/(3V)}[I_\nu(z)+I_{-\nu}(z)] + \frac{i}{\sqrt{\pi}}\,{\rm e}^{1/(3V)}K_\nu(z)\right\}, \quad (27)$$

where $z \equiv \frac{1}{2}a^2$ and $\nu = (p-1)/4$.

The problem with the definition of $\Psi_{\rm TV}$ is now manifest, since, as $a \to 0$ [23],

$$K_0(\tfrac{1}{2}a^2) \sim -\ln(\tfrac{1}{2}a^2) \quad (28)$$

and

$$K_{(p-1)/4}(\tfrac{1}{2}a^2) \sim \tfrac{1}{2}\Gamma\left(\frac{|p-1|}{4}\right)\left(\frac{2}{a}\right)^{|p-1|/2} \quad (29)$$

for $p \neq 1$, and so the product $a^{-(p-1)/2}K_{(p-1)/4}(\frac{1}{2}a^2)$ diverges for $p \geqslant 1$. In fact, it is quite clear that if we are to have a regular wave function for operator orderings with $p \geqslant 1$, then the only solution to Eq. (11) which will yield a regular wave function in the limit $a \to 0$ is Eq. (16). That is to say, if regularity of the wave function is important, then any consistent boundary condition for the wave function must coincide with that of Hartle and Hawking [5] in the context of this minisuperspace model for $p \geqslant 1$. Any boundary condition which includes a contribution from the $(+)$ mode of Eq. (8) in the WKB limit will match onto the $K_{(p-1)/4}(\frac{1}{2}a^2)$ solution of Eq. (11) in the $a^2V \ll 1$ limit, and this diverges as $a \to 0$. For $p \leqslant 0$ the divergence is regulated by the prefactor in Eq. (10), but for $p \geqslant 1$ the problem is unavoidable. Our conclusion thus applies to $\Psi_{\rm TL}$ as well as to $\Psi_{\rm TV}$.

For operator orderings with $p<1$, Eq. (27) is well defined as $a \to 0$ and thus a normalization condition can be set in this limit to fix $\mathcal{A}_-(\phi)$. Vilenkin chose $\Psi_{\rm TV} \to 1$ in the $p=-1$ case [7]. However, a choice $|\Psi_{\rm TV}| \to 1$ might be more appropriate here to preserve the real and imaginary parts of Eq. (27). In either case, if $V \ll 1$, then

$$\mathcal{A}_-(\phi) \propto \exp\left(\frac{-1}{3V(\phi)}\right) \quad (30)$$

as previously anticipated in Eq. (9). Only in this manner can the ϕ dependence in the prefactor of the oscillatory WKB wave function (23) be constrained. The oscillatory WKB solution (23) can be matched in the large a limit to solutions of Eqs. (12), (13) expressed in the combination of a Hankel function, similarly to Eq. (20) for $\Psi_{\rm NB}$.

V. PROBABILITY AMPLITUDES

We now wish to point out that the issue of the regularity of the wave function is crucial in discussions using probability measures in minisuperspace. While the question of the definition of a suitable probability measure in quantum cosmology is a tricky one [9–12] it can be argued [27] that in the semiclassical limit the ordinary ''Klein-Gordon'' type conserved probability current

$$\mathcal{J} = -\tfrac{1}{2}i(\bar{\Psi}\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\bar{\Psi}) \quad (31)$$

leads to a well-defined probability measure for trajectories peaked around particular WKB modes, even though $\mathcal{J}$ is not positive definite in general. The resulting probability density

$${\rm d}\mathcal{P} = \mathcal{J}^A {\rm d}\Sigma_A \quad (32)$$

can be integrated over a hypersurface in minisuperspace to answer questions of conditional probability such as, given that a classical universe nucleates, what is the probability that it inflates sufficiently ($\sim 60-65\,e$-folds)? Ideally, the hypersurface Σ here should be chosen in the oscillatory region, close to the boundary of the tunneling region, but for potentials satisfying the ''de Sitter minisuperspace approximation'' it is assumed [7,9–14] that this surface can be approximated by an $a=$ const hypersurface. (See Fig. 1.) In this limit the probability for sufficient inflation is then assumed to be [7,9–14]

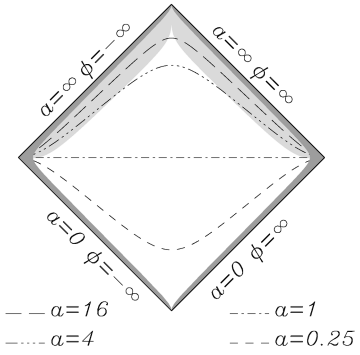

FIG. 1. Conformal diagram for $V=0.04\phi^2$. The oscillatory region, given roughly by $a^2V>1$, is lightly shaded. Lines $a=$const are superimposed. For very large values of ϕ these lie almost entirely in the oscillatory region. The region of ϕ values excluded by a Planck scale cutoff is darkly shaded.

$$\mathcal{P}(\phi_0>\phi_{\rm suff}|\phi_1<\phi_0<\phi_2)=\frac{\int_{\phi_{\rm suff}}^{\phi_2}{\rm d}\phi_0\exp\left(\frac{\pm 2}{3V(\phi_0)}\right)}{\int_{\phi_1}^{\phi_2}{\rm d}\phi_0\exp\left(\frac{\pm 2}{3V(\phi_0)}\right)}, \tag{33}$$

where ϕ_0 is the value of ϕ at nucleation, $\phi_{\rm suff}$ is the minimum value for sufficient inflation, ϕ_1 is the minimum value for a universe to nucleate and ϕ_2 a Planck scale cutoff, suggested by the approximations used. In Fig. 1, ϕ_1 and ϕ_2 correspond roughly to the points of intersection of a suitable $a=$const hypersurface with the tunneling (white) and Planck cutoff (dark) regions respectively.

According to the assumed wisdom the $(+)$ sign in Eq. (33) corresponds to $\Psi_{\rm NB}$ and the $(-)$ to $\Psi_{\rm TV}$, and the resulting probability is more likely to give $\mathcal{P}\simeq 1$ for $\Psi_{\rm TV}$ in the presence of a Planck scale cutoff [7]. This is considered to be a problem for the ''no boundary'' proposal. However, Eq. (33) arises from evaluating $\Psi_{\rm NB}$ and $\Psi_{\rm TV}$ when peaked around the $(-)$ WKB mode of Eq. (7) on an $a=$const hypersurface, so that

$${\rm d}\mathcal{P}\propto|\Psi|^2{\rm d}\phi\propto\mathcal{A}_-(\phi)^2{\rm d}\phi; \tag{34}$$

i.e., in the oscillatory region the phase is unimportant when calculating $|\Psi|^2$, and it is the prefactor which counts. Our analysis shows, however, that for $\Psi_{\rm TV}$ the quantity $\mathcal{A}_-$ cannot be normalized for operator orderings $p\geqslant 1$. The problem is thus not merely a mathematical subtlety, but spells serious problems for the tunneling proposal in terms of its predictive power.

Of course, it is possible to ''save'' Vilenkin's proposal in its present form [7,26] if there is some justification as to why operator orderings with $p<1$ correspond to a natural quantization. Unfortunately, we know of no such justification. In fact, the only operator ordering which has ever been claimed to be ''natural'' to date is the ''D'Alembertian ordering'' $p=1$ [19,28]. Louko [28] has made a detailed analysis of this point in minisuperspace models, showing that the ''D'Alembertian ordering'' is preferred if a scale-invariant measure is chosen when calculating the prefactor by zeta function regularization.

An alternative approach has been pursued by Barvinsky [29], who argues that the operator ordering question should be determined by demanding unitarity of the wave function. While the issue of unitarity is clearly open to question in a quantum cosmological setting [27], it does provide strong physical grounds on which operator-ordering questions could be debated. Barvinsky [29] has pursued this question in superspace at the 1-loop quantum level. In this context, the ''D'Alembertian ordering'' is again picked out, this time by the criterion of ensuring Hermiticity of relevant operators and closure of an appropriate algebra for the 1-loop quantum constraints.

It is not our intention to focus on the merits of any particular operator ordering, as any debate must obviously involve questions about Planck scale physics about which we have, as yet, no direct understanding. However, we believe that the very fact that a consistent definition of the semiclassical probability is operator-ordering dependent, unless particular boundary conditions are chosen, does raise some important questions which have been overlooked in the previous literature.

VI. WAVE FUNCTION DISCORD OR CONCORD?

We will now discuss the implications of the result of the previous sections in terms of the debate about the relative merits of proposals for the boundary conditions of the wave function of the Universe.

First, as mentioned above, Linde's wave function, $\Psi_{\rm TL}$, also suffers from problems similar to Vilenkin's for operator orderings with $p\geqslant 1$. However, we consider criticism about the stability of matter fields in quantum field theory under a Wick rotation with the ''wrong'' sign, $t\rightarrow+i\tau$, as restated most recently by Hawking and Turok [3], as being a much more serious indictment of Linde's proposal. We will not therefore discuss $\Psi_{\rm TL}$ further.

There are two levels of criticism which have been put forward by parties to the debate about $\Psi_{\rm NB}$ versus $\Psi_{\rm TV}$. One common criticism of Vilenkin's proposal is that since its intuition is so closely tied to the WKB approximation in particular minisuperspace models, it is difficult to suitably generalize it to superspace. This is due to the difficulty of rigorously defining the notions of ''outgoing waves'' and the ''boundary of superspace'' which form the basis of the tunneling proposal [6,7,26]. Vilenkin has given arguments to suggest how the tunneling proposal might be put on a firmer footing, through consideration of the implications of topology change and other issues [26]. However, the discussion remains speculative. On the other hand, the no-boundary proposal is not completely well defined in a superspace setting either. For example, metrics which are neither of purely Euclidean nor purely Lorentzian signature must be included in the path integral to make it converge. Such metrics can make significant contributions even in relatively simple minisuperspace models, and there is no obvious unique way in which to define the integration contour through such

saddle points [30]. One must attempt to find a sense in which the Hartle-Hawking proposal can be reformulated in terms of geometries which are ''approximately'' Euclidean [31]. Using a momentum representation in which the wave function depends on the second fundamental form, as proposed recently by Bousso and Hawking [32], may be a way forward, but much work remains to be done.

It is not our intention to debate the superspace formulation here, as the main purpose of this paper is to comment on the other level of the wave function debate, which involves the predictions of quantum cosmology. It has become common in recent papers to simply state that the no-boundary proposal does not ''predict'' sufficient inflation, whereas Vilenkin's tunneling proposal does so more easily. However, this has not always been the assumed perception, and it is useful to review how this popular perception arose.

In Hawking and Page's original analysis [19] no Planck scale cutoff was taken in evaluating in the nucleation probability: they set $\phi_2=\infty$ in Eq. (33), so that the integrals are dominated by the values of ϕ above the Planck scale, and $\mathcal{P}\simeq 1$ even for $\Psi_{\rm NB}$. This argument was then criticized by Vilenkin [7], who argued that because Planck-scale physics goes beyond the semiclassical approximation, then a Planck-scale cutoff must be introduced. Of course, one might still argue, as Page does [33], that such a choice is simply an *ad hoc* guess about unknown physics, and the Hawking-Page answer could be the correct one. However, the use of a Planck scale cutoff for $\Psi_{\rm NB}$ does seem to be justified by calculations which suggest that the wave function is damped for values of ϕ above the Planck scale by 1-loop effects [15,29]. The introduction of a Planck scale cutoff has the consequence that $\Psi_{\rm NB}$ does not predict sufficient inflation, at least in terms of the simple models which have been studied to date [7,13,14].

What we wish to stress here, however, is that if one wishes to consistently exclude predictions based arbitrarily on Planck scale physics from the discussion, it is not simply good enough to exclude values of ϕ above the Planck cutoff from the $a=$const integration slice through minisuperspace; one must also exclude any choices forced by Planck scale physics in the limit $a\rightarrow 0$. While it may of course be possible to use conditional probabilities in a way that avoids the need to normalize the wave function [19], the fact remains that the particular chain of argument that leads to the particular probability measures (1), (2) for the minisuperspace model we have studied does rely on the requirement of normalizing the wave function as $a\rightarrow 0$. Thus arbitrary choices about Planck scale physics via preferred operator orderings enter Vilenkin's proposal as soon as we require that it make predictions. This point was unfortunately missed at the time that Vilenkin first discussed the predictions of the probability of inflation [7] because his analysis at that stage was restricted to the $p=-1$ model, despite his earlier remark about the $p=1$ case [6]. In Ref. [7] Vilenkin stated that since the Hawking-Page derivation of sufficient inflation from $\Psi_{\rm NB}$ relied on contributions from Planck scale energies, the semiclassical approximation on which the derivation of the no-boundary semiclassical probability density was based ''could not be trusted in this regime,'' and therefore [7] ''My conclusion is that at this stage inflation cannot be claimed as one of the predictions of the Hartle-Hawking approach.'' However, since a consistent derivation of the semiclassical tunneling probability density also requires arbitrary choices at the Planck scale, by similar logic we would have to conclude that at this stage inflation cannot be claimed as one of the predictions of Vilenkin's approach either. Since the ease of prediction of sufficient inflation is widely regarded as the principal advantage of $\Psi_{\rm TV}$ over $\Psi_{\rm NB}$, we regard this as a rather serious problem for Vilenkin's proposal.

The strongest claims for the prediction of sufficient inflation from the tunneling wave function have been made from consideration of 1-loop effects [16], similar to those leading to the Planck scale cutoff mentioned above [15,29]. The claim is that, in the context of a model with the inflaton non-minimally coupled to gravity, 1-loop effects lead not only to a suppression of values of ϕ beyond the Planck scale, but also enhance the bare probability in such a way as to provide a narrow peak in the probability distribution, thereby leading to sufficient inflation for the tunneling wave function even though the corresponding tree-level probability does not [16]. We believe that our findings place such claims in doubt for two reasons. First, such calculations [15,16,29] have been restricted to quantum corrections in ϕ about classical backgrounds with $\Psi\propto {\rm e}^{\mp I_{\rm cl}}$ and do not address the question of $O(\hbar)$ corrections to a in the limit $a\rightarrow 0$, which were the basis of our investigation here. Second, 1-loop calculations require a choice of operator ordering: the actual choice of Refs. [15,16,29] is the ''D'Alembertian'' ordering, chosen for the requirement of 1-loop unitarity [29] as discussed above, but this choice is at odds with a consistent definition of the tunneling wave function, as we have seen.

The other arena of predictions made from quantum cosmology, which has been the focus of some debate [17,18], is the question of primordial black hole production and the stability of de Sitter space. Our findings here certainly support the argument of Garriga and Vilenkin [18] that $\Psi_{\rm TV}$ cannot in general be associated with the probability density (1), and thus criticisms of $\Psi_{\rm TV}$ based on such a loose association [17] are aiming wide of the target. However, we believe a far better defense of the tunneling wave function would be to find some physical model to which one could confidently say that $\Psi_{\rm TV}$ did apply, with definitive predictions. As discussed above, in our opinion the prediction of inflation does not enjoy such a status, and we do not know of a physical process which does. While our hopes for a finding a suitable minisuperspace model for discussing the primordial black hole issue are more optimistic than the view expressed by Garriga and Vilenkin, there are many other issues to be considered, such as whether different horizon volumes are nucleated independently, as these authors have discussed [18]. However, since the relevant discussion of Ref. [18] again appealed to the probabilities (1), (2), but this time in relation to inflation (which the authors of [18] considered to be justified but which we do not), we believe that many issues need to be very carefully reconsidered before the debate of Refs. [17,18] could be said to have been put on a firm footing.

Some general comments about the use of probability mea-

sures in quantum gravity are in order. It is common simply to use the bare probability densities (1), often in a saddle-point approximation corresponding to an instanton, in which both the prefactor and the integration of the probability density over a hypersurface (or region) of (mini)superspace are neglected. It is certainly possible to ignore the effects of integration over a hypersurface if there is a cutoff at a finite scale, such as the Planck scale, so that the integral is dominated by field values which dominate the probability density. What is perhaps less well appreciated is that in considering ''tunneling from nothing,'' whether via $\Psi_{\rm NB}$, $\Psi_{\rm TV}$ or otherwise, one is placing a boundary condition at $a \to 0$ and Planck scale physics cannot be ignored in this regime. In the discussion of the simple model here we have seen evidence of this in the important role played by the prefactor. In more sophisticated treatments there might be other problems.

We consider that the use of instantons as approximations to the calculation of the amplitude for processes such as pair production of black holes on classical spacetime backgrounds is well justified since *both* the initial and final states of the system are classical. However, the nucleation of the Universe is a different problem in a fundamental sense. To this extent we sympathize with the sentiment expressed by Linde who likened the semiclassical approach to quantum cosmology to the problem of the harmonic oscillator, with the comment [2] that the ''wave function simply describes the probability of deviations of the harmonic oscillator from its equilibrium. It certainly does not describe quantum creation of a harmonic oscillator.''

While our findings concerning the prefactor and operator ordering could be taken as support for Linde's statement in the absence of a preferred quantization, we will refrain from suggesting, as a hard-nosed skeptic might, that the conclusion to be drawn is that semiclassical quantum cosmology does not predict anything. Rather we believe that all parties must face up to the fact that boundary conditions at the beginning of the Universe do entail Planck scale physics by default. In the case of Vilenkin's proposal this fact is somewhat disguised because the ''boundary'' condition is set in the later Lorentzian regime—however, as we have argued, Planck scale physics enters at the moment we wish to make a prediction. If semiclassical quantum cosmology is to have any pretensions to make predictions about the nucleation of the actual Universe, then boundary conditions for the wave function of the Universe must be robust when confronted by the Planck scale. While it remains technically possible that the no-boundary proposal could suffer from other problems at higher orders in perturbation theory or in other minisuperspace models, we believe that of the current boundary condition proposals the prospects for $\Psi_{\rm NB}$ remain the best, on account of the fact that the underlying mathematical intuition in the no-boundary proposal is one of geometrical smoothness. The ''robustness'' of $\Psi_{\rm NB}$ as compared with $\Psi_{\rm TV}$ and $\Psi_{\rm TL}$ in the simple minisuperspace we have considered could thus well be more than an accident.

While the results here seem to favor the no-boundary wave function, or at least to provide some justification for the use of Eqs. (1), (2) as the relevant nucleation probability for $\Psi_{\rm NB}$ in semiclassical calculations, there are still a number of important outstanding issues to be resolved in the Hartle-Hawking approach, both on the technical and interpretational sides. Some of these problems have been mentioned above. Another major problem is the breakdown of the WKB approximation, which has been observed to occur in the model with $V(\phi) = m^2\phi^2$ since the solutions to the Wheeler-DeWitt equation (5) with $p = 1$ exhibit deterministic chaos [34].

In terms of the question about the semiclassical probability densities, the most glaring problem which has been glossed over in the preceding discussion is the fact that the semiclassical probability current (31) is in fact identically zero for $\Psi_{\rm NB}$, and to arrive at Eq. (33) a decoherence mechanism to the $(-)$ WKB mode of Eq. (7) has usually been invoked. If such a mechanism can be found, then of course the appropriate mode describing the Universe is outgoing in Vilenkin's sense. The absence of any well-defined mechanism to describe this decoherence is one of the greatest outstanding problems for cosmological predictions in the Hartle-Hawking approach. Since decoherence to a mode that very much resembles $\Psi_{\rm TV}$ seems to be what is ultimately desired of the no-boundary approach, one might hope that a synthesis of the Hartle-Hawking and Vilenkin approaches might be possible and indeed advantageous. The recent paper of Bousso and Hawking [32] could provide a promising start in this direction, because it suggests a means of distinguishing between the ingoing and outgoing modes of the wave function, thereby suggesting a natural choice of a contour of integration through complex saddle points in superspace without having to appeal arbitrarily to decoherence. While we believe the issue of probabilities in Bousso and Hawking's approach may require more care than they have exercised, their approach could provide a bridge between $\Psi_{\rm NB}$ and $\Psi_{\rm TV}$, and maybe even eventual concordance.

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