Overlap formula for the chiral multiplet

Tatsumi Aoyama* and Yoshio Kikukawa†
Department of Physics, Kyoto University, Kyoto 606-8502, Japan
(Received 24 March 1998; published 8 February 1999)

The vacuum overlap formalism is extended to describe the supersymmetric multiplet of a Weyl fermion, a complex scalar boson and an auxiliary field in the case without an interaction, based on the fact that supersymmetry can be maintained up to quadratic terms by introducing bosonic species doublers. We also obtain a local action which describes the chiral multiplet and discuss its symmetry structure. It is shown that, in addition to the manifest supersymmetry, the action possesses a chiral symmetry of the type given by Lüscher and analogous bosonic symmetries which may be regarded as independent infinitesimal rotations of complex phases of the scalar and the auxiliary fields. This implies that the $U(1)\times U(1)_R$ symmetry of the chiral multiplet can be exact on the lattice. [S0556-2821(98)05419-8]

PACS number(s): 11.15.Ha, 11.30.Pb, 11.30.Rd

I. INTRODUCTION

The vacuum overlap formula [1,2] provides a well-defined lattice regularization of the chiral determinant. It can reproduce the known features of the chiral determinant in the continuum theory: the one-loop effective action of the background gauge field [3] including the consistent anomaly [4], topological charges and fermionic zero modes associated with the topologically nontrivial gauge fields [1,2,5], the SU(2) global anomaly [6,7], and so on. These properties of this formalism allow the description of fermion number violation on the lattice. Several numerical applications [8] have been performed. Their results strongly suggest that the overlap formalism can actually be a promising building block for the construction of lattice chiral gauge theories.

When applied to vectorlike theories like QCD, the overlap formalism also provides a quite satisfactory description of the massless Dirac fermion. It has been shown by Neuberger [9,10] that the vacuum overlaps for the massless Dirac fermion can be written as a single determinant of a Dirac operator which satisfies the Ginsparg-Wilson relation [11]. Moreover, Lüscher has shown that the action given by the Dirac operator possesses an exact chiral symmetry [12]. For the free fermion, the Dirac operator defines a local action. The Ginsparg-Wilson relation is the clue to escape the Nielsen-Ninomiya theorem [13].

The use of the overlap formalism to describe the supersymmetric Yang-Mills theory was also suggested in [2]. It has been further considered in [14,6,10].

In this paper, we discuss an extension of the vacuum overlap formalism to describe the chiral multiplet [15] of a single Weyl fermion and a complex scalar boson in the case without an interaction. We first formulate a supersymmetric version of the domain-wall system, based on the fact that supersymmetry can be maintained up to quadratic terms by introducing bosonic species doublers. Then we derive the overlap formula of the partition function of the chiral multiplet. In order to examine the symmetry structure of the theory of the chiral multiplet so obtained, we next derive a local action with which a functional integral reproduces the partition function of the chiral multiplet. We find that, besides the manifest supersymmetry, the action possesses a chiral symmetry of the type given by Lüscher and analogous bosonic symmetries which may be regarded as independent infinitesimal rotations of complex phases of the scalar and the auxiliary fields. This means that all the symmetries of the continuum theory can be manifest on the lattice in this formalism.

II. BOSONIC SPECIES DOUBLERS, LATTICE SUPERSYMMETRY, AND THE BREAKDOWN OF $U(1)\times U(1)_R$ SYMMETRY

It is known that there are several difficulties in formulating supersymmetric theories on the lattice [16–22]. One of the difficulties is that the Leibniz rule breaks down on the lattice:

$$\sum_n \{\partial_\mu A_n\cdot B_n\cdot C_n + A_n\cdot\partial_\mu B_n\cdot C_n + A_n\cdot B_n\cdot\partial_\mu C_n\}\neq 0, \tag{1}$$

where ∂_μ is taken as the symmetric lattice derivative

$$\partial_\mu=\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu). \tag{2}$$

∇_μ and $\bar{\nabla}_\mu$ are the nearest-neighbor forward and backward difference operators, respectively. This breakdown of the Leibniz rule causes difficulty in introducing a local cubic interaction in a supersymmetric manner.

On the other hand, since the lattice derivative is anti-Hermitian, the quadratic rule holds true:

$$\sum_n \{\partial_\mu A_n\cdot B_n + A_n\cdot\partial_\mu B_n\}=0. \tag{3}$$

Then free lattice theories may possess manifest supersymmetry. In fact, it is known that, if species doublers are also introduced for the boson, the free lattice theory of a single Weyl fermion and a complex scalar boson has a manifest supersymmetry.

*Email address: aoyama@gauge.scphys.kyoto-u.ac.jp
†Email address: kikukawa@gauge.scphys.kyoto-u.ac.jp

0556-2821/99/59(5)/054507(10)/$15.00

This fact can easily be understood by considering the cancellation of the free energy. The action of a free Weyl fermion is given on a lattice by[1]

$$S_F=\sum_n \psi_n^\dagger\sigma_\mu\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\psi_n. \tag{4}$$

The free energy of this fermion is given by

$$F_F=-\ln\left[\prod_p \det i\sigma_\mu \sin p_\mu\right]=-\ln\left[\prod_p \sin^2 p_\mu\right]. \tag{5}$$

We can see that there are 15 species doublers. On the other hand, the action of the free complex boson is usually taken as the following form:

$$S_B=-\sum_n \phi_n^\dagger\bar{\nabla}_\mu\nabla_\mu\phi_n, \tag{6}$$

and its free energy is given by

$$F_B=\ln\left[\prod_p 4\sin^2\frac{p_\mu}{2}\right]. \tag{7}$$

Since there is no species doubler for the boson, the bosonic free energy cannot cancel the fermionic one. However, if we adopt the following action for the boson,

$$S_B=-\sum_n \phi_n^\dagger\left\{\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\right\}^2\phi_n, \tag{8}$$

then its free energy is given by

$$F_B=\ln\left[\prod_p \sin^2 p_\mu\right]. \tag{9}$$

Fifteen species doublers for the boson appear and their free energies exactly cancel the free energies of species doublers of the fermion, as long as both the fermion and the boson are subject to the same boundary condition.[2]

Thus we are led to consider the lattice theory whose action is given by

$$\begin{aligned}S_0=\sum_n &\left\{\bar{\psi}_n\sigma_\mu\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\psi_n\right.\\ &\left.-\phi_n^*\left[\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\right]^2\phi_n-F_n^*F_n\right\}.\end{aligned} \tag{10}$$

The supersymmetry transformation is then defined by

$$\delta\phi_n=-\sqrt{2}\,\xi^T\epsilon\psi_n, \tag{11}$$

$$\delta\phi_n^*=+\sqrt{2}\,\bar{\psi}_n\epsilon\,\bar{\xi}^T, \tag{12}$$

$$\delta\psi_n=\sqrt{2}\,\sigma_\mu^\dagger\,\epsilon\,\bar{\xi}^T\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\phi_n+\sqrt{2}\,\xi F_n, \tag{13}$$

$$\delta\bar{\psi}_n=\sqrt{2}\,\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\phi_n^*\,\xi^T\epsilon\,\sigma_\mu^\dagger-\sqrt{2}\,\bar{\xi}F_n^*, \tag{14}$$

$$\delta F_n=-\sqrt{2}\,\bar{\xi}\sigma_\mu\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\psi_n, \tag{15}$$

$$\delta F_n^*=-\sqrt{2}\,\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\bar{\psi}_n\sigma_\mu\xi. \tag{16}$$

ξ and $\bar{\xi}$ are the parameters of the supertransformation. $\epsilon=\epsilon_{\alpha\beta}$ stands for the invariant antisymmetric tensor in spinor space with the convention $\epsilon_{12}=1$.

The degeneracy due to the species doublers of both the fermion and the boson can be resolved by adding the Majorana-type Wilson mass term

$$M_w=\left(m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right), \tag{17}$$

in a supersymmetric manner:

$$\begin{aligned}S_{MW}=\sum_n &\left\{\bar{\psi}_n\sigma_\mu\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\psi_n\right.\\ &\left.-\phi_n^*\left[\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\right]^2\phi_n-F_n^*F_n\right\}\\ &+\sum_n\frac{1}{2}\{\psi_n^T\epsilon\,M_w\psi_n+\phi_nM_wF_n+F_nM_w\phi_n\\ &-\psi_n^\dagger\epsilon\,M_w\psi_n^*+\phi_n^*M_wF_n^*+F_n^*M_w\phi_n^*\}.\end{aligned} \tag{18}$$

Note, however, that because of the introduction of the Majorana-Wilson term, the chiral symmetry of the fermion,

$$\psi_n\rightarrow\exp(i\alpha)\psi_n, \tag{19}$$

and the symmetry associated with the coherent rotation of the complex phases of the scalar and the auxiliary fields,

$$\phi_n\rightarrow\exp(i\beta)\phi_n, \tag{20}$$

$$F_n\rightarrow\exp(i\beta)F_n, \tag{21}$$

are lost. This means that $U(1)\times U(1)_R$ symmetry of the chiral multiplet cannot be maintained by this procedure. In the following, we will formulate the same theory through the vacuum overlap formalism and will find that it can maintain all the required symmetries.

III. SUPERSYMMETRIC DOMAIN-WALL SYSTEM

Since the domain-wall fermion [23] can be regarded as a collection of an infinite flavors of the Wilson fermion with a specific mass matrix [24,25], it is then rather straightforward to formulate the supersymmetric version of the domain-wall system. The action may be written in the form

[1] In our convention, σ_μ matrices are defined by $\sigma_\mu=(\mathbb{1},i\sigma_k)$.

[2] In the following discussion, we will assume that this is the case.

$$
\begin{aligned}
S = a\sum_{ns} & \left\{ \overline{\psi}_{1ns}\sigma_\mu \frac{1}{2}(\nabla_\mu+\overline{\nabla}_\mu)\psi_{1ns} - \phi^\dagger_{1ns}\left[\frac{1}{2}(\nabla_\mu+\overline{\nabla}_\mu)\right]^2\phi_{1ns} - F^*_{1ns}F_{1ns}\right\} \\
& + a\sum_{ns} \left\{ \overline{\psi}_{2ns}\sigma_\mu \frac{1}{2}(\nabla_\mu+\overline{\nabla}_\mu)\psi_{2ns} - \phi^\dagger_{2ns}\left[\frac{1}{2}(\nabla_\mu+\overline{\nabla}_\mu)\right]^2\phi_{2ns} - F^*_{2ns}F_{2ns}\right\} \\
& + a\sum_{nst} \{\psi^T_{1ns}\epsilon\, M(s,t)\psi_{2nt} + \phi_{1ns}M(s,t)F_{2nt} + F_{1ns}M(s,t)\phi_{2nt} \\
& - \overline{\psi}_{2ns}\epsilon\, M^\dagger(s,t)\overline{\psi}^T_{1nt} + \phi^*_{2ns}M^\dagger(s,t)F^*_{1nt} + F^*_{2ns}M^\dagger(s,t)\phi^*_{1nt}\},
\end{aligned}
\tag{22}
$$

where

$$
M(s,t) = \left[-m_0\,\mathrm{sgn}\left(s+\frac{1}{2}\right) - \frac{1}{2}\nabla_\mu\overline{\nabla}_\mu\right]\delta_{st} + \frac{1}{a}(\delta_{st}-\delta_{s+1,t}).
\tag{23}
$$

We only keep the lattice spacing of the fifth dimension, which we denote by a, for convenience in deducing the Hamiltonian. Hereafter we will suppress the indexes for four-dimensional lattices, n,m.

IV. TRANSFER MATRIX OF THE BOSONIC SECTOR

The transfer matrix of the bosonic part can be obtained by the standard method, which is quite analogous to the fermionic case given by Narayanan and Neuberger in [1]. It is also possible to obtain it starting from the vectorlike formulation [26], which may be truncated at finite flavors just as discussed by Neuberger in [25]. Here we follow the latter method for simplicity. We will summarize the derivation of the transfer matrix by the former method in Appendix A.

Following [25], instead of considering the domain-wall system, we simply take the k flavor pairs of free chiral multiplets from the negative mass region of $-m_0$. We introduce the bosonic field variable defined as

$$
\Phi_s = \begin{pmatrix} F_{2s} \\ \phi_{2s} \\ \phi^*_{1s} \\ F^*_{1s} \end{pmatrix}.
\tag{24}
$$

Then the bosonic part of the action may be written as

$$
S_B = \Phi^\dagger_s \mathcal{D}_B(s,t)\Phi_t ,
\tag{25}
$$

where

$$
\mathcal{D}_B = \begin{pmatrix}
\sigma_1\tilde{C}\sigma_1 & B\mathbb{1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
B\mathbb{1} & \tilde{C} & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & -1 & \sigma_1\tilde{C}\sigma_1 & B\mathbb{1} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & B\mathbb{1} & \tilde{C} & -1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & -1 & \sigma_1\tilde{C}\sigma_1 & B\mathbb{1} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & B\mathbb{1} & \tilde{C} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \sigma_1\tilde{C}\sigma_1 & B\mathbb{1} \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & B\mathbb{1} & \tilde{C}
\end{pmatrix}.
\tag{26}
$$

This is the $4k\times 4k$ matrix-valued operator acting on the lattice index, which we have suppressed for simplicity. $\tilde{C}$ is a 2×2 matrix defined by

$$\tilde{C}=\begin{pmatrix} -\left\{\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\right\}^2 & 0 \\ 0 & -1 \end{pmatrix}. \tag{27}$$

$\mathbb{1}$ is a 2×2 unit matrix and the factor B is defined by

$$B=1+a\left(-m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right). \tag{28}$$

The partition function of the bosonic part reads

$$Z_B=\int\prod_s \mathcal{D}\Phi_s\mathcal{D}\Phi_s^\dagger \exp(-S_B)=(\det\mathcal{D}_B)^{-1}. \tag{29}$$

In evaluating the determinant, we can apply the formula given in the Appendix of [25]. The result is

$$\det\mathcal{D}_B=(-)^{qk}(\det B\mathbb{1})^k\det[1+e^{kaH_B}]\times\det\left[\frac{1+\Gamma_5\tanh\left(\frac{k}{2}aH_B\right)}{2}\right], \tag{30}$$

where the transfer matrix is identified as

$$e^{-aH_B}=\begin{pmatrix} \frac{1}{B}\mathbb{1} & -a\frac{1}{B}\tilde{C} \\ a\sigma_1\tilde{C}\sigma_1\frac{1}{B} & -a^2\sigma_1\tilde{C}\sigma_1\frac{1}{B}\tilde{C}+B\mathbb{1} \end{pmatrix}, \tag{31}$$

$$\Gamma_5=\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{32}$$

The result of the fermionic sector has been obtained by Neuberger in [25]:

$$\det\mathcal{D}_F=(-)^{qk}(\det B\mathbb{1})^k\times\det[1+e^{kaH_F}]\det\left[\frac{1+\gamma_5\tanh\left(\frac{k}{2}aH_F\right)}{2}\right], \tag{33}$$

where the transfer matrix is identified as

$$e^{-aH_F}=\begin{pmatrix} \frac{1}{B}\mathbb{1} & a\frac{1}{B}C \\ aC^\dagger\frac{1}{B} & a^2C^\dagger\frac{1}{B}C+B\mathbb{1} \end{pmatrix} \tag{34}$$

and

$$C=\sigma_\mu\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu). \tag{35}$$

Because of supersymmetry, this determinant of the fermionic part is identical to the determinant of the bosonic part:

$$\det\mathcal{D}_F=\det\mathcal{D}_B. \tag{36}$$

This is also true if we take the antiperiodic boundary condition in the fifth dimension (flavor space), which implies that

$$(-)^{qk}(\det B\mathbb{1})^k\det[1+e^{kaH_F}]=(-)^{qk}(\det B\mathbb{1})^k\det[1+e^{kaH_B}]. \tag{37}$$

Therefore, the supersymmetric relation implies that the total partition function of the chiral multiplets is unity, but it may be written as

$$Z=\det\left[\frac{1+\gamma_5\tanh\left(\frac{k}{2}aH_F\right)}{2}\right]\times\frac{1}{\det\left[\frac{1+\Gamma_5\tanh\left(\frac{k}{2}aH_B\right)}{2}\right]}. \tag{38}$$

Taking the limit of the infinite extent of the fifth dimension (infinite number of flavors) and the continuum limit in the fifth dimension (flavor space), we obtain a formula for the partition function of the ''vectorlike'' pair of the chiral multiplets:

$$Z=\det\left[\frac{1+\gamma_5\epsilon(H_F)}{2}\right]\frac{1}{\det\left[\frac{1+\Gamma_5\epsilon(H_B)}{2}\right]}, \tag{39}$$

where

$$\epsilon(H)=\frac{H}{\sqrt{H^2}}, \tag{40}$$

$$H_F=\begin{pmatrix} \left(-m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right)\mathbb{1} & C \\ C^\dagger & -\left(-m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right)\mathbb{1} \end{pmatrix}, \tag{41}$$

$$H_B=\begin{pmatrix} \left(-m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right)\mathbb{1} & \tilde{C} \\ -\sigma_1\tilde{C}\sigma_1 & -\left(-m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right)\mathbb{1} \end{pmatrix}. \tag{42}$$

Note also that

$$H_F^2=H_B^2=X^\dagger X\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{43}$$

$$X^\dagger X = \left[-\left\{ \frac{1}{2}(\nabla_\mu + \overline{\nabla}_\mu) \right\}^2 + \left(-m_0 - \frac{1}{2}\nabla_\mu \overline{\nabla}_\mu \right)^2 \right]. \quad (44)$$

V. OVERLAP FORMULA FOR THE CHIRAL MULTIPLET

Next we discuss the factorization property of the partition function of the ''vectorlike'' pair of the chiral multiplets given by Eq. (39) into overlaps. We follow the argument given by Narayanan [27]. H_F is a Hermitian matrix and can be diagonalized by a certain unitary matrix:

$$H_F U = U \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \quad (45)$$

where $\lambda_\pm$ are the diagonal matrices which consist of the positive eigenvalues and negative eigenvalues of H_F, respectively. We denote U in the chiral basis as

$$U = \begin{pmatrix} U_{R+} & U_{R-} \\ U_{L+} & U_{L-} \end{pmatrix}. \quad (46)$$

For the free theory, it is possible to fix the phases of the eigenvectors so that the determinant of U would be unity:

$$\det U = 1. \quad (47)$$

Using U, the Dirac operator can be written as

$$\frac{1+\gamma_5 \epsilon(H_F)}{2} = \begin{pmatrix} U_{R+} & 0 \\ 0 & U_{L-} \end{pmatrix} U^\dagger, \quad (48)$$

and we obtain

$$\det\left[\frac{1+\gamma_5 \epsilon(H_F)}{2}\right] = \det U_{R+} \det U_{L-}. \quad (49)$$

In the bosonic case, H_B is a real nonsymmetric matrix. But it can be diagonalized by a certain matrix:

$$H_B O = O \begin{pmatrix} \lambda'_+ & 0 \\ 0 & \lambda'_- \end{pmatrix}, \quad (50)$$

where $\lambda'_\pm$ are the diagonal matrices which consist of the positive eigenvalues and negative eigenvalues of H_B, respectively. It is also possible to normalize the eigenvectors so that the determinant of O would be unity.

$$\det O = 1. \quad (51)$$

If we denote O in the ''chiral basis'' with respect to Γ_5 as

$$O = \begin{pmatrix} O_{R+} & O_{R-} \\ O_{L+} & O_{L-} \end{pmatrix}, \quad (52)$$

we have

$$\frac{1+\Gamma_5 \epsilon(H_B)}{2} = \begin{pmatrix} O_{R+} & 0 \\ 0 & O_{L-} \end{pmatrix} O^{-1}. \quad (53)$$

Therefore we obtain

$$\det\left[\frac{1+\Gamma_5 \epsilon(H_B)}{2}\right] = \det O_{R+} \det O_{L-}. \quad (54)$$

From these factorization properties, we may define the partition function of the single chiral multiplet by the following overlaps:

$$Z_{chiral} = \det U_{L+} \frac{1}{\det O_{L+}}. \quad (55)$$

In fact, if U and O are chosen as given in Appendix B, the overlaps are evaluated as

$$\det U_{L+} = \det O_{L+} = \det\left[\frac{\left(\sqrt{X^\dagger X} - m_0 - \frac{1}{2}\nabla\overline{\nabla} \right)}{2\sqrt{X^\dagger X}} \right] \quad (56)$$

and the supersymmetric relation holds true.

VI. ACTION OF THE CHIRAL MULTIPLET

We will next discuss a local action which describes the chiral multiplet. It can be obtained by reducing the half degrees of freedom of the field variables in the vectorlike theory with the Majorana condition. Once the local action is given, we can discuss the symmetry structure of the theory.

The partition function of the vectorlike pair of the chiral multiplets can be expressed as a functional integral based with a local action:

$$Z = \int \mathcal{D}\Psi \mathcal{D}\overline{\Psi} \mathcal{D}\Phi \mathcal{D}\overline{\Phi} \exp(-\overline{\Psi} D_F \Psi - \overline{\Phi} D_B \Phi), \quad (57)$$

where

$$D_F = 1 + \gamma_5 H_F \frac{1}{\sqrt{H_F^2}}, \quad (58)$$

$$D_B = 1 + \Gamma_5 H_B \frac{1}{\sqrt{H_B^2}}. \quad (59)$$

Ψ and Φ are the field variables which describe the vectorlike pair of the chiral multiplets and may be written in terms of components as

$$\Psi_n = \begin{pmatrix} \psi_{2n} \\ \psi_{1n} \end{pmatrix}, \qquad \overline{\Psi}_n = (\overline{\psi}_{1n} \quad \overline{\psi}_{2n}), \quad (60)$$

$$\Phi_n = \begin{pmatrix} F_{2n} \\ \phi_{2n} \\ \phi^*_{1n} \\ F^*_{1n} \end{pmatrix}, \quad \overline{\Phi}_n = (\phi_{1n} \quad F_{1n} \quad F^*_{2n} \quad \phi^*_{2n}). \quad (61)$$

As shown by Neuberger, D_F satisfies the Ginsparg-Wilson relation

$$D_F \gamma_5 + \gamma_5 D_F = D_F \gamma_5 D_F. \quad (62)$$

Furthermore, as shown by Lüscher, the fermionic action possesses the symmetry under the chiral transformation

$$\delta\Psi=\gamma_5\left(1-\frac{D_F}{2}\right)\Psi,\quad \delta\bar{\Psi}=\bar{\Psi}\left(1-\frac{D_F}{2}\right)\gamma_5. \tag{63}$$

It is interesting to note that the bosonic operator D_B satisfies an analogous relation

$$D_B\Gamma_5+\Gamma_5 D_B=D_B\Gamma_5 D_B. \tag{64}$$

Accordingly, the bosonic action is symmetric under the transformation

$$\delta\Phi=\Gamma_5\left(1-\frac{D_B}{2}\right)\Phi,\quad \delta\bar{\Phi}=\bar{\Phi}\left(1-\frac{D_B}{2}\right)\Gamma_5. \tag{65}$$

The action of the chiral multiplet can be obtained by reducing the half degrees of freedom of the field variables in Φ and Ψ with the Majorana condition. The Majorana condition on the fermion implies that ψ_{2n} is identified with the conjugate of ψ_{1n} as[3]

$$\psi_{2n}=-\epsilon\bar{\psi}_{1n}^T,\bar{\psi}_{2n}=\psi_{1n}^T\epsilon. \tag{68}$$

For the boson, we mean by the Majorana condition that ϕ_{2n} and F_{2n} are identified with the complex conjugates of ϕ_{1n} and F_{1n}, respectively:

$$\phi_{2n}=\phi_{1n}^*,\quad F_{2n}=F_{1n}^*. \tag{69}$$

Then we may express Φ and Ψ in terms of a two-component Weyl spinor, a complex scalar, and an auxiliary field as

$$\Psi_n=\frac{1}{\sqrt{2}}\begin{pmatrix}-\epsilon\bar{\psi}_n^T\\ \psi_n\end{pmatrix},\quad \bar{\Psi}_n=\frac{1}{\sqrt{2}}(\bar{\psi}_n\quad \psi_n^T\epsilon), \tag{70}$$

$$\Phi_n=\frac{1}{\sqrt{2}}\begin{pmatrix}F_n\\ \phi_n\\ \phi_n^*\\ F_n^*\end{pmatrix},\quad \bar{\Phi}_n=\frac{1}{\sqrt{2}}(\phi_n\quad F_n\quad F_n^*\quad \phi_n^*). \tag{71}$$

Accordingly, the above action reduces to

$$\begin{aligned}S=\sum_n\Bigg\{&\bar{\psi}_n\sigma_\mu\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\frac{1}{\sqrt{X^\dagger X}}\psi_n\\
&-\phi_n^*\left[\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\right]^2\frac{1}{\sqrt{X^\dagger X}}\phi_n-F_n^*\frac{1}{\sqrt{X^\dagger X}}F_n\Bigg\}\\
&+\sum_n\frac{1}{2}\Bigg\{\psi_n^T\epsilon\left(1+\frac{M_w^-}{\sqrt{X^\dagger X}}\right)\psi_n+\phi_n\left(1+\frac{M_w^-}{\sqrt{X^\dagger X}}\right)F_n\\
&+F_n\left(1+\frac{M_w^-}{\sqrt{X^\dagger X}}\right)\phi_n\\
&-\bar{\psi}_n\epsilon\left(1+\frac{M_w^-}{\sqrt{X^\dagger X}}\right)\bar{\psi}_n^T+\phi_n^*\left(1+\frac{M_w^-}{\sqrt{X^\dagger X}}\right)F_n^*\\
&+F_n^*\left(1+\frac{M_w^-}{\sqrt{X^\dagger X}}\right)\phi_n^*\Bigg\},\end{aligned} \tag{72}$$

$$M_w^-=\left(-m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right). \tag{73}$$

This action should be compared to Eq. (18).

The remarkable point about this action is that it possesses as many symmetries as the target continuum theory. First of all, it possesses the manifest supersymmetry under the supertransformation of Eq. (11). Second, it possesses a chiral symmetry of the type given by Lüscher, which corresponds to the transformation (63). In terms of the single Weyl spinor variable, the transformation reads

$$\begin{aligned}\delta\psi_n=&-\frac{1}{2}\psi_n+\frac{1}{2}\frac{1}{\sqrt{X^\dagger X}}\left[\left(-m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right)\psi_n\right.\\
&\left.-\sigma_\mu^\dagger\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\epsilon\bar{\psi}^T\right],\\
\delta\bar{\psi}_n=&+\frac{1}{2}\bar{\psi}_n-\frac{1}{2}\left[\bar{\psi}_n\left(-m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right)\right.\\
&\left.+\psi^T\epsilon\sigma_\mu^\dagger\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\right]\frac{1}{\sqrt{X^\dagger X}}.\end{aligned} \tag{74}$$

Third, the bosonic part of the action possesses symmetries associated with independent infinitesimal rotations of complex phases of the scalar and the auxiliary fields. It is easily

---

[3]By this identification, the kinetic term of ψ_{2n} is transformed to that of ψ_{1n} as follows:

$$\begin{aligned}&\sum_n\bar{\psi}_{2n}\sigma_\mu^\dagger\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\psi_{2n}\\
&\to\sum_n\psi_{1n}^T\epsilon\sigma_\mu^\dagger\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)(-\epsilon)\bar{\psi}_{1n}^T\\
&=\sum_n\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\bar{\psi}_{1n}(\epsilon\sigma_\mu^\dagger\epsilon)^T\psi_{1n}\\
&=-\sum_n\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\bar{\psi}_{1n}\sigma_\mu\psi_{1n}.\end{aligned} \tag{66}$$

At the last step, we used the identity, $\epsilon\sigma_\mu^\dagger\epsilon=-\sigma_\mu^T$. Similarly, the Dirac mass term among ψ_{1n} and ψ_{2n} is transformed to the Majorana mass term of ψ_{1n} as

$$\bar{\psi}_{2n}\psi_{1n}+\bar{\psi}_{1n}\psi_{2n}\to\psi_{1n}^T\epsilon\psi_{1n}-\bar{\psi}_{1n}\epsilon\bar{\psi}_{1n}^T. \tag{67}$$

seen that the action is invariant under the asymmetric rotation of the complex phases of the scalar and the auxiliary fields:

$$\delta\phi_n = \phi_n, \tag{75}$$

$$\delta F_n = -F_n. \tag{76}$$

Moreover, corresponding to the transformation (65), the action possesses the symmetry under the transformation

$$\delta\phi_n = \frac{1}{2}\phi_n - \frac{1}{2}\frac{1}{\sqrt{X^\dagger X}}\left[\left(-m_0 - \frac{1}{2}\nabla_\mu\overline{\nabla}_\mu\right)\phi_n - F_n^*\right], \tag{77}$$

$$\delta F_n = \frac{1}{2}F_n - \frac{1}{2}\frac{1}{\sqrt{X^\dagger X}}\left[\left(-m_0 - \frac{1}{2}\nabla_\mu\overline{\nabla}_\mu\right)F_n - \left\{\frac{1}{2}(\nabla_\mu + \overline{\nabla}_\mu)\right\}^2\phi_n^*\right]. \tag{78}$$

This transformation can be regarded as the coherent infinitesimal rotation of the complex phases of the scalar and the auxiliary fields. These latter two symmetries imply that the $U(1)\times U(1)_R$ symmetry of the chiral multiplet can be exact on the lattice, if we formulate it through the vacuum overlap formalism.

VII. DISCUSSION

In summary, we have seen that the vacuum overlap formalism can provide a natural description of the supersymmetric chiral multiplet on the lattice: all the symmetries of the target continuum theory can be manifest in this formalism. This feature may be useful in attempts at the construction of supersymmetric theories on the lattice.

When the gauge field is introduced, the phase of the fermionic overlap should be fixed by the Wigner-Brillouin phase convention [1]. For the bosonic overlap, we would also need to fix the phase in a similar manner and this procedure would cause the explicit gauge symmetry breaking. On the other hand, it is straightforward to introduce the gauge field in the overlap formula of the partition function of the vectorlike pair of the chiral multiplet, Eq. (39). In any way, for the construction of the supersymmetric gauge theory, we must consider how to introduce the cubic interaction with the gaugino.

The next issue would be to formulate a lattice Wess-Zumino model by constructing local and supersymmetric cubic interactions among chiral multiplets, although it seems a difficult issue in view of the problem discussed in Sec. II.

It may be interesting to formulate the chiral multiplet discussed in this paper in terms of the superfield [28]. In particular, the structure of the lattice superspace should be examined. This is because the lattice counterpart of the $U(1)\times U(1)_R$ symmetry transformation, which consists of the transformations of Eqs. (74) and (77), mixes the field variables with their conjugates. This suggests that the structure of the lattice superspace could be different from the continuum theory counterpart.

ACKNOWLEDGMENTS

The authors would like to thank H. Neuberger and T. Kugo for enlightening discussions.

APPENDIX A: TRANSFER MATRIX OF THE BOSONIC PART

In this appendix, we explain the derivation of the transfer matrix for the bosonic part of the supersymmetric domain-wall system in some detail. It turns out that the following holomorphic variables may be regarded to consist of the canonical variables:

$$z_{1s} = \sqrt{B_s}F_{2s}, \tag{A1}$$

$$\overline{z}_{1s} = \sqrt{B_s}F_{2s}^*, \tag{A2}$$

$$z_{2s} = \sqrt{B_s}\phi_{2s}, \tag{A3}$$

$$\overline{z}_{2s} = \sqrt{B_s}\phi_{2s}^* \tag{A4}$$

and

$$z_{1s}^* = \sqrt{B_s}\phi_{1s}, \tag{A5}$$

$$\overline{z}_{1s}^* = \sqrt{B_s}\phi_{1s}^*, \tag{A6}$$

$$z_{2s}^* = \sqrt{B_s}F_{1s}, \tag{A7}$$

$$\overline{z}_{2s}^* = \sqrt{B_s}F_{1s}^*, \tag{A8}$$

where

$$B_s = 1 + a\left[-m_0\mathrm{sgn}\left(s+\frac{1}{2}\right) - \frac{1}{2}\nabla_\mu\overline{\nabla}_\mu\right]. \tag{A9}$$

Then the bosonic part of the partition function can be written in the form

$$Z_B = \int \mathcal{D}\phi_1\mathcal{D}\phi_1^\dagger\mathcal{D}F_1\mathcal{D}F_1^\dagger\mathcal{D}\phi_2\mathcal{D}\phi_2^\dagger\mathcal{D}F_2\mathcal{D}F_2^\dagger\, e^{-S_B} = \int\prod_s (\det B_s)^4 dZ_s dZ_s^* e^{-Z_s^* Z_s}\, T_s(Z_s^*, Z_{s+1}), \tag{A10}$$

where

$$T_s(Z_s^*, Z_{s+1}) = L_s(Z_s^*)\, K_{s,s+1}(Z_s^*, Z_{s+1})\, R_{s+1}(Z_{s+1}), \tag{A11}$$

$$K_{s,s+1}(Z_s^*,Z_{s+1}) = \exp\left(z_{1s}^* \frac{1}{\sqrt{B_s}} \frac{1}{\sqrt{B_{s+1}}} z_{1s+1} + \overline{z}_{1s}^* \frac{1}{\sqrt{B_s}} \frac{1}{\sqrt{B_{s+1}}} \overline{z}_{1s+1} + z_{2s}^* \frac{1}{\sqrt{B_s}} \frac{1}{\sqrt{B_{s+1}}} z_{2s+1} + \overline{z}_{2s}^* \frac{1}{\sqrt{B_s}} \frac{1}{\sqrt{B_{s+1}}} \overline{z}_{2s+1} \right), \tag{A12}$$

$$L_s(Z_s^*) = \exp\left(a\ \overline{z}_{1s}^* \frac{1}{\sqrt{B_s}} \left\{ \frac{1}{2}(\nabla_\mu + \overline{\nabla}_\mu) \right\}^2 \frac{1}{\sqrt{B_s}} z_{1s}^* - a\ \overline{z}_{2s}^* \frac{1}{B_s} z_{2s}^* \right), \tag{A13}$$

$$R_{s+1}(Z_{s+1}) = \exp\left(a\ \overline{z}_{2s+1} \frac{1}{\sqrt{B_{s+1}}} \left\{ \frac{1}{2}(\nabla_\mu + \overline{\nabla}_\mu) \right\}^2 \times \frac{1}{\sqrt{B_{s+1}}} z_{2s+1} - a\ \overline{z}_{1s+1} \frac{1}{B_{s+1}} z_{1s+1} \right). \tag{A14}$$

Note that we have made use of the following notation:

$$Z = \{z_1, \overline{z_1}, z_2, \overline{z_2}\}, \tag{A15}$$

$$Z^* = \{z_1^*, \overline{z_1^*}, z_2^*, \overline{z_2^*}\}, \tag{A16}$$

and

$$dZdZ^* e^{-Z^*Z} = dz_1 dz_1^* d\overline{z_1} d\overline{z_1^*} dz_2 dz_2^* d\overline{z_2} d\overline{z_2^*} \times e^{-z_1^* z_1 - z_2^* z_2 - \overline{z_1^*}\,\overline{z_1} - \overline{z_2^*}\,\overline{z_2}}. \tag{A17}$$

Next we introduce the bosonic operators which satisfy the canonical commutation relation:

$$[a_{1n}, a_{1m}^\dagger] = \delta_{nm}, \tag{A18}$$

$$[\overline{a}_{1n}, \overline{a}_{1m}^\dagger] = \delta_{nm}, \tag{A19}$$

$$[a_{2n}, a_{2m}^\dagger] = \delta_{nm}, \tag{A20}$$

$$[\overline{a}_{2n}, \overline{a}_{2m}^\dagger] = \delta_{nm}. \tag{A21}$$

Using these operators, the transition amplitudes are translated to operators as follows:

$$\hat{K}_{s,s+1} = \exp\left(a_1^\dagger \ln\left[\frac{1}{\sqrt{B_s}} \frac{1}{\sqrt{B_{s+1}}} \right] a_1 + \overline{a}_1^\dagger \ln\left[\frac{1}{\sqrt{B_s}} \frac{1}{\sqrt{B_{s+1}}} \right] \overline{a}_1 + a_2^\dagger \ln\left[\frac{1}{\sqrt{B_s}} \frac{1}{\sqrt{B_{s+1}}} \right] a_2 + \overline{a}_2^\dagger \ln\left[\frac{1}{\sqrt{B_s}} \frac{1}{\sqrt{B_{s+1}}} \right] \overline{a}_2 \right), \tag{A22}$$

$$\hat{L}_s = \exp\left(a\ \overline{a}_1^\dagger \frac{1}{\sqrt{B_s}} \left\{ \frac{1}{2}(\nabla_\mu + \overline{\nabla}_\mu) \right\}^2 \frac{1}{\sqrt{B_s}} a_1^\dagger - a\ \overline{a}_2^\dagger \frac{1}{B_s} a_2^\dagger \right), \tag{A23}$$

$$\hat{R}_s = \exp\left(a\ \overline{a}_2 \frac{1}{\sqrt{B_s}} \left\{ \frac{1}{2}(\nabla_\mu + \overline{\nabla}_\mu) \right\}^2 \frac{1}{\sqrt{B_s}} a_2 - a\ \overline{a}_1 \frac{1}{B_s} a_1 \right). \tag{A24}$$

Then the partition function can be expressed as

$$Z_B = \langle bc+ | \prod_s (\det B_s)^4 \hat{L}_s \hat{K}_{s,s+1} \hat{R}_{s+1} | bc- \rangle. \tag{A25}$$

By introducing another set of the canonical bosonic operators defined by

$$\hat{b} = \begin{pmatrix} a_1 \\ a_2 \\ \overline{a}_1^\dagger \\ \overline{a}_2^* \end{pmatrix}, \quad \hat{b}^\dagger = \begin{pmatrix} a_1^\dagger & a_2^\dagger & -\overline{a}_1 & -\overline{a}_2 \end{pmatrix}, \tag{A26}$$

we obtain

$$\hat{K}_{s,s+1} = (\det B_s)(\det B_{s+1}) \exp(\hat{b}^\dagger D_s \hat{b}) \exp(\hat{b}^\dagger D_{s+1} \hat{b}), \tag{A27}$$

$$\hat{L}_s = \exp(\hat{b}^\dagger Q_L \hat{b}), \tag{A28}$$

$$\hat{R}_s = \exp(\hat{b}^\dagger Q_R \hat{b}), \tag{A29}$$

where

$$\exp(D_s) = \begin{pmatrix} \frac{1}{\sqrt{B_s}} \mathbb{1} & 0 \\ 0 & \sqrt{B_s} \mathbb{1} \end{pmatrix}, \tag{A30}$$

$$\exp(Q_{Ls}) = \begin{pmatrix} 1 & -a \frac{1}{\sqrt{B_s}} \tilde{C} \frac{1}{\sqrt{B_s}} \\ 0 & 1 \end{pmatrix}, \tag{A31}$$

$$\exp(Q_{Rs}) = \begin{pmatrix} 1 & 0 \\ a \frac{1}{\sqrt{B_s}} \sigma_1 \tilde{C} \sigma_1 \frac{1}{\sqrt{B_s}} & 1 \end{pmatrix}. \tag{A32}$$

C is a 2×2 matrix defined by

$$\tilde{C} = \begin{pmatrix} -\left\{ \frac{1}{2}(\nabla_\mu + \overline{\nabla}_\mu) \right\}^2 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A33}$$

Then we define the transfer matrix by

$$\hat{T}_s=\exp(-a\hat{b}^\dagger H_s\hat{b}), \tag{A34}$$

where

$$\exp(-aH_s)=\exp(D_s)\exp(Q_{Rs})\exp(Q_{Ls})\exp(D_s) \tag{A35}$$

$$=\begin{pmatrix} \dfrac{1}{B_s} & -a\dfrac{1}{B_s}\tilde{C} \\ a\sigma_1\tilde{C}\sigma_1\dfrac{1}{B_s} & -a^2\sigma_1\tilde{C}\sigma_1\dfrac{1}{B_s}\tilde{C}+B_s \end{pmatrix}. \tag{A36}$$

For $s\geq 0$, the mass parameter is homogeneous and negative. We denote the quantities in this region with the subscript ‘‘$-$.’’ For $s\leq -1$, it is homogeneous and positive. We denote the quantities in that region with the subscript ‘‘$+$.’’ The partition function now can be written as

$$Z_B=\prod_s (\det B_s)^2 \langle bc'+|\prod_{s\leq -1} T_+ \prod_{s\geq 0} T_- |bc'-\rangle. \tag{A37}$$

Note that we have redefined the boundary states as

$$\langle bc'+|=\langle bc+|\hat{L}_+\exp(\hat{b}^\dagger D_+\hat{b}), \tag{A38}$$

$$|bc'-\rangle=\exp(\hat{b}^\dagger D_-\hat{b})\hat{R}_-|bc-\rangle. \tag{A39}$$

In the continuum limit in the fifth direction ($a\to 0$), the ‘‘Hamiltonian’’ operator is obtained as follows:

$$\hat{H}_\mp=\hat{b}^\dagger\begin{pmatrix} \left(\mp m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right)\mathbb{1} & \tilde{C} \\ -\sigma_1\tilde{C}\sigma_1 & -\left(\mp m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right)\mathbb{1} \end{pmatrix}\hat{b}. \tag{A40}$$

We should note that this operator is not Hermitian.

APPENDIX B: DIAGONALIZATION OF HAMILTONIANS

In this appendix, we give explicit forms of the matrices of U and O, which diagonalize H_F and H_B, respectively. They are given by the Fourier transforms of the following matrices:

$$U=\frac{1}{\sqrt{2\lambda\left(\lambda-m_0-\frac{1}{2}\nabla\bar{\nabla}\right)}}\begin{pmatrix} \left(\lambda-m_0-\frac{1}{2}\nabla\bar{\nabla}\right)\mathbb{1} & C \\ C^\dagger & -\left(\lambda-m_0-\frac{1}{2}\nabla\bar{\nabla}\right)\mathbb{1} \end{pmatrix}, \tag{B1}$$

$$O=\frac{1}{\sqrt{2\lambda\left(\lambda-m_0-\frac{1}{2}\nabla\bar{\nabla}\right)}}\begin{pmatrix} \left(\lambda-m_0-\frac{1}{2}\nabla\bar{\nabla}\right)\mathbb{1} & \tilde{C} \\ -\sigma_1\tilde{C}\sigma_1 & -\left(\lambda-m_0-\frac{1}{2}\nabla\bar{\nabla}\right)\mathbb{1} \end{pmatrix}, \tag{B2}$$

where

$$\lambda=\sqrt{X^\dagger X}=\sqrt{-\left\{\frac{1}{2}(\nabla_\mu+\bar{\nabla}_\mu)\right\}^2+\left(-m_0-\frac{1}{2}\nabla_\mu\bar{\nabla}_\mu\right)^2}. \tag{B3}$$

[1] R. Narayanan and H. Neuberger, Nucl. Phys. **B412**, 574 (1994); Phys. Rev. Lett. **71**, 3251 (1993).

[2] R. Narayanan and H. Neuberger, Nucl. Phys. **B443**, 305 (1995).

[3] S. Aoki and R. B. Levien, Phys. Rev. D **51**, 3790 (1995); S. Randjbar-Daemi and J. Strathdee, Nucl. Phys. **B466**, 335 (1996).

[4] S. Randjbar-Daemi and J. Strathdee, Phys. Lett. B **348**, 543 (1995); Nucl. Phys. **B443**, 386 (1995).

[5] R. Narayanan and P. Vranas, Nucl. Phys. **B506**, 373 (1997); R. Narayanan and R. L. Singleton, Jr., Nucl. Phys. B (Proc. Suppl.) **63**, 555 (1998).

[6] Y. Kikukawa and H. Neuberger, Nucl. Phys. **B513**, 735 (1998).

[7] H. Neuberger, hep-lat/9803011.

[8] R. Narayanan, H. Neuberger, and P. Vranas, Phys. Lett. B **353**, 507 (1995); Nucl. Phys. B (Proc. Suppl.) **47**, 596 (1995); R. Narayanan and H. Neuberger, Nucl. Phys. **B477**, 521 (1996); Nucl. Phys. B (Proc. Suppl.) **53**, 661 (1997); R. Narayanan and P. Vranas, Nucl. Phys. **B506**, 373 (1997); R. Narayanan and H. Neuberger, Phys. Lett. B **393**, 360 (1997); Y. Kikukawa, R. Narayanan, and H. Neuberger, *ibid.* **399**, 105 (1997); Phys. Rev. D **57**, 1233 (1998).

[9] H. Neuberger, Phys. Lett. B **417**, 141 (1998).

[10] H. Neuberger, Phys. Lett. B **427**, 353 (1998).

[11] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D **25**, 2649 (1982).

[12] M. Lüscher, Phys. Lett. B **428**, 342 (1998).

[13] H. B. Nielsen and M. Ninomiya, Phys. Lett. B **105**, 219 (1981). Nucl. Phys. **B185**, 20 (1981); **B195**, 541(E) (1982) **B193**, 173 (1981).
[14] N. Maru and J. Nishimura, Int. J. Mol. Phys. A **13**, 2841 (1998).
[15] J. Wess and B. Zumino, Nucl. Phys. **B70**, 39 (1974); Phys. Lett. **49B**, 52 (1974).
[16] P. Dondi and H. Nicolai, Nuovo Cimento A **41**, 1 (1977).
[17] N. Sakai and M. Sakamoto, Nucl. Phys. **B229**, 173 (1983).
[18] J. Bartles and G. Krammer, Z. Phys. C **20**, 159 (1983).
[19] T. Banks and P. Windy, Nucl. Phys. **B229**, 226 (1981); S. Elitzur, E. Rabinovici, and A. Schwimmer, Phys. Lett. **119B**, 165 (1982); R. Nakayama and Y. Okada, *ibid.* **134B**, 241 (1984); I. Ichinose, *ibid.* **122B**, 68 (1983).
[20] J. Bartels and J. B. Bronzan, Phys. Rev. D **28**, 818 (1983).
[21] S. Nojiri, Prog. Theor. Phys. **74**, 819 (1985).
[22] G. Curci and G. Veneziano, Nucl. Phys. **B292**, 555 (1987).
[23] D. B. Kaplan, Phys. Lett. B **288**, 342 (1992).
[24] R. Narayanan and H. Neuberger, Phys. Lett. B **302**, 62 (1993).
[25] H. Neuberger, Phys. Rev. D **57**, 5417 (1998).
[26] Y. Shamir, Nucl. Phys. **B406**, 90 (1993); V. Furman and Y. Shamir, *ibid.* **B439**, 54 (1995).
[27] R. Narayanan, Phys. Rev. D **58**, 097501 (1998).
[28] A. Salam and J. Strathdee, Nucl. Phys. **B76**, 477 (1974); S. Ferrara, J. Wess, and B. Zumino, Phys. Lett. **51B**, 239 (1974).