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Remarks on two-loop free energy in $\mathcal{N}=4$ supersymmetric Yang-Mills theory at finite temperature

A. Fotopoulos and T. R. Taylor
Department of Physics, Northeastern University, Boston, Massachusetts 02115

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The strong coupling behavior of finite temperature free energy in $\mathcal{N}=4$ supersymmetric SU(N) Yang-Mills theory has been recently discussed by Gubser, Klebanov, and Tseytlin in the context of anti–de Sitter-SU(N) Yang-Mills theory correspondence. In this Rapid Communication, we focus on the weak coupling behavior. As a result of a two-loop computation we obtain, in the large N 't Hooft limit, $F(g^2N\to 0)\approx -(\pi^2/6)N^2V_3T^4[1-(3/2\pi^2)g^2N]$. Comparison with the strong coupling expansion provides further indication that free energy is a smooth monotonic function of the coupling constant. [S0556-2821(99)50106-9]

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Finite temperature effects break supersymmetry [1]. By switching on nonzero temperature, one can interpolate between supersymmetric and nonsupersymmetric theories. For instance in gauge theories, one can interpolate between the supersymmetric case and a theory which contains pure Yang-Mills (YM) as the massless sector, with some additional thermal excitations. In the infinite temperature limit, the time dimension decouples and, at least formally, one obtains a nonsupersymmetric Euclidean gauge theory. If no phase transition occurs when the YM gas is heated up, then the dynamics of realistic gauge theories such as QCD is smoothly connected to their supersymmetric relatives.

Maldacena's conjecture [2], which relates the large N limit of $\mathcal{N}=4$ supersymmetric SU(N) Yang-Mills theory (SYM) to type IIB superstrings propagating on $AdS_5\times S^5$ (where AdS is anti–de Sitter), provides a very promising starting point towards QCD. On the superstring side, a nonzero temperature can be simulated by including Schwarzschild black holes embedded in AdS spacetime [3], which describe the near-horizon geometry of nonextremal D-brane solutions [4]. The classical geometry of black holes with Hawking temperature T does indeed encode correctly many qualitative features of large N gauge theory heated up to the same temperature. At the quantitative level though, the comparison between SYM and supergravity becomes rather subtle because the supergravity side merely provides the strong coupling expansion for physical quantities, while most finite temperature computations in SYM are limited to the perturbative, weak coupling expansion. In this note, we comment on the computation of free energy.

The SYM thermodynamics was first compared with the thermodynamics of D-branes in Ref. [5]. The free energy F obtained in Ref. [5] describes the limit of infinitely strong coupled SYM theory. More recently, the AdS-SYM correspondence has been employed for computing the subleading term in the strong coupling expansion (in $\lambda\equiv g^2N$) [6,7]:[1]

$$F(\lambda\to\infty)\approx -\frac{\pi^2}{6}N^2V_3T^4\left(\frac{3}{4}+\frac{45}{32}\zeta(3)(2\lambda)^{-3/2}\right). \quad (1)$$

The comparison with the limiting free-theory value,

$$F(\lambda=0)=-\frac{\pi^2}{6}N^2V_3T^4, \quad (2)$$

indicates that the exact answer has the form

$$F(\lambda)=-\frac{\pi^2}{6}N^2V_3T^4f(\lambda), \quad (3)$$

where the function $f(\lambda)$ interpolates smoothly between the asymptotic values $f(0)=1$ and $f(\infty)=3/4$ [5]. The sign of the subleading correction $\mathcal{O}[(2\lambda)^{-3/2}]$ in Eq. (1) indicates that f decreases monotonically from 1 to 3/4.

The question whether free energy interpolates smoothly between weak and strong coupling limits deserves careful investigation, especially in view of the recent claim in favor of a phase transition at finite λ [8].[2] There is, however, a place to look for further hints on the properties of free energy: the subleading terms in the weak coupling expansion. Surprisingly enough, they cannot be found in the existing literature. In order to fill this gap, we calculated the two-loop correction to free energy. The result is

[1]In this context, the gauge coupling g is related to the type IIB superstring coupling g_s: $g^2=2\pi g_s$.

[2]It is beyond the scope of this note to review the arguments of Ref. [8], however we would like to point out that they involve certain assumptions on the convergence properties of perturbative expansions. The proposed $2\pi^2$ convergence radius does not seem realistic after one looks at the two-loop correction, see Eq. (4).

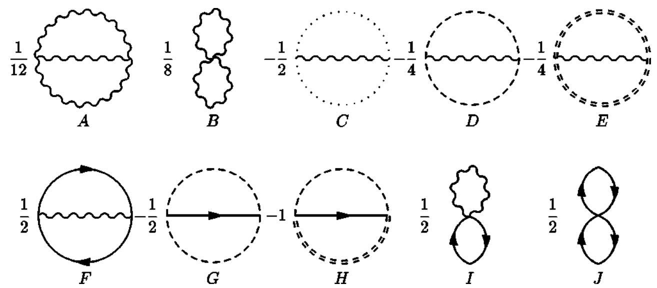

FIG. 1. Two-loop diagrams contributing to free energy. Gauge bosons are represented by wiggly lines, ghosts by dotted lines, fermionic (Majorana) components of chiral multiplets by dashed lines, scalars by solid lines and $\mathcal{N}=1$ gauginos by double-dashed lines.

$$F(\lambda\to 0)\approx -\frac{\pi^2}{6}N^2V_3T^4\left(1-\frac{3}{2\pi^2}\lambda\right). \tag{4}$$

The (relative) negative sign of the two-loop correction provides further indication that the free energy is a smooth, monotonic function of the 't Hooft coupling λ. In the following part of this note, we present some details of the two-loop computation leading to Eq. (4).

For the purpose of diagrammatic computations, it is convenient to use the $\mathcal{N}=1$ decomposition of $\mathcal{N}=4$ SYM [9], with one Majorana fermion corresponding to the gaugino, and the three remaining Majorana fermions combined with scalars in three $\mathcal{N}=1$ chiral multiplets. The two-loop diagrams are displayed in Fig. 1, together with the combinatorial/statistics factors. The two-loop integrals can be readily performed by using techniques described in Refs. [10]. In the table below, we list results for individual diagrams:[3]

A	B	C	D	E	F	G	H	I	J
$-\frac{9}{4}\alpha$	3α	$\frac{1}{4}\alpha$	$\frac{3}{4}\beta$	$\frac{1}{4}\beta$	$-\frac{9}{2}\alpha$	$\frac{3}{2}\beta$	$\frac{3}{2}\beta$	12α	$\frac{15}{2}\alpha$

(5)

where

$$\alpha=g^2c_AdV_3\left(\frac{T^4}{144}+\frac{T^2}{12(2\pi)^3}\int\frac{d^3\vec{k}}{|\vec{k}|}\right), \tag{6}$$

$$\beta=g^2c_AdV_3\left(\frac{5T^4}{144}-\frac{T^2}{3(2\pi)^3}\int\frac{d^3\vec{k}}{|\vec{k}|}\right), \tag{7}$$

with d denoting the dimension of the gauge group and c_A the Casimir operator in the adjoint representation. Note that individual diagrams contain ultraviolet divergences. After combining all contributions, we obtain the (finite) result:

$$F_{2\text{-loop}}=16\alpha+4\beta=\frac{1}{4}g^2c_AdV_3T^4. \tag{8}$$

Specified to the case of SU(N), with $d=N^2-1$ and $c_A=N$, in the leading large N order the above result yields Eq. (4).

Finally, we would like to make a few remarks on the structure of higher-order perturbative corrections. The computation of higher-order terms requires reorganizing the perturbation theory to account for Debye screening and yields terms nonanalytic in λ such as $\mathcal{O}(\lambda^{3/2})$ and $\mathcal{O}(\lambda^2\ln\lambda)$ [10,11]. The full $\mathcal{O}(\lambda^2)$ term requires a three-loop calculation [12] and a full accounting of Debye screening at three loops would produce the $\mathcal{O}(\lambda^{5/2})$ terms. However, perturbation theory is believed to be incapable of pushing the calculation to any higher order due to infrared problems associated with magnetic confinement and the presence of nonperturbative $\mathcal{O}(\lambda^3)$ contributions [10,11]. It would be very interesting to analyze from this point of view the strong coupling expansion.

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