Resummation of cactus diagrams in the clover improved lattice formulation of QCD

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We extend to the clover improved lattice formulation of QCD the resummation of cactus diagrams, i.e., a certain class of tadpole-like gauge invariant diagrams. Cactus resummation yields an improved perturbative expansion. We apply it to the lattice renormalization of some two-fermion operators improving their one-loop perturbative estimates. [S0556-2821(99)03905-3]

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In a previous work [1] we showed how to perform a resummation of a certain class of gauge invariant diagrams, termed cactus diagrams, in the Wilson formulation (for both gluons and fermions) of lattice QCD. The resummation of such diagrams led to an improved perturbative expansion, essentially by dressing the one-loop calculation of the lattice renormalizations. Applied to a number of cases of interest, this expansion yielded a remarkable improvement when compared with the available nonperturbative estimates. In this paper we extend such calculations to the case of the clover improved action formulation of lattice QCD [2], which is widely used in numerical simulations in order to reduce scaling corrections. In the following we will heavily refer to Ref. [1] for notation and many analytical results.

Cactus diagrams are tadpole diagrams which become disconnected if any one of their vertices is removed (see Fig. 1). Our original motivation was the well known observation of ''tadpole dominance'' in lattice perturbation theory. Indeed tadpoles diagrams are often largely responsible for lattice artifacts. This observation has already inspired many proposals to improve lattice pertubation theory, see e.g. [3,4]. Of course the contribution of standard tadpole diagrams is not gauge invariant. So we need to further specify the class of gauge invariant diagrams we are considering.

Let us write the so-called clover improved action

$$S_L=\frac{1}{g_0^2}\sum_{x,\mu\nu}\mathrm{Tr}\,[1-U_{x,\mu\nu}]+\sum_f\sum_x(4+m_{f,0})\overline{\psi}_x^f\psi_x^f$$
$$-\frac{1}{2}\sum_f\sum_{x,\mu}[\overline{\psi}_x^f(1-\gamma_\mu)U_{x,\mu}\psi_{x+\hat{\mu}}^f+\overline{\psi}_{x+\hat{\mu}}^f$$
$$\times(1+\gamma_\mu)U_{x,\mu}^\dagger\psi_x^f]+c_{\mathrm{SW}}\sum_f\sum_{x,\mu\nu}\overline{\psi}_x^f\frac{i}{4}\sigma_{\mu\nu}\hat{F}_{x,\mu\nu}\psi_x^f, \tag{1}$$

where f is a flavor index; $U_{x,\mu\nu}$ is the usual product of link variables $U_{x,\mu}$ along the perimeter of a plaquette originating at x in the positive μ-ν directions:

$$\hat{F}_{x,\mu\nu}=\frac{1}{8}(Q_{x,\mu\nu}-Q_{x,\nu\mu}), \tag{2}$$

$$Q_{x,\mu\nu}=U_{x,\mu}U_{x+\hat{\mu},\nu}U^\dagger_{x+\hat{\nu},\mu}U^\dagger_{x,\nu}$$
$$+U_{x,\nu}U^\dagger_{x-\hat{\mu}+\hat{\nu},\mu}U^\dagger_{x-\hat{\mu},\nu}U_{x-\hat{\mu},\mu}$$
$$+U^\dagger_{x-\hat{\mu},\mu}U^\dagger_{x-\hat{\mu}-\hat{\nu},\nu}U_{x-\hat{\mu}-\hat{\nu},\mu}U_{x-\hat{\nu},\nu}$$
$$+U^\dagger_{x-\hat{\nu},\nu}U_{x-\hat{\nu},\mu}U_{x+\hat{\mu}-\hat{\nu},\nu}U^\dagger_{x,\mu}. \tag{3}$$

The improvement coefficient c_{SW} can be calculated in perturbation theory as a function of g_0^2. Its tree-order value is $c_{\mathrm{SW}}=1$; in this case only the leading log scaling corrections of $O(a)$ are eliminated. More recently a nonperturbative determination has also been performed, which allows to completely cancel the $O(a)$ corrections [5,6].

By the Baker-Campbell-Hausdorff (BCH) formula we have

$$U_{x,\mu\nu}=e^{ig_0A_{x,\mu}}\,e^{ig_0A_{x+\mu,\nu}}\,e^{-ig_0A_{x+\nu,\mu}}\,e^{-ig_0A_{x,\nu}}$$
$$=\exp\{ig_0(A_{x,\mu}+A_{x+\mu,\nu}-A_{x+\nu,\mu}-A_{x,\nu})+\mathcal{O}(g_0^2)\}$$
$$=\exp\{ig_0F^{(1)}_{x,\mu\nu}+ig_0^2F^{(2)}_{x,\mu\nu}+\mathcal{O}(g_0^4)\}. \tag{4}$$

The diagrams that we propose to resum to all orders are the cactus diagrams made of vertices containing $F^{(1)}_{x,\mu\nu}$. Terms of this type come from the pure gluon and clover parts of the lattice action.

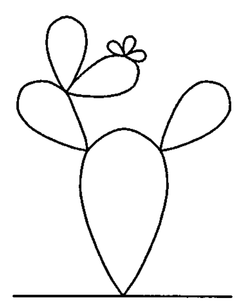

FIG. 1. A cactus.

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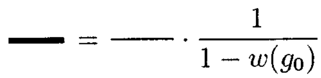
In Ref. [1] we showed how these diagrams dress the gluon propagator and the gluon vertices [we denote by a thick (thin) solid line the transverse dressed (bare) gluon propagator]:

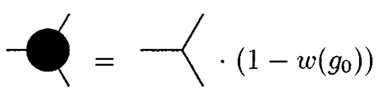


$$\cdot \frac{1}{1-w(g_0)} \qquad (5)$$

where the function $w(g_0)$ can be extracted by an appropriate algebraic equation that has been derived in Ref. [1] and that can be easily solved numerically; for $SU(3)$, $w(g_0)$ satisfies

$$ue^{-u/3}[u^2/3-4u+8]=2g_0^2, \quad u(g_0)\equiv\frac{g_0^2}{4(1-w(g_0))}. \qquad (6)$$

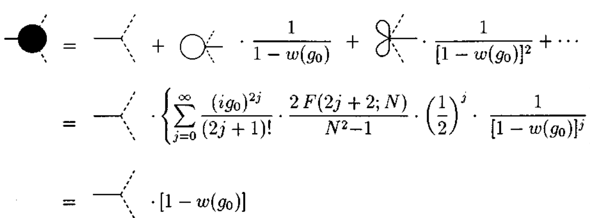
The 3-point vertex dresses as



$$\cdot (1-w(g_0)) \qquad (7)$$

and similarly for other vertices. Contributions to vertices coming from the standard Wilson fermionic action stay unchanged, since their definition contains no plaquettes on which to apply the linear BCH formula. In the clover improved action formulation plaquettes appear in the new fermionic term; thus in this case one should also dress the new fermion-gluon vertices originating from this term.

Let us now prove that the fermion-gluon three-point vertex coming from the clover term gets dressed as the three-gluon vertex, cf. Eq. (7). Proceeding as in Ref. [1] [cf. Eq. (20) and Appendix B therein], we write for the fermion-gluon three-point vertex:



$$\cdot \frac{1}{1-w(g_0)} + \cdots \cdot \frac{1}{[1-w(g_0)]^2}+\cdots$$
$$= \cdots \cdot \left\{\sum_{j=0}^{\infty}\frac{(ig_0)^{2j}}{(2j+1)!}\cdot\frac{2F(2j+2;N)}{N^2-1}\cdot\left(\frac{1}{2}\right)^j\cdot\frac{1}{[1-w(g_0)]^j}\right\}$$
$$= \cdots \cdot [1-w(g_0)] \qquad (8)$$

[solid (dashed) lines represent gluons (fermions)]. No other dressed vertices are necessary in most of the interesting applications, that essentially amount to a dressing of the perturbative one-loop calculation. In these cases the dressing of the fermion-gluon three-point vertex in the one-loop calculation is equivalent to a rescaling of the constant c_{SW}:

$$c_{SW}\rightarrow\overline{c_{SW}}\equiv c_{SW}\cdot(1-w(g_0)). \qquad (9)$$

One can apply the resummation of cactus diagrams to the calculation of the renormalizations of lattice operators. Approximate expressions are obtained by dressing the corresponding one-loop calculations. In the case of operators whose anomalous dimension is zero in the modified minimal subtraction ($\overline{\mathrm{MS}}$) renormalization scheme, a consistent means of implementing the cactus dressing is to apply it to the one-loop difference between lattice and continuum contributions that determine the finite renormalization. Cases with nonzero anomalous dimension can be dealt with in an analogous manner, by setting the scale $\mu=1/a$ and dressing the finite renormalization coefficients as before.

In the following we present a few examples of lattice renormalizations for which non-pertubative evaluations are available in the literature. Let us consider the non-singlet vector and axial currents $V_\mu^a=\bar{\psi}\lambda^a\gamma_\mu\psi$ and $A_\mu^a=\bar{\psi}\lambda^a\gamma_\mu\gamma_5\psi$ and the renormalization of their lattice counterparts. So far, essentially three nonperturbative methods have been successfully implemented in the computation of such renormalizations: (i) use of the Ward identities [7] (WI); (ii) nonperturbative renormalization on external quark and gluon states [8] (NP); (iii) use of finite size scaling techniques [9] (FSS). The major source of systematic error in these calculations is due to $O(a)$ scaling violations. Already for the tree clover improved action they turn out to be rather small at $g_0^2\simeq 1$, where simulations are actually done. So nonperturbative estimates are quite reliable. In the case of the tree clover improved action, scaling corrections are estimated to be less than 5% at $g_0^2\simeq 1$ using the WI approach [10,11]. Lattice renormalizations can be also calculated in perturbation theory. Most perturbative calculations have been performed to one loop. Thus their use as approximation of the lattice renormalizations introduces $O(g_0^4)$ errors in the final estimates of physical quantities [to be compared with the $O(a)$ scaling corrections of the nonperturbative methods]. Many recipes of improvement have been proposed (see e.g. [4], and [11] for a review of them) that essentially consist in a better choice of the expansion parameter. Among them we mention the so-called tadpole improvement [4] (MFI) motivated by mean-field arguments, in which one scales the link variable with $u_0(g_0^2)\equiv\langle(1/N)\mathrm{Tr}\,U_{x,\mu\nu}\rangle^{1/4}$ as measured in the Monte Carlo simulation. Accordingly one rescales the coupling constant: $g_0^2\rightarrow g_{\mathrm{mf}}^2=g_0^2/u_0^4$. Thus, if at one loop: $Z=1+z_1g_0^2+O(g_0^4)$, one obtains a mean-field improved expansion by

$$Z=u_0\left[1+g_{\mathrm{mf}}^2\left(z_1+\frac{1}{12}\right)+O(g_{\mathrm{mf}}^4)\right]. \qquad (10)$$

For example, for $SU(3)$ in the quenched approximation and at $g_0^2=1$ one finds $u_0\simeq 0.878$ and $g_{\mathrm{mf}}^2\simeq 1.68$. A more naive and simple recipe of improvement consists just in the change of variable $g_0\rightarrow g_{\mathrm{mf}}$ in the standard perturbative expansion (NMFI).

In the context of the clover action, the following improved lattice operators have been considered [12]:

$$\bar{\psi}\left[1+\frac{1}{4}(\gamma_\alpha\overleftarrow{D}_\alpha-m_0)\right]\lambda^a\Gamma\left[1-\frac{1}{4}(\gamma_\beta\overrightarrow{D}_\beta+m_0)\right]\psi \qquad (11)$$

where $\Gamma=\gamma_\mu,\gamma_\mu\gamma_5$ for V_μ^a and A_μ^a respectively, and D_μ is the symmetric lattice covariant derivative. Their one-loop renormalization is known [13]

TABLE I. Some estimates of Z_V and Z_A for the operators (11) and the tree-improved clover action at $g_0^2=1$ $(\beta=6)$.

Method	Z_V	Z_A
PT	0.90	0.98
CI	0.86	1.00
MFI	0.85	0.97
NMFI	0.83	0.97
VWI [11]	0.82	
AWI [11]	0.80(2)	1.11(2)
NP [8]	0.84(1)	1.06(8)

$$Z_{V,A}=1+z_{V,A}g_0^2+O(g_0^4), \tag{12}$$

where

$$z_V(c_{\rm SW})=\frac{c_F}{16\pi^2}\,(-14.36+3.30c_{\rm SW}-0.75c_{\rm SW}^2), \tag{13}$$

$$z_A(c_{\rm SW})=\frac{c_F}{16\pi^2}\,(6.87-12.54c_{\rm SW}+3.55c_{\rm SW}^2). \tag{14}$$

The cactus dressing of the above one-loop expressions can be simply obtained by using the dressed transverse gluon propagator (5) and by rescaling $c_{\rm SW}$ according to Eq. (9). We thus obtain the following approximate expressions:

$$Z_{V,A}\approx 1+g_0^2\frac{z_{V,A}(\overline{c}_{\rm SW})}{1-w(g_0^2)}. \tag{15}$$

Nonperturbative numerical calculations of $Z_{V,A}$ for the tree-improved clover action (i.e. $c_{\rm SW}=1$) and at $g_0^2=1$ have been obtained in quenched theory by imposing vector (V) and axial (A) WI's and by nonperturbative renormalization on quark states (NP). In Table I we list these results and compare them with the one-loop perturbative calculation (PT), our cactus dressing (CI) of the one-loop expression, the mean-field inspired improvement (MFI) and the result that one obtains just by substituting g_0^2 with $g_{\rm mf}^2$ (NMFI). Results from other recipes can be found in Ref. [11]. In the case of Z_V all improved perturbative estimates get closer to the nonperturbative results, thus improve PT. On the other hand, in the case of Z_A the simple change of coupling from g_0 to $g_{\rm mf}$ (NMFI) does not help. Since $g_{\rm mf}>g_0$, it increases the one-loop perturbative correction that has the ''wrong'' sign, thus worsening the plain one-loop estimate. Similarly, a change of coupling and momentum scale, in the manner of Lepage and Mackenzie [4], also worsen the PT estimate as the corresponding $g(q^*)$ (defined in [4]) turns out to be larger than g_0. In the case of Z_A the only procedure improving PT is cactus resummation, but its estimate is still relatively far from the nonperturbative result.

In Ref. [9] the lattice renormalizations of two further lattice operators corresponding to V_μ and A_μ have been calculated nonperturbatively employing finite size scaling techniques and using the nonperturbative estimate of $c_{\rm SW}$. The lattice operators were

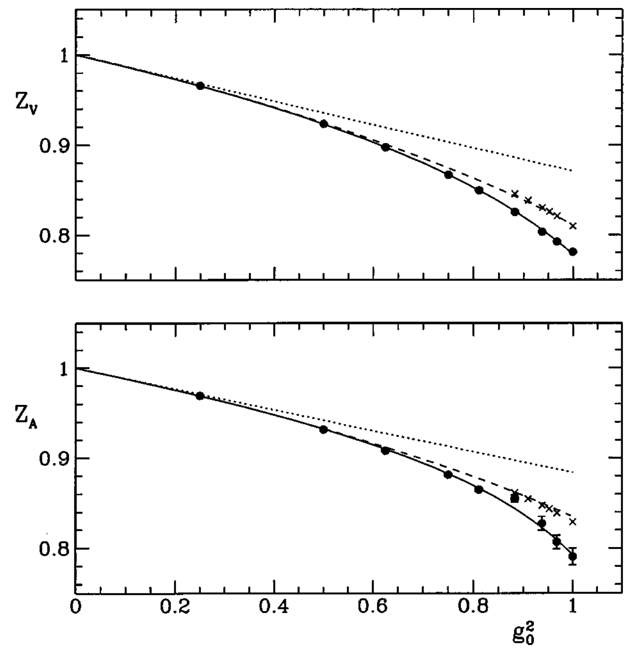


FIG. 2. Results for Z_V and Z_A (from Ref. [9]), coming from numerical simulations (filled circles, fitted by a solid line), bare perturbation theory (dotted lines) and ''mean field improved'' perturbation theory (crosses). The dashed lines superimposed on these figures are our results from cactus dressing.

$$V_\mu^L=\overline{\psi}\lambda^a\gamma_\mu\psi+c_V\frac{1}{2}(\Delta_\mu^-+\Delta_\mu^+)i\overline{\psi}\lambda^a\sigma_{\mu\nu}\psi \tag{16}$$

$$A_\mu^L=\overline{\psi}\lambda^a\gamma_\mu\gamma_5\psi+c_A\frac{1}{2}(\Delta_\mu^-+\Delta_\mu^+)\overline{\psi}\lambda^a\gamma_5\psi, \tag{17}$$

where $c_{V,A}$ are $O(g_0^2)$ constants, and the corresponding terms serve to obtain on-shell improved operators. Their perturbative renormalization is given by formula (12) with [14]

$$z_V(c_{\rm SW})=\frac{c_F}{16\pi^2}(-20.62+4.75c_{\rm SW}+0.54c_{\rm SW}^2), \tag{18}$$

$$z_A(c_{\rm SW})=\frac{c_F}{16\pi^2}(-15.80-0.25c_{\rm SW}+2.25c_{\rm SW}^2). \tag{19}$$

In Fig. 2 we compare the nonperturbative calculations of Ref. [9] with the one-loop and the dressed one-loop calculations. A remarkable improvement is observed. In this case the cactus resummation performs as the mean-field inspired boosted perturbation theory (MFI). As already noted in Ref. [9], the nonperturbative data are best reproduced by NMFI.

It is clear that nonperturbative methods are in general preferable to approximations based on perturbative calculations, due to their better controlled systematic errors [$O(a)$

against $O(g_0^n)$]. However, improved perturbative estimates are still quite useful. They indeed provide important consistency checks. Further, in those cases where nonperturbative methods are difficult to implement, perturbative methods remain the only source of quantitative information. In this report we have shown how to extend to the clover improved lattice formulation of QCD the resummation of cactus diagrams, which represents a direct implementation of the idea of tadpole dominance. The examples considered here and in Ref. [1] show that the resummation of cactus diagrams leads to a general improvement in the evaluation of the lattice renormalizations based on perturbation theory. The comparison with the corresponding nonperturbative calculations is globally satisfactory. Of course, cactus resummation may also be applied to the lattice renormalizations of other operators without further complications.

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