Yang-Mills analogues of the Immirzi ambiguity

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We show that the somewhat controversial ''Immirzi ambiguity'' of the Ashtekar-like formulation of canonical quantum gravity has strong similarities with other ambiguities that appear in Yang-Mills theories, such as the θ ambiguity. This clarifies the role of the ambiguities. We also discuss other ambiguities in the Maxwell case, and the implications for the loop quantization of these theories. [S0556-2821(99)01004-8]

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I. INTRODUCTION

Several new insights into the canonical quantization of general relativity have been acquired using Ashtekar-like variables [1]. Originally, this consisted in basing the theory on a canonical pair formed by a set of (densitized) triads $\tilde{E}^a_i$, and a (complex) SU(2) connection $A^i_a \equiv \Gamma^i_a + iK^i_a$, where Γ^i_a is the spin connection compatible with the triads and $K^i_a = K_{ab}E^{bi}$, where K_{ab} is the extrinsic curvature. In terms of these variables, the constraints of the theory became a Yang-Mills-like Gauss law, plus expressions for the traditional vector and Hamiltonian constraints

$$D_a \tilde{E}^a_i = 0, \tag{1}$$

$$\tilde{E}^a_i F^i_{ab} = 0, \tag{2}$$

$$\epsilon^{ijk}\tilde{E}^a_i \tilde{E}^b_j F^k_{ab} = 0. \tag{3}$$

It was noted by Barbero [2] that a reasonably similar structure could be achieved in terms of a one-parameter family of variables. If one considers a connection of the form $A^i_a = \Gamma^i_a + \beta K^i_a$, with β an arbitrary complex number, it can be shown that the vector and Gauss-law constraints retain exactly the same form as Eqs. (1),(2), provided one rescales the triads by an overall $1/\beta$ factor. The form of the Hamiltonian constraint changes. Immirzi [3] noted that the availability of this one-parameter family of connections led to apparently puzzling results. Because of the complexity of the Hamiltonian constraint (3), a significant portion of the work on canonical quantum gravity has up to now concentrated on ''kinematics.'' This refers to the study of features that only depend on the structure of the Gauss law and vector constraints (1),(2). Examples of this kind of work are the quantization of area and volume [4,5]. These results have a direct impact on more attractive ''physical'' issues such as the recent attempts to compute black hole entropy in nonperturbative quantum gravity [6–8]. What Immirzi noticed is that in spite of the fact that different values of β leave the constraints (1),(2) invariant, the spectra of certain quantum operators depend on β. An example of this property is the area operator, whose spectra in terms of spin network states depends on an overall β factor. Rovelli and Thiemann [9] noted that the different conjugate pairs $(\tilde{E}^a_i, A^i_a)$ constructed with different β differed by a canonical transformation. However, this canonical transformation was not being unitarily implemented in the quantum theory. Thus, the changes in the spectra of physical operators. The fact that the change in spectra had direct impact in ''observable'' computations, such as the entropy of a black hole, motivates trying to understand better the role that the β parameter has in canonical quantum gravity. The purpose of this paper is to discuss this. We will note that the role of the β parameter in canonical quantum gravity is analogous in various senses to that of the θ parameter that describes the different sectors associated to the topological structure of large gauge transformations in Yang-Mills theory. In particular we will notice that loop representations appear only capture one such ''sector'' at a time.

The organization of this paper is as follows. In the next section we discuss the Immirzi ambiguity, in Sec. III we draw a parallel with the θ ambiguity of Yang-Mills theories and in Sec. IV we study the case of Maxwell theory.

II. THE IMMIRZI AMBIGUITY

In the gravitational case, the Immirzi ambiguity arises as a canonical transformation that is not implemented unitarily in the quantum theory in terms of the loop representation. In such case one is using a basis of states (diffeomorphism invariant functions of loops) that is invariant under small gauge and diffeomorphism transformations. If one writes Barbero's Hamiltonian in terms of loops it would be β dependent and the physical quantities, such as the area, are also β dependent. To emphasize the analogy with the Yang-Mills case, let us write the action for general relativity in a Palatini

form in terms of tetrads, but also add to it a term that vanishes on-shell, as suggested in Ref. [10]:

$$S=\frac{1}{2}\int \mathrm{Tr}(\Sigma\wedge R)-\frac{1}{\beta}\mathrm{Tr}(\Sigma\wedge *R), \tag{4}$$

where $\Sigma=e\wedge e$, e^I_a being a tetrad and R^{IJ}_{ab} is the curvature associated with the spin connection compatible with the tetrad. $*R^{IJ}_{ab}\equiv\epsilon^{IJ}_{KL}R^{IJ}_{ab}$. It is well known that the added term vanishes on-shell, this was the key idea that launched the original Ashtekar new variables (which are obtained taking $\beta=i$), allowing us to use a complex action to describe a real theory without adding new equations since the imaginary part of the action is topological in nature. If one performs a canonical decomposition of this action, the canonically conjugate pair is given by a densitized triad $\tilde{E}^a_i$, playing the analogous role to the electric field in a Yang-Mills theory and a connection $A^i_a=\Gamma^i_a+\beta K^i_a$.

The Gauss law and the vector constraint are not β dependent (strictly speaking, this means that one can always find linear combinations of these constraints that are β independent). This is suggested at the level of the action by the fact that the action is diffeomorphism and gauge independent for all values of β. The Hamiltonian constraint, however, is β dependent:

$$H=\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j F^k_{ab}-2(1+\beta^2)\tilde{E}^{[a}_i\tilde{E}^{b]}_j K^i_a K^j_b=0, \tag{5}$$

where $K^i_a=(A^i_a-\Gamma^i_a)/\beta$ is related to the extrinsic curvature.

The β dependence of the Hamiltonian shows that the resulting physics of quantum gravity will be β dependent in general. Therefore, one could fix the value of the parameter β ''experimentally.'' What is more surprising, is that physical quantities that do not have to do with the Hamiltonian, also end up being β dependent. A typical example is the area operator. If one considers a surface S and computes the quantum operator in the loop representation for the area of such surface one finds that in the basis of spin networks the operator is given by [8]

$$\hat{A}|\Gamma\rangle=8\pi\beta l^2_{\mathrm{Planck}}\sum_p \sqrt{j_p(j_p+1)}, \tag{6}$$

where j_p are the valences of the p lines of the spin network that cross the surface S.

The fact that the Immirzi parameter appears multiplied times Newton's constant in Eq. (6) has sometimes been suggested as a proof of the unobservability of β, just having the role of rescaling Newton's constant. The ultimate proof of this will be when more observable consequences of the theory are worked out in a detailed way. However, it appears unlikely that β will always appear multiplying Newton's constant. For instance, one could follow the proposal of Rovelli and Smolin [11] to introduce a scalar field as a time variable in gravity. In such a context, the resulting Hamiltonian is integral of the square root of Eq. (5), and is an observable. The explicit presence of β^2 in the Hamiltonian suggests that the spectrum of the operator will be β (and not βG) dependent.

It is also worthwhile mentioning that the Immirzi ambiguity in gravity does not change if one couples the theory to matter. If one considers non-Fermionic matter, there is no contribution to the (gravitational) Gauss law and the contributions to the vector constraint do not involve the connection and therefore are β independent. For Fermions, there is a contribution to the Gauss law proportional to $\psi^\dagger\psi$, but the gravitational part of it is β independent. For the vector constraint, the gravitational part and the Fermionic piece are β dependent, but one can see that the portion depending on β is proportional to the Gauss law.

The rather surprising features of the ambituity raise the question of what the nature of it is and if similar ambiguities are present in other theories. As we mentioned in the Introduction, these ambiguities correspond to canonical transformations that are not being unitarily implemented in the quantum theory. We also may add that the transformations preserve the form of the ''kinematical'' constraints of the theory. We will see in the following sections that similar ambiguities may arise in gauge theories. From studying their behavior we can learn suitable lessons for the Immirzi ambiguity in gravity.

III. THE SU(2) YANG-MILLS CASE

Let us briefly recall the θ ambiguity in Yang-Mills theory (for a more complete discussion see Refs. [12,13]). If one starts from the Yang-Mills action, $S=(1/4g^2)\mathrm{Tr}[\int F\wedge *F]$ and performs a canonical formulation of the theory, one finds that the quantum Gauss law constraint ensures invariance of the wave function under gauge transformations connected with the identity. Wave functions in general are not invariant under large gauge transformations, characterized by a winding number n. We denote by $\hat{\Omega}_n$ the generator of large gauge transformations $\hat{\Omega}_n\Psi[A]=\Psi[g\cdot A\cdot g^{-1}+g\partial g^{-1}]$, where g is the gauge transformation matrix for a gauge transformation with winding number n. $\hat{\Omega}_n$ is a unitary operator that commutes with the Hamiltonian of the theory.

One can therefore construct a basis of common eigenstates of $\hat{\Omega}_n$ and the Hamiltonian, labeled by the eigenvalues of $\hat{\Omega}_n$:

$$\hat{\Omega}_n\Psi_\theta[A]=\exp(i\theta n)\Psi_\theta[A], \tag{7}$$

$$\hat{H}\Psi_\theta[A]=E_\theta\Psi_\theta[A]. \tag{8}$$

We therefore see that the quantum theory contains an infinite number of disjoint sectors labeled by the continuous angle θ. If one is working in the connection representation, as we have done up to now, one is able to describe simultaneously all the disjoint sectors. However, if one wishes to consider the loop representation, things are different. Since the basis of Wilson loops is invariant under large gauge transformations, it can only give rise to functions that are invariant under large gauge transformations, or in terms of Eq. (7), to the sector $\theta=0$. That is, the loop representation only captures one of the θ sectors of the theory [14,15].

If one now considers a new action for the theory, obtained by adding the Pontryagin topological term to the ordinary Yang-Mills action

$$S=\frac{1}{2g^2}\mathrm{Tr}\left[\int F\wedge *F\right]+\frac{\theta_0}{16\pi^2}\mathrm{Tr}\left[\int F\wedge F\right], \qquad (9)$$

the classical theory is unchanged since one added a total divergence to the action. The added term only contributes a Chern-Simons type term evaluated on the boundary of the manifold, and is invariant under gauge tranformations connected with the identity, changing by an integer value for large gauge transformations.

If one constructs a canonical formulation starting from action (9), the resulting electric field is related to that of the original action by $E=E_{\mathrm{orig}}+(\theta_0 g^2/8\pi^2)B_{\mathrm{orig}}$ where B is the magnetic field. The resulting theory has the same physical predictions as the one we considered before. There is a relationship between the description of both given by $\Psi[A]=\exp(iW[A]\theta_0)\Psi[A]_{\mathrm{orig}}$, where $W[A]=(-1/16\pi^2)\mathrm{Tr}[\int F\wedge A-2/3A\wedge A\wedge A]$ is the integral of the Chern-Simons form. The new theory has the same θ structure for the vacua as the one originally considered, the θ angles being shifted by θ_0, in the sense that

$$\Omega_n\Psi[A]=\exp[i(\theta-\theta_0)n]\Psi_\theta[A], \qquad (10)$$

$$\hat{H}_{\theta_0}\Psi_\theta[A]=E_\theta\Psi_\theta[A]. \qquad (11)$$

If we now consider the loop representation, the Hamiltonian of the theory is θ_0 dependent and so are its eigenvalues. The loop representation still captures a single θ sector of the theory, but now for a different value, given by the parameter θ_0. Therefore, it is clear that the canonical transformation we just introduced is not being unitarily implemented in the loop representation, since the spectra of the Hamiltonian changes.

We see that there are clear parallels (and distinctions) between the θ ambiguity and the Immirzi ambiguity. In both cases, one finds physical quantities that depend on the ambiguity. The ambiguity is ''resolved experimentally'' when one considers the full dynamics of the theory, since in both cases the Hamiltonians depend on the parameters in question. In both cases the ambiguities correspond to canonical transformations at a classical level.

The main difference between both ambiguities is due to the extra term in the action one adds in both cases is of a different nature. In the θ ambiguity it is a total divergence. This allows a deeper understanding of the θ sectors as related to the topological structure of large gauge transformations, and the identification of the corresponding θ sectors. Such understanding is lacking in the case of the Immirzi ambiguity, which is generated by a term in the action that vanishes on shell, but is not a total divergence.

IV. MAXWELL THEORY

It has been noticed by Corichi and Krasnov [16] that free Maxwell theory has an Immirzi-like ambiguity consisting of rescaling the electric field and vector potential by a constant ϵ in such a way as to preserve the canonical commutation relations. If one constructs a loop representation in terms of the connection A_a/ϵ one can see that one can define an operator representing the charge enclosed by a surface that works in an analogous way as the area operator in quantum gravity. Its spectrum is rescaled by $1/\epsilon$. Therefore there is a parallel with the gravitational case, the charge playing the role of the area observable. It is worthwhile noticing, however, that the above ambiguity does not survive the coupling of the theory to matter. If one adds electric charge, Gauss' law implies that one cannot rescale the electric field unless one changes the charge in the theory. Another way of seeing this is to consider the theory coupled to Fermions and build a loop representation. If one does so, requiring that the holonomy with Fermions inserted at its ends be a gauge invariant quantity uniquely fixes the ϵ parameter. We would therefore like to concentrate on other types of ambiguities in Maxwell theory that would survive the inclusion of matter. As we mentioned in Sec. II, the Immirzi ambiguity in gravity survives the inclusion of matter. Sometimes it is argued that this ambiguity, tantamount to a rescaling of the charge, suggests that the Immirzi ambiguity in gravity is tantamount to a rescaling of the mass (or Newton's constant). As we discussed in Sec. II, it is easy to construct situations where this is not the case.

One can introduce ambiguities in Maxwell theory that survive the inclusion of matter by considering θ ambiguities. The discussion goes through very much as in the SU(2) case, but with one important difference: in $3+1$ dimensions there are no large gauge transformations associated with the U(1) group, so for all practical purposes the ambiguity is not there. One can add to the action a θ term, but physical quantities do not change their spectra. Loop representations can be built and although their appearance is different, one can see that they are unitarily related. This is accomplished by noticing that for the Abelian case one can find an expression for the Chern-Simons factor in the loop representation, built using the connection and loop derivatives [17], since there is no problem with large gauge transformations.

One can construct an analogue of the θ ambiguity for the Maxwell theory in $1+1$ dimensions, and the situation is completely analogous to the Yang-Mills case in higher dimensions, see Ref. [18].

There is a different type of ambiguity that arises in Maxwell theory. This is slightly different from the theta ambiguity and has parallels with the Immirzi case. This arises from the fact that one can introduce more than one connection[1] for Maxwell theory. This was first noticed by Ashtekar and Rov-

[1]For Maxwell theory one can introduce more than one connection and also more than one electric field. This is easily seen in the analogy with the harmonic oscillator in the Bargmann representation, where one can take as canonical pairs $(z,\bar{z})$ with $z=q+ip$, or (q,z) or (p,z), etc. Ashtekar and Rovelli choose mixed variables for both the connection and the electric field in their treatment of the Maxwell theory.

elli [19], by drawing an analogy with the Bargmann quantization. They considered as canonical variables for Maxwell theory the positive frequency connection

$$ {}^{+}A_a = \frac{1}{\sqrt{2}}\left(A_a^T(x) + i\,\frac{1}{\Delta^{1/2}} E_T^a(x) \right) \tag{12} $$

with T standing for transverse, and Δ is minus the Laplacian operator. If we now consider a more general connection

$$ {}^{\beta}A_a = \frac{1}{\sqrt{2}}\left(A_a^T(x) + \beta\,\frac{1}{\Delta^{1/2}} E_T^a(x) \right) \tag{13} $$

we can construct a family of quantum theories. The transformation is clearly a canonical transformation. Yet, if one goes to the loop representation they are not necessarily implemented unitarily, as we will immediately see.

An interesting aspect that is worthwhile pointing out is that in this context certain values of β are preferred purely from mathematical considerations. If one considers β real, and one tries to construct a loop representation, one ends up with the same problems as the first attempts found [20,17]. Namely, the Fock space wavefunctions are not well implemented in the loop representation. This difficulty was circumvented by Ashtekar and Rovelli by considering $\beta = i$. One can see that the problem does not arise for $\mathrm{Im}(\beta)$ nonzero. Clearly these two representations cannot be unitarily connected. It is worthwhile pointing out that this ambiguity survives the inclusion of matter, it is perfectly possible to discuss Maxwell theory coupled to Fermions in terms of these variables without fixing the value of β.

V. CONCLUSIONS

In this paper we have pointed out that ambiguities similar to the one Immirzi encountered in gravity exist in other theories, in particular in Yang-Mills theory. This confirms what was pointed out by Rovelli and Thiemann, in the sense that one ''needs two connections'' for Immirzi-like ambiguities to arise. What we see is that through the addition of θ terms one accomplishes essentially the same by having ''two electric fields,'' and introducing a canonical transformation that preserves the Gauss law constraint. For Maxwell theory, one can take advantage of the simplification in Gauss' law that arises in the Abelian case to again introduce ''two electric fields'' or ''two connections'' (or combinations thereof), and end up with ambiguities. We see that for the Maxwell case the ambiguity can be eliminated partially in the loop representation by requiring that the Fock space structure be properly represented. It is worthwhile considering if a similar selection based on purely mathematical criteria might be present in the case of quantum gravity.

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