PHYSICAL REVIEW D, VOLUME 59, 046003

Bulk versus boundary dynamics in anti–de Sitter spacetime

Vijay Balasubramanian*
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

Per Kraus†
California Institute of Technology, Pasadena, California 91125

Albion Lawrence‡
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 12 August 1998; published 25 January 1999)

We investigate the details of the bulk-boundary correspondence in Lorentzian signature anti–de Sitter space. Operators in the boundary theory couple to sources identified with the boundary values of non-normalizable bulk modes. Such modes do not fluctuate and provide classical backgrounds on which bulk excitations propagate. Normalizable modes in the bulk arise as a set of saddlepoints of the action for a fixed boundary condition. They fluctuate and describe the Hilbert space of physical states. We provide an explicit, complete set of both types of modes for free scalar fields in global and Poincaré coordinates. For AdS_3, the normalizable and non-normalizable modes originate in the possible representations of the isometry group $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ for a field of given mass. We discuss the group properties of mode solutions in both global and Poincaré coordinates and their relation to different expansions of operators on the cylinder and on the plane. Finally, we discuss the extent to which the boundary theory is a useful description of the bulk spacetime. [S0556-2821(99)07502-5]

PACS number(s): 11.25.Hf, 04.65.+e, 11.15.Pg

I. INTRODUCTION

The description of certain charged black holes as D-branes in string theory implies a connection between the low-energy gauge dynamics on the brane and the low energy supergravity in spacetime.[1] Recently, Maldacena has proposed decoupling limits in which the brane gauge dynamics is dual to string theory on the near-horizon anti–de Sitter (AdS) geometry of the corresponding black hole [2].

A more precise definition of this duality was developed in [3,4]. We associate with the string compactification on $AdS_{d+1}\times\mathcal{M}$ a conformal field theory (CFT) residing on a space conformal to the d-dimensional boundary $\mathcal{B}$ of the AdS factor. To each field Φ_i there is a corresponding local operator $\mathcal{O}^i$ in the conformal field theory. The relation between string theory in the bulk and field theory on the boundary is

$$Z_{\rm eff}(\Phi_i)=e^{iS_{\rm eff}(\Phi_i)}=\left\langle T\exp\left(i\int_{\mathcal{B}}\Phi_{b,i}\mathcal{O}^i\right)\right\rangle. \qquad (1)$$

Here $S_{\rm eff}$ is the effective action in the bulk, $\Phi_{b,i}$ is the field Φ_i restricted to the boundary, and T is the time-ordering symbol in the field theory on $\mathcal{B}$. The expectation value on the right hand side is taken in the boundary field theory, with $\Phi_{b,i}$ treated as a source term. In the classical supergravity limit, given a boundary field we solve for the corresponding bulk field and use it to relate the bulk effective action to boundary correlation functions.[2]

In Euclidean AdS space this proposal used the absence of normalizable solutions to the field equations, and the resulting unique extension of a boundary field $\Phi_{b,i}$ into the bulk [4]. We are interested in issues of spacetime causal structure and dynamics: so we would also like a Hamiltonian formulation of the bulk theory in Lorentzian signature AdS spaces. The standard construction of quantum field theory depends on the existence of a complete set of normalizable modes, which is in tension with the unique extendibility of AdS boundary conditions into the bulk. Indeed, several consistent quantizations have been found [5,6], involving particular choices of boundary conditions for AdS spacetimes and the resulting set of normalizable modes. On the other hand, the bulk-boundary correspondence demands the ability to tune the boundary conditions in order to describe the appropriate boundary correlation functions. In this paper we resolve these tensions by arguing that the bulk-boundary correspondence as formulated in [2,3,4] demands the inclusion of both normalizable and non-normalizable modes. The former propagate in the bulk and correspond to physical states while the latter serve as classical, non-fluctuating backgrounds and encode the choice of operator insertions in the boundary theory.

We begin in Sec. II by reviewing the computation of boundary correlation functions, and providing a prescription for computing the bulk effective action in a Hamiltonian

*Email address: vijayb@pauli.harvard.edu
†Email address: perkraus@theory.caltech.edu
‡Email address: lawrence@string.harvard.edu

[1]Among many other works, see [1].

[2]By ''boundary correlation functions'' we mean the correlation functions of the CFT.

formulation. We will argue that specifying the boundary conditions involves turning on non-normalizable modes which do not fluctuate. Including such non-fluctuating modes may seem strange, but several well-known examples exist in other contexts: a classic example is a field theory which undergoes spontaneous symmetry breaking, and more recent related examples arise in [7,8,9]. Normalizable solutions to the wave equation are then used in the mode expansion of operators and the construction of a Fock space. The resulting Hilbert space of states is identified with the Hilbert space of the boundary theory [10,4]. In Sec. III we make this more explicit by studying solutions to the wave equation for free scalars of arbitrary mass. We find that for general masses the field equations have both normalizable and non-normalizable solutions and the latter couple to the boundary.

In Sec. IV we specialize to AdS_3, whose bulk isometry group and boundary conformal group are both $G = SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$. We show that the normalizable modes transform in unitary irreducible representations of G while the non-normalizable modes transform in non-unitary reducible representations which contain a highest weight module. The normalizable and non-normalizable highest weight representations are built on states with $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ weights $h_L = h_R = h_\pm = (1/2)(1 \pm \sqrt{m^2\Lambda^2+1})$.[3] Interesting subtleties arise for small masses and for integral $\nu = (1/2)\sqrt{m^2\Lambda^2+1}$. We carry out the analysis in both global and Poincaré coordinates in order to discuss conformal field theories both on the cylinder and the plane. The latter case has some curious features because Poincaré coordinates only cover a patch of the global spacetime. We conclude the paper with a discussion of the utility and limitations of the bulk-boundary correspondence for describing bulk spacetime physics via the boundary gauge theory.

II. BOUNDARY CORRELATORS FROM THE BULK

A. Euclidean formulation

Specifying the boundary behavior of a field Φ in Euclidean AdS space leads to a unique solution to the equations of motion, given some regularity conditions (see [4] for a discussion and references). So Eq. (1) is unambiguously interpreted by evaluating the AdS_{d+1} effective action on the unique bulk extension of the boundary field. For a free scalar Φ with boundary value Φ_b,

$$S_{\rm eff} = \int_{AdS_{d+1}} d^{d+1}x\sqrt{g}(g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2). \qquad (2)$$

We relate this to the right hand side of Eq. (1) by integrating by parts. Since Φ is a solution to the equations of motion, the bulk contribution vanishes. There is, however, a boundary term which is non-vanishing for the relevant solutions to the Euclidean wave equation:

$$S_{\rm eff} = \lim_{r\to 0}\int d\tau d^{d-1}\vec{x}\sqrt{g}g^{rr}\Phi\partial_r\Phi. \qquad (3)$$

[3]Here Λ is the inverse of the cosmological constant of AdS_3.

(Here $r=0$ is the boundary of spacetime in the Poincaré coordinates defined in the Appendix.) As $r\to 0$ this is quadratic in Φ_b and can be thought of as a quantity in the boundary theory. In order to compare to the right-hand side of Eq. (1), we must identify the boundary operator coupling to Φ. The dimension of the operator is determined by the growth of Φ at the boundary of AdS space [3,4]. In particular, suppose that the boundary behavior of Φ is

$$\Phi \overset{r\to 0}{\to} r^{-\lambda}\Phi_0(\vec{x}). \qquad (4)$$

Then the corresponding operator has mass dimension $d+\lambda$. In fact, this procedure is not completely well-defined: in general, Eq. (3) blows up and some regularization is required in order to extract the correlator [11]. For interacting theories, we can calculate $S_{\rm eff}$ perturbatively by integrating over fluctuations away from the classical solution. These fluctuations should have finite action and vanish appropriately at the boundary.

B. Lorentzian formulation

In Lorentzian signature AdS spacetime the relation in Eq. (1) is more subtle, because there are normalizable solutions to the wave equation [5,6,12] which do not affect the leading boundary behavior (4). So the boundary value Φ_b does not uniquely specify the bulk field. Furthermore, the normalizable modes cannot be projected out since they are needed to expand quantum operators, build a Fock space, and compute Green's functions in the bulk.

The bulk effective action appearing in Eq. (1) can be written as

$$Z_{\rm eff}(\Phi_i) = \int \mathcal{D}\Phi_i e^{iS(\Phi)} \qquad (5)$$

where the path integral[4] is taken over fields Φ_i that take the boundary value $\Phi_{b,i}$. The measure $\mathcal{D}\Phi_i$ can be normalized by requiring that $Z_{\rm eff}=1$ when the boundary condition is trivial ($\Phi_{b,i}=0$). In general there is a set of saddle points that must be summed over; these correspond to the normalizable modes that are solutions to the equations of motion with a fixed boundary condition. For a free theory in Minkowski space, the normalizable modes indicate that there is a manifold of flat directions that must be integrated over for fixed boundary behavior. For a free theory the calculations in Refs. [3,4] are not affected as the flat directions lead to an overall volume factor that is divided out when we appropriately normalize the path integral.

In order to define the path integral in Eq. (5) we must specify the behavior of fields at the AdS boundary. We will see that boundary conditions can be specified by including certain modes that are generically not normalizable or at least perturb the asymptotic geometry too violently. In fact,

[4]Here we are restricting attention to those low energy processes for which a field theory path integral makes sense. More generally, the full string theory partition function is implied.

these are precisely the modes that are identified in [3,4] with sources coupling to boundary operators. As such, these modes should be locked or non-fluctuating. Such non-fluctuating modes arise in a variety of contexts. One example is the homogeneous mode of a scalar field that takes a nonzero value through spontaneous symmetry breaking. Another example is Euclidean field theory on spaces with constant negative curvature; the authors of Ref. [7] note that the large volume at infinity keeps modes with certain boundary conditions from fluctuating. Our discussion is particularly reminiscent of Liouville theory, where local operators create non-normalizable wave functions when inserted in a disk amplitude [8,9]. This analogy is especially attractive after the work in [13] and was a motivation for the present form of this paper; indeed, for AdS_3 the gravity sector of the bulk has a Chern-Simons action which reduces to a boundary Liouville theory [14,15]. In particular Ref. [13] relates the asymptotic behavior of solutions to the wave equation to the gravitational dressing of chiral operators in the boundary CFT.

For an interacting theory, our prescription provides the analog of an S-matrix for field theories in AdS space, an issue raised in Ref. [16]. In standard flat-space field theory, one specifies boundary conditions at infinity by turning off the interactions and physically separating the initial excitations, so that we can sensibly discuss asymptotic states. In our picture, even though the bulk excitations may not be separable and asymptotic states may not exist, we can sensibly discuss the dependence of the partition function on boundary conditions for locked fields at infinity.

C. Hamiltonian quantization

Once we have specified the boundary conditions, we must ask what fluctuating modes to keep in a Hamiltonian formalism. The point is that quantization requires a complete, normalizable set of field modes, which in AdS space requires some sort of boundary condition at infinity. Several consistent choices have been suggested in the past; for the bulk-boundary correspondence to make sense the choice should be unambiguously determined by the physics of the situation. Our prescription is heavily motivated by the work of Breitenlohner and Freedman [6] (which covers AdS_4): so we will review their discussion for scalars of arbitrary mass m in AdS_{d+1} (also see [17]).

Fix a spacelike slice $\Sigma \subset AdS_{d+1}$ with coordinates x and an orthogonal, timelike coordinate t. Given two solutions u_1, u_2 to the scalar wave equation, define the inner product:

$$(u_1,u_2)=i\int_{\Sigma} d^dx\,\sqrt{g}g^{tt}(u_1^*\partial_t u_2-\partial_t u_1^* u_2). \tag{6}$$

If $u_1=u_2$, this is the integral of the time component of the current

$$j^{\mu}=ig^{\mu\nu}(u^*\partial_\nu u-\partial_\nu u^* u). \tag{7}$$

Here j^t is only time-independent up to boundary terms coming from the timelike boundary of anti–de Sitter space; it is easy to see, using the equations of motion, that

$$\partial_t j^t=\int_{\partial\Sigma} dA_k j^k, \tag{8}$$

where $\partial\Sigma$ is the boundary of the spacelike slice. The authors of [6] find complete sets of modes by demanding, first, that the flux normal to the boundary vanish at infinity. They also demand that no energy is exchanged with the boundary:

$$F=\int_{\partial\Sigma} dA_k\sqrt{g}T_0^k=0. \tag{9}$$

The authors of [6,17] considered the relative merits of the canonical and improved stress tensors in this equation. This potential ambiguity in the stress tensor arises because the curvature is constant, and its possible coupling to a scalar field would appear as an effective mass term. In our case the choice is dictated by the underlying string theory and the supergravity effective action that descends from it. We will simply treat scalar fields of mass squared m^2 with the understanding that the mass term may contain a contribution from a curvature coupling.

We will find that for masses $m^2 \geqslant 1-d^2/4$, simply requiring normalizability is sufficient to isolate a suitable class of fluctuating solutions, and the more refined discussion in Ref. [6] is unnecessary. For $-d^2/4<m^2<1-d^2/4$ (stability requires $m^2 \geqslant -d^2/4$), there are two sets of normalizable solutions and some criterion is needed to distinguish them. References [6,17] show that for any given field propagating on $AdS_{d\geqslant 4}$, conservation of the inner product (6) and the vanishing of F at the boundary requires either but not both sets. Furthermore, in the case of AdS_4, the authors of [6] show that the two sets of modes are built on representations of the conformal algebra with different lowest energy states and that supersymmetry generically requires one to take both types of mode into account. The clearest example is that of the scalar and pseudoscalar in the gravity supermultiplet of gauged $N=4$ supergravity [6]. Both are conformally coupled, and supersymmetry requires that the modes of the scalar lie in one representation while the modes of the pseudoscalar lie in the other. Further criteria are needed, however, to decide which assignment is realized. In this particular case, Hawking[5] [18] imposed the requirement that the metric be asymptotically anti–de Sitter (in a sense defined in that work) to find particular boundary conditions on the linearized metric perturbations. Supersymmetry then requires the scalar mode to reside in the representation whose highest-weight state has the lowest energy. It would be nice if similar criteria could be applied to general modes in the range $-d^2/4<m^2<1-d^2/4$, independently of supersymmetry; we will not investigate this point, however.

D. Summary

The lesson is that we need to keep both normalizable and non-normalizable modes in AdS spacetimes. The non-

[5]We would like to thank S. Ross for pointing out this reference and its relevance.

normalizable modes correspond to operator insertions in the boundary gauge theory; from the AdS point of view they provide non-trivial boundary conditions. The normalizable modes fluctuate in the bulk; quanta occupying such modes have a dual description in the boundary Hilbert space. In path integral language, Eq. (1) can be understood via the background field expansion: we compute the effective action by expanding the path integral of the bulk theory as

$$\Phi = \Phi_{cl} + \delta\Phi. \tag{10}$$

Φ_{cl} is a classical, non-normalizable solution to the equations of motion, corresponding to an operator insertion at infinity and a particular choice of boundary conditions. Then $\delta\Phi$ is the fluctuating piece over which we integrate to get the partition function. The normalizable modes appear as stationary points of the action given the background $\Phi_{\rm cl}$.

III. EXPLICIT MODES FOR SCALARS OF ARBITRARY MASS

In this section we show that a scalar of arbitrary mass in AdS spacetime has both fluctuating and non-fluctuating solutions that implement the bulk-boundary correspondence as advocated above. Generically, the non-fluctuating solutions are not normalizable in the norm (6).

A. Solutions in Poincaré coordinates

We begin with solutions in Poincaré coordinates, which allows for a direct comparison with Refs. [3,4]. It is easy to separate variables in these coordinates by writing the scalar of mass squared m^2/Λ^2 as

$$\Phi = e^{-i\omega\tau + i\vec{k}\cdot\vec{x}} r^{d/2}\chi(r). \tag{11}$$

χ then satisfies the equation

$$r^2\partial_r^2\chi + r\partial_r\chi - \left[\left(m^2 + \frac{d^2}{4}\right) + (\vec{k}^2 - \omega^2)r^2\right]\chi = 0. \tag{12}$$

For $q^2 = \vec{k}^2 - \omega^2 > 0$, the solution is[6] [3,11]

$$\Phi^{\rm s.l.} = e^{-i\omega t + i\vec{k}\cdot\vec{x}} r^{d/2} K_\nu(qr), \tag{13}$$

where $\nu = \frac{1}{2}\sqrt{d^2 + 4m^2}$. This solution is non-normalizable at the boundary at infinity but well-behaved in the interior. The second, independent solution I_ν is very badly behaved (i.e. blows up exponentially) in the interior and is therefore eliminated. If we consider anti–de Sitter space as the near-horizon geometry of a brane, then in the asymptotically flat region $\Phi^{\rm s.l.}$ would have imaginary momentum perpendicular to the brane.

For $q^2 < 0$, there are two possible solutions which are regular in the interior. If ν is not integral,

$$\Phi^{(\pm)} = e^{-i\omega t + i\vec{k}\cdot\vec{x}} r^{d/2} J_{\pm\nu}(|q|r) \tag{14}$$

are two independent solutions. If ν is integral, $J_{\pm\nu}$ are equivalent, and the two independent solutions are $\Phi^{(+)}$ in Eq. (14) and

$$\Phi^{(-)} = e^{-i\omega t + i\vec{k}\cdot\vec{x}} r^{d/2} Y_\nu(|q|r). \tag{15}$$

[Note that many of the scalar modes arising from the Kaluza-Klein (KK) reduction of string theory down to AdS_{d+1} have integral ν [4,20,21,22,23,24,25]: so this is a ''special case'' of particular importance.] The boundary term (3) will vanish for $\Phi^{(+)}$; for $\Phi^{(-)}$ it will either go to a constant (for $m^2 = 0$, $d = 2$) or blow up. $\Phi^{\rm s.l.}$ and $\Phi^{(-)}$ both behave as

$$\Phi \sim r^{2h_-}\Phi_0 \tag{16}$$

as discussed in [3,4], where

$$h_\pm = \frac{d}{4} \pm \frac{1}{4}\sqrt{d^2 + 4m^2}. \tag{17}$$

For $\nu = 0$, $\Phi^{(-)}$ behaves as

$$\phi \sim r^{d/2}\ln r. \tag{18}$$

As in [3,4], $\Phi^{(-,{\rm s.l.})}$ couple to operators of dimension $2h_+$.

The asymptotic behavior of Bessel functions is well-known: so criteria for selecting fluctuating and non-fluctuating modes are easy to apply. For $\nu > 1$, only $\Phi^{(+)}$ is normalizable and $\Phi^{(-)}$ must therefore act as the non-fluctuating source term in Eq. (1). For $\nu < 1$ the story is more complicated as both of $\Phi^{(\pm)}$ are normalizable and, as pointed out in Refs. [6,7], both kinds of modes are necessary for supersymmetry. Indeed, for compactifications of M-theory on $\mathrm{AdS}_4 \times S^7$, a conformally coupled scalar and pseudoscalar appear in the KK spectrum [20,21,25,26].[7] Both the analysis of the KK spectrum and its relation to operators in the dual 3D CFT [25,26], and the arguments in [8], show that the fluctuating modes for the scalar are of the form $\Phi^{(-)}$ and the fluctuating modes for the pseudoscalar are of the form $\Phi^{(+)}$. Furthermore, the ''conjugate'' solutions $\Phi^{(+)}$ and $\Phi^{(-)}$ couple to operators with dimension h_- and h_+, respectively [25,26].[8] One may worry legitimately that for the scalar mode, interpreting Eq. (1) is problematic because the surface term (3) diverges for this mode and not for the ''source'' modes coupling to the boundary. Perhaps the answer is that when computing correlation functions, one does not perturb the background with classical normalizable modes. At present we will leave the resolution of this issue for future work.

[6]We use the notation in [19] for Bessel functions.

[7]We would like to thank O. Aharony for pointing out this example, and for discussing the $\nu < 1$ case in general.

[8]Note that the fact that the scalar mode has dimension $1 < d/2$ contradicts the statement made in Ref. [4] for Euclidean theories that the dimension of operators in the CFT is bounded below by $d/2$. It however satisfies the unitarity condition $h \geqslant \frac{1}{2}(d-2)$ for scalar operators in the dual CFT [27].

It is also worth noting that the borderline non-normalizable modes, $\Phi^{(-)}$ for $h_-=\frac{1}{2}(d-2)$, contain the singleton [34,35].

B. Solutions in global coordinates

Let us examine a scalar of mass m^2/Λ^2. Make the substitution

$$\Phi=e^{-i\omega t}Y_{l,\{m\}}(\Omega)\chi(\rho). \tag{19}$$

Here Y_l are the l^{th} spherical harmonics on S^{d-1}, for which

$$\nabla^2_{S^{d-1}}Y_l=-l(l+d-2)Y_l \tag{20}$$

with $l\geqslant 0$. The wave equation in global coordinates (see the Appendix) is

$$\frac{1}{(\tan\rho)^{d-1}}\partial_\rho[(\tan\rho)^{d-1}\partial_\rho]\chi +[\omega^2-l(l+d-2)\csc^2\rho-m^2\sec^2\rho]\chi=0. \tag{21}$$

We have chosen Φ with which we can build an arbitrary configuration on $S^{d-1}\times\mathbb{R}$ for a given ρ; Eq. (21) then gives us the ρ dependence. For general m^2, ω the equation of motion is easily converted into a hypergeometric equation which can naturally be expanded in variables that vanish either at the origin $(\rho=0)$ or the boundary $(\rho=\pi/2)$. For maintaining regularity at the origin, it will be easier to examine the solutions as functions of $\sin^2\rho$. For examining the boundary behavior, solutions as functions of $\cos^2\rho$ will be more convenient. We will display both sets of solutions explicitly. Of course one can pass between them using standard formulas relating hypergeometric functions as arguments of z and $1-z$ [19,28,29]; these permit the enforcement of regularity conditions at the origin and the boundary, thereby imposing quantization conditions on the spectrum of the theory.

Begin by substituting

$$\chi(\rho)=(\cos\rho)^{2h}(\sin\rho)^{2b}f(\rho). \tag{22}$$

Let $y=\sin^2\rho$. It is then easy to see that

$$y(1-y)\partial_y^2 f+\left[2b+\frac{d}{2}-(2h+2b+1)y\right]\partial_y f -\left[(h+b)^2-\frac{\omega^2}{4}\right]f=0, \tag{23}$$

where

$$h\left(h-\frac{d}{2}\right)=\frac{m^2}{4},\quad 2b(2b+d-2)=l(l+d-2). \tag{24}$$

The equations for h,b each have two solutions:[9]

$$h_\pm=\frac{d\pm\sqrt{d^2+4m^2}}{4} \tag{25}$$

$$b=\frac{l}{2},\quad \frac{1}{2}(2-d-l). \tag{26}$$

The hypergeometric equation will have two independent solutions, corresponding to the two solutions of the indicial equation for b. One solution will be logarithmic if $l+d/2$ is an integer, i.e., if d is even.

Choosing instead $x=\cos^2\rho$, one gets the hypergeometric equation

$$x(1-x)\partial_x^2 f+\left[2h+1-\frac{d}{2}-(2h+2b+1)x\right]\partial_x f +\left[(h+b)^2-\frac{\omega^2}{4}\right]f=0, \tag{27}$$

with h,b as before. Again, the hypergeometric equation has two independent solutions; this time they correspond to the two solutions of the indicial equation for h. One solution will be logarithmic if $\nu=\frac{1}{2}\sqrt{d^2+4m^2}$ is an integer. In fact, if we wish to transform solutions as functions of $\sin^2\rho$ to solutions as functions of $\cos^2\rho$, the relevant formulas are modified in the case of integral ν; thus we will examine the two cases separately.

Behavior at the origin. The behavior at the origin is conveniently analyzed by studying solutions as a function of $\sin^2\rho$. Choose, without loss of generality, $h=h_+$. The first solution as a function of $\sin^2\rho$ is[10]

$$\Psi^{(1)}=e^{-i\omega t}Y_l(\Omega)(\cos\rho)^{2h_+}(\sin\rho)^l\,{}_2F_1\left(h_++\frac{1}{2}(l+\omega),h_++\frac{1}{2}(l-\omega),l+\frac{d}{2};\sin^2\rho\right). \tag{28}$$

The second solution depends on whether d is even or odd. If d is odd, then

[9]Let λ^{bf} be what Ref. [6] calls λ and $\lambda^w_\pm$ be what Ref. [4] calls λ. Then h is related to these as $h_\pm=\frac{1}{2}\lambda^{\text{bf}}_\pm=-\frac{1}{2}\lambda^w_\mp$.

[10]We use the notation in [19].

$$\Psi^{(2)}=e^{-i\omega t}Y_l(\Omega)(\cos\rho)^{2h_+}(\sin\rho)^{2-d-l}\,{}_2F_1\left(h_++\frac{1}{2}(2-d-l+\omega),h_++\frac{1}{2}(2-d-l-\omega),2-l-\frac{d}{2};\sin^2\rho\right). \tag{29}$$

If d is even, the second solution is logarithmic:

$$\Psi^{(2)}=\Psi^{(1)}\ln\sin^2\rho+\sum_{k=1}^{\infty}\sin^{2k}\rho\frac{[h_++\frac{1}{2}(l+\omega)]_k[h_++\frac{1}{2}(l-\omega)]_k}{(l+d/2)_k k!}\left[h\left(0,l+\frac{d}{2}-1,\nu+1,\omega,k\right)-h\left(0,l+\frac{d}{2}-1,\nu+1,\omega,0\right)\right]$$
$$-\sum_{k=1}^{l+\frac{1}{2}d-1}\frac{(k-1)!(1-l-d/2)_k}{[1-h_+-\frac{1}{2}(l+\omega)]_k[1-h_+-\frac{1}{2}(l+\omega)]_k}\sin^{-2k}\rho, \tag{30}$$

where

$$h(e,f,g,\omega,k)=\psi\left(\frac{1}{2}\left[\frac{1}{2}e+f+g+\omega\right]+k\right)+\psi\left(\frac{1}{2}\left[\frac{1}{2}e+f+g-\omega\right]+k\right)-\psi(1+f+k)-\psi(1+k), \tag{31}$$

following Ref. [29], and

$$(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)},\quad \psi(x)=\frac{d}{dx}\ln\Gamma(x). \tag{32}$$

We must impose a regularity condition at the origin because in order for Eq. (1) to make sense, we should not have contributions to correlation functions coming from the interior. So we will only keep solutions for which the boundary term of the classical action vanishes at the origin $\rho=0$:

$$S_{\rm origin}=\lim_{\rho\to 0}\int_{\rho\ \rm fixed} dt d\Omega\,\sqrt{g}g^{\rho\rho}\Phi\,\partial_\rho\Phi\to 0. \tag{33}$$

It is easy to show that this means that only the first solution $\Psi^{(1)}$ is allowed.

1. ν nonintegral

Behavior at the boundary. To study the behavior at the boundary it is most convenient to work with solutions as a function of $\cos^2\rho$:

$$\Phi^{(+)}=e^{-i\omega t}Y_{l,\{m\}}(\Omega)(\cos\rho)^{2h_+}(\sin\rho)^l\,{}_2F_1\left(h_++\frac{1}{2}(l+\omega),h_++\frac{1}{2}(l-\omega),2h_++1-\frac{d}{2};\cos^2\rho\right) \tag{34}$$

and

$$\Phi^{(-)}=e^{-i\omega t}Y_{l,\{m\}}(\Omega)(\cos\rho)^{2h_-}(\sin\rho)^l\,{}_2F_1\left(h_-+\frac{1}{2}(l+\omega),h_-+\frac{1}{2}(l-\omega),2h_-+1-\frac{d}{2};\cos^2\rho\right). \tag{35}$$

In general the regular solution at the origin ($\Psi^{(1)}$) is a linear combination of $\Phi^{(\pm)}$.

We can see directly that the leading behavior at the boundary of $\Phi^{(\pm)}$ is $(\cos\rho)^{2h_\pm}$. For $\nu>1$ the norm (6) of $\Phi^{(-)}$ diverges at $\rho=\pi/2$, while the norm of $\Phi^{(+)}$ converges. Thus, up to regularity at the origin, $\Phi^{(+)}$ is our candidate normalizable mode. Similarly, a combination of $\Phi^{(+)}$ and $\Phi^{(-)}$ is a candidate non-normalizable mode. Again, its behavior at infinity indicates that it will couple to operators of dimension h_+ in (1).

For $\nu<1$ both modes are well-behaved at infinity and further examination is required to select the relevant fluctuating modes, just as in our discussion of the solutions in Poincaré coordinates. Again, fields which have the solutions $\Phi^{(\pm)}$ as their fluctuating modes will have solutions $\Phi^{(\mp)}$ which act as source terms in Eq. (1) for the related operator of dimension $h_\pm$ in the dual CFT.

Quantization condition for normalizable modes. As we have discussed, regularity at the origin requires the choice of $\Psi^{(1)}$. This solution can be written as a linear combination of $\Phi^{(\pm)}$:

$$\Psi^{(1)} = C^{(+)}\Phi^{(+)} + C^{(-)}\Phi^{(-)}, \tag{36}$$

where

$$C^{(+)} = \frac{\Gamma(l+d/2)\Gamma(-\nu)}{\Gamma(h_- + \frac{1}{2}(l+\omega))\Gamma(h_- + \frac{1}{2}(l-\omega))}$$
$$C^{(-)} = \frac{\Gamma(l+d/2)\Gamma(\nu)}{\Gamma(h_+ + \frac{1}{2}(l+\omega))\Gamma(h_+ + \frac{1}{2}(l-\omega))}. \tag{37}$$

(Note that ν is assumed to be non-integral here.) For $\nu > 1$, $C^{(-)}$ must vanish for a fluctuating solution because the norm of $\Phi^{(-)}$ diverges at the boundary. This will happen if one of the gamma functions in the denominator has zero or a negative integer as its argument, i.e., if

$$\omega = \pm(2h_+ + l + 2n), \quad n = 0,1,2,\ldots . \tag{38}$$

So this is the spectrum of normalizable modes $\Phi^{(+)}$ for $\nu > 1$. For the same range of ν, non-fluctuating modes do not have a quantization condition. For a special set of frequencies

$$\omega = \pm(2h_- + l + 2n), \quad n = 0,1,2,\ldots, \tag{39}$$

the non-fluctuating modes are purely of the $\Phi^{(-)}$ type. We will see in the next section that such modes are in a highest weight representation. In general, however, the non-normalizable modes are simply the linear combination appearing in Eq. (37).[11]

For $\nu < 1$, both $\Phi^{(+)}$ and $\Phi^{(-)}$ are potentially normalizable because both norms are well-behaved at the boundary. The quantization condition (38) gives the spectrum of normalizable modes of the form $\Phi^{(+)}$. Normalizable modes of the form $\Phi^{(-)}$ are obtained by imposing instead:

$$\omega = \pm(2h_- + l + 2n), \quad n = 0,1,2,\ldots . \tag{40}$$

After picking one of these towers as the fluctuating modes, the remaining modes that are regular at the origin should be locked and mediate the bulk-boundary correspondence. As we have already discussed, the case when $\nu < 1$ and $\Phi^{(-)}$ is the fluctuating mode is confusing since the locked mode falls off faster at the boundary than the fluctuating mode. As a result, for the non-fluctuating mode to couple as source to an operator in the boundary theory with dimension h_-, it must not contain any terms which behave as $(\cos\rho)^{2h_-}$ at the boundary. Thus, its frequency must be quantized according to Eq. (38); it seems that in this case one cannot write a classical source with arbitrary behavior in global time. If we simply interpret the left-hand side of Eq. (1) as a path integral over field configurations which fall off as $(\cos\rho)^{2h_+}$, then there will be no saddle points in the path integral when the time dependence of the boundary configuration is arbitrary. We leave the resolution of this conundrum for future work.

Upon imposing the conditions (38) and (40) for $\Phi^{(+)}$ and $\Phi^{(-)}$ respectively, the solutions may be written in terms of Jacobi polynomials:[12]

$$\Phi^{(\pm)} = e^{-i\omega t} Y_l (\cos\rho)^{2h_\pm} \times (\sin\rho)^l P_n^{(l+d/2-1, 2h_\pm - d/2)}(\cos 2\rho). \tag{41}$$

It is easy to show that these quantized $\Phi^{(+)}$ can be made orthonormal under the norm (6). For $\nu < 1$ these quantized $\Phi^{(-)}$ can be made orthonormal as well.

2. $\nu \in \mathbb{Z}^+ \cup \{0\}$

As functions of $\sin^2\rho$ the solutions $\Psi^{(1,2)}$ are the same as before. Because ν is the difference of the roots of the indicial equation, however, the solutions expanded near infinity are a little different than those listed above. $\Phi^{(+)}$ is the same as before:

$$\Phi^{(+)} = e^{-i\omega t} Y_{l,\{m\}}(\Omega)(\cos\rho)^{2h_+}(\sin\rho)^l\, {}_2F_1\left(h_+ + \frac{1}{2}(l+\omega), h_+ + \frac{1}{2}(l-\omega), 1+\nu; \cos^2\rho\right). \tag{42}$$

Again, the norm (6) is well-behaved at infinity. For $\nu = 0$ $\Phi^{(-)}$ becomes

[11]In a previous unpublished version of this work, we imposed a quantization condition on the non-normalizable solutions also, by demanding that the be purely of the $\Phi^{(+)}$ type. In fact, there is no need to impose such a condition. Since the AdS-CFT correspondence should work when time is noncompact, we should be allowed sources with arbitrary time dependence. We would like to thank J. Maldacena for emphasizing this to us.

[12]See Ref. [28] for notation.

$$\Phi^{(-)}=e^{-i\omega t}Y_l(\Omega)(\cos\rho)^{d/2}(\sin\rho)^l\Bigg[{}_2F_1\left(\frac{d}{4}+\frac{1}{2}(l+\omega),\frac{d}{4}+\frac{1}{2}(l-\omega),1;\cos^2\rho\right)\ln\cos^2\rho$$
$$+\sum_{k=1}^{\infty}(\cos\rho)^{2k}\frac{[d/4+\frac{1}{2}(l+\omega)]_k[d/4+\frac{1}{2}(l-\omega)]_k}{k!}[h(d,0,l,\omega,k)-h(d,0,l,\omega,0)]\Bigg]. \tag{43}$$

Here $h(e,f,g,\omega,k)$ is defined in Eq. (31). Note that this has a well-behaved norm at the boundary. For $\nu>0$ the second solution is

$$\Phi^{(-)}=e^{-i\omega t}Y_l(\Omega)(\cos\rho)^{d/2+\nu}(\sin\rho)^l\Bigg[{}_2F_1\left(\frac{d}{4}+\frac{1}{2}(\nu+l+\omega),\frac{d}{4}+\frac{1}{2}(\nu+l-\omega),1+\nu;\cos^2\rho\right)\ln\cos^2\rho$$
$$+\left(\sum_{k=1}^{\infty}(\cos\rho)^{2k}\frac{[d/4+\frac{1}{2}(\nu+l+\omega)]_k[d/4+\frac{1}{2}(\nu+l-\omega)]_k}{(1+\nu)_k k!}[h(d,\nu,l,\omega,k)-h(d,\nu,l,\omega,0)]\right)$$
$$-\sum_{k=1}^{\nu}\frac{(k-1)!(-\nu)_k}{[1-d/4-\frac{1}{2}(\nu+l+\omega)]_k[1-d/4-\frac{1}{2}(\nu+l-\omega)]_k}(\cos\rho)^{-2k}\Bigg]. \tag{44}$$

The norm of these solutions blows up at the boundary.

Quantization condition. Once again we can start with the solution $\Psi^{(1)}$ which is regular at the origin and examine its behavior at the boundary. The transformation laws are modified when ν is integral, so that

$$\Psi^{(1)}=e^{-i\omega t}Y_l(\Omega)(\sin\rho)^l$$
$$\times\Bigg\{\frac{\Gamma(\nu)\Gamma(d/2+l)}{\Gamma(h_++\frac{1}{2}(l+\omega))\Gamma(h_++\frac{1}{2}(l-\omega))}(\cos\rho)^{2h_-}\sum_{k=0}^{\nu-1}\frac{[h_-+\frac{1}{2}(l+\omega)]_k[h_-+\frac{1}{2}(l-\omega)]_k}{k!(1-\nu)_k}(\cos\rho)^{2k}$$
$$-\frac{(-1)^\nu\Gamma(d/2+l)}{\Gamma(h_-+\frac{1}{2}(l+\omega))\Gamma(h_-+\frac{1}{2}(l-\omega))}(\cos\rho)^{2h_+}\sum_{k=0}^{\infty}\frac{[h_++\frac{1}{2}(l+\omega)]_k[h_++\frac{1}{2}(l-\omega)]_k}{k!(k+\nu)!}(\cos\rho)^{2k}$$
$$\times\left[\ln\cos^2\rho-\psi(k+1)-\psi(\nu+k+1)+\psi\left(h_++\frac{1}{2}(l+\omega)+k\right)+\psi\left(h_++\frac{1}{2}(l-\omega)+k\right)\right]\Bigg\}. \tag{45}$$

Once again, in order to isolate normalizable modes which fall off as $(\cos\rho)^{2h_+}$ at the boundary we must impose Eq. (38). At these frequencies, the gamma functions in the denominator of the coefficient of the final sum have negative integer argument; so only terms in the sum which have compensating poles will survive. Such poles will come from one of the final two ψ functions; thus, the logarithmic term drops out and the solution falls off at the boundary as desired, giving a series of modes built from $\Phi^{(+)}$. Equivalently, one can easily show that when Eq. (38) holds, $\Phi^{(+)}$ is regular at the origin. As before, for general ω we have non-fluctuating modes which are well-behaved at the origin and which have a logarithmic part at infinity. A particularly interesting set of non-normalizable solutions occurs when

$$\omega=2h_-+l+2n,\quad n=0,1,\ldots,\nu-1. \tag{46}$$

For such frequencies the final sum in Eq. (45) vanishes and the result is, as in the case of non-integral ν, a rational function in $\cos\rho$. We will see in the next section that these solutions are part of a special highest weight representation that exists for integral ν.

IV. AdS_3 AND $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$

The zoo of solutions that we have described in global coordinates should fall in various representations of the spacetime-isometry–boundary-conformal group. The fluctuating modes should clearly fall in unitary representations, and the boundary operators should create states (in sectors for which the state-operator map is one-to-one) which fall in such unitary representations as well. The non-normalizable modes do not have to fall in unitary representations of the conformal group, but we will see that they do lie in linear representations. Since they couple to primary boundary operators and their conformal descendants, such representations are also important; in order to understand the coupling of descendants we need to understand how the various representations combine.

We will discuss mode solutions in AdS_3, for which the representations of the isometry group are well-known; see especially [30] for a discussion and references. The highest-weight unitary representations in global coordinates and the continuous representations in Poincaré coordinates were discussed in [31]. As we will see, the results of the previous section provide explicit expressions for the wave functions for all of the linear representations.[13]

As discussed in the Appendix, subsection 1, AdS_3 is obtained as the hyperboloid $-\Lambda^2=-U^2-V^2+X^2+Y^2$ embedded in $\mathcal{R}^{2,2}$ with metric $ds^2=-dU^2-dV^2+dX^2+dY^2$. The isometry group of $\mathcal{R}^{2,2}$ is clearly $SO(2,2)$ generated by

$$J_{01}=V\partial_U-U\partial_V,\quad J_{02}=X\partial_V+V\partial_X,\quad J_{03}=Y\partial_V+V\partial_Y,$$
$$J_{23}=X\partial_Y-Y\partial_X,\quad J_{12}=X\partial_U+U\partial_X,\quad J_{13}=Y\partial_U+U\partial_Y. \tag{47}$$

We can construct two commuting $SL(2,\mathbb{R})$ factors from these generators as $SL(2,\mathbb{R})_L=\{L_1=(J_{01}+J_{23})/2, L_2=(J_{02}-J_{13})/2, L_3=(J_{12}+J_{03})/2\}$ and $SL(2,\mathbb{R})_R=\{\bar{L}_1=(J_{01}-J_{23})/2, \bar{L}_2=(J_{02}+J_{13})/2, \bar{L}_3=(J_{12}-J_{03})/2\}$. With these definitions,

$$[L_1,L_2]=-L_3,\quad [L_1,L_3]=L_2,\quad [L_2,L_3]=L_1 \tag{48}$$

and similarly for the $\bar{L}$. These generators preserve the hyperboloid that embeds AdS_3 in $\mathcal{R}^{2,2}$ and so are also isometries of AdS_3.

From $\{L_1,L_2,L_3\}$ it is easy to construct linear combinations $\{L_0,L_\pm\}$ that satisfy the algebra

$$[L_0,L_\pm]=\mp L_\pm,\quad [L_+,L_-]=2L_0. \tag{49}$$

The quadratic Casimir of $SL(2,\mathbb{R})_L$ is then

$$L^2=\frac{1}{2}(L_+L_-+L_-L_+)-L_0^2. \tag{50}$$

Representations can be built by starting with a state $|\psi\rangle$ with L_0 eigenvalue E and acting on it with powers of L_+ or L_-. The commutation relations as defined imply that L_- (L_+) raises (lowers) the L_0 eigenvalue by one unit. Highest (lowest) weight representations contain a state $|h\rangle$ that is annihilated by L_+ (L_-). The entire representation can be built by acting on this state with arbitrary powers of L_- (L_+).

A. Review of the representations of SL(2,ℝ)

The irreducible representations of the $SL(2,\mathbb{R})$ algebra are well known [33]. Barut and Fronsdal [30] derived a set of linear representations which contain them; we will follow their discussion as we will find that this more general set is important for our purposes.[14] The representations are indexed by 2 invariants. The first is h (called $-\Phi$ in [30]) which is related to the quadratic Casimir operator:

$$L^2=h(h-1). \tag{51}$$

We will see that this is the same h as defined in the previous section; $h_\pm$ are the two solutions for a given value of the Casimir operator. The second invariant is the fractional part E_0 of the spectrum of L_0 for a given representation (here we use the same notation as in [30]). Representations are filled out by starting with a given vector in the representation and acting on it an arbitrary number of times with $L_\pm$. The resulting representations are:

$\mathcal{D}(L^2,E_0)$. This an irreducible, infinite-dimensional representation but does not have a highest or lowest weight state. h and E_0 are not related; the only condition is that $h\pm E_0$ is not an integer. E does not even have to be real, but since we wish to describe stable modes in spacetime we will not consider complex values. We can define $-\frac{1}{2}<E_0\leqslant\frac{1}{2}$ without losing generality; the spectrum is $L_0=E_0+n$ for n an arbitrary integer. For fixed L^2 and E_0, the representations for each branch of Eq. (51) are equivalent. The non-unitary representations will correspond to non-normalizable modes for which the energies are not related to the mass, i.e. for which ω is not quantized in even integers above $2h_-+l$. Imposing unitarity restricts $\mathcal{D}(L^2,E_0)$ to two types of representations:

(1) $\mathcal{D}_P$—the "principal series" occurs for $L^2<-\frac{1}{4}$; thus $h=\frac{1}{2}+i\lambda$. $\lambda\neq 0$ will correspond to unstable modes in spacetime as noted in [6,4].
(2) $\mathcal{D}_S$—the "supplementary series" occurs for $L^2>-\frac{1}{4}$. Here h is real and $|h-\frac{1}{2}|<\frac{1}{2}-|E_0|$. This occurs in the range $0<\nu<1$ discussed in the previous section.

$\mathcal{D}^+(h,E_0)$. This is an irreducible, infinite-dimensional highest weight representation and exists for $2h\notin\mathbb{Z}^-\cup 0$. Here $E_0=h$ and the spectrum is $L_0=E_0+n$ for integral $n\geqslant 0$; the highest weight $n=0$ state is annihilated by L_+. The representation will be unitary if $h>0$. These states correspond to the solutions in global coordinates quantized according to Eq. (38) for general ν. For non-integral ν solutions quantized according to Eq. (40) also transform in this representation. The solutions $\Phi^{(+)}$ have $h=h_+>0$ and reside in a unitary positive energy representation. The solutions $\Phi^{(-)}$ have $h=h_-$; they reside in a non-unitary representation for $\nu>1$ and in a unitary representation for $\nu<1$. The derivation of this representation in [30] shows that it can be imbedded in a reducible, nondecomposable representation where n is an arbitrary integer. In this representation we can reach $\mathcal{D}^+$ by starting with negative n states and acting on

[13]For previous discussions of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ structure in solutions to the wave equation in black hole backgrounds, see the review in [32] and references therein.

[14]We use generators with a slightly different normalization than [30]. The generators in [30] are called L_{12} and $M^\pm$, and the Casimir operator is called Q. In this notation our generators are $L_0=L_{12}$ and $L_\pm=i\sqrt{2}M^\mp$; our expression for the Casimir operator is $L^2=-Q$.

them repeatedly with L_-; however, once we examine states in $\mathcal{D}^+$ we cannot leave the irreducible representation (irrep) with actions of $L_\pm$ because of the highest weight state.

$\mathcal{D}^-(h,E_0)$. This is an irreducible, infinite-dimensional lowest weight representation and again exists for $2h \notin \mathbb{Z}^- \cup \{0\}$. Here $E_0 = -h$ and the representation is unitary for $h>0$. The spectrum is $L_0 = E_0 - n$ for integral $n \geqslant 0$. The lowest weight $n=0$ state is annihilated by L_-. These are the negative energy modes corresponding to $\mathcal{D}^+$. Reference [30] embeds this in a reducible, nondecomposable representation which contains energies larger than that of the lowest weight (highest energy) state.

$\mathcal{D}(h)$. This is an irreducible, finite-dimensional representation. It occurs when $2h \in \mathbb{Z}^- \cup \{0\}$ and $E_0 = 0$. Its spectrum is $L_0 = h + n$ for integral $0 \leqslant n \leqslant -2h$. The representation is only unitary in the case $h=0$, i.e. for the identity representation, also known as the singleton [34,12,35]. It is contained in a reducible nondecomposable representation for which n is arbitrary. $\mathcal{D}(h)$ arises in AdS_3 as $\Phi^{(-)}$ for integral ν. In global coordinates, the case $l=0$, $\omega = 2h_- - \cdots - 2h_-$ for integral ω corresponds to a tensor product $\mathcal{D}(h_-) \times \mathcal{D}(h_-)$ transforming under $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$. For arbitrary l nondecomposable representations containing this irrep can occur.

In the next subsection we will explicitly discuss how these representations are realized as solutions to the wave equation in AdS_3 in global coordinates.

B. Global coordinates

In global coordinates the AdS_3 metric is

$$ds^2 = \Lambda^2[-\cosh^2\mu\, dt^2 + d\mu^2 + \sinh^2\mu\, d\theta^2] \tag{52}$$

and a scalar field of mass m has a wave equation $(\Box - m^2\Lambda^2)\phi = 0$ where

$$\Box = \partial_\mu^2 + \frac{2\cosh(2\mu)}{\sinh(2\mu)}\partial_\mu + \frac{1}{\sinh^2\mu}\partial_\theta^2 - \frac{1}{\cosh^2\mu}\partial_t^2. \tag{53}$$

In these coordinates, a convenient basis for $SL(2,\mathbb{R})_L$ is

$$L_0 = iL_1, \quad L_+ = (L_2 + iL_3), \quad L_- = -(L_2 - iL_3). \tag{54}$$

Starting with generators in Eq. (47) and using the coordinate patch in the Appendix, subsection 1, that yields the global metric, it is easy to work out the explicit representation

$$L_0 = i\partial_w \tag{55}$$

$$L_- = ie^{-iw}\left[\frac{\cosh(2\mu)}{\sinh(2\mu)}\partial_w - \frac{1}{\sinh(2\mu)}\partial_{\bar{w}} + \frac{i}{2}\partial_\mu\right] \tag{56}$$

$$L_+ = ie^{iw}\left[\frac{\cosh(2\mu)}{\sinh(2\mu)}\partial_w - \frac{1}{\sinh(2\mu)}\partial_{\bar{w}} - \frac{i}{2}\partial_\mu\right] \tag{57}$$

where $w = t + \theta$ and $\bar{w} = t - \theta$. The generators $\bar{L}_0$, $\bar{L}_\pm$ of $SL(2,\mathbb{R})_R$ are obtained by exchanging w and $\bar{w}$ in Eqs. (55)–(57). Clearly $L_0 \pm \bar{L}_0$ generate time translations and rotations.

Discussion of explicit solutions. The d'Alembertian for scalar fields is given in terms of the left and right Casimir operators as

$$\Box\Phi = -2(L_L^2 + \overline{L_R^2})\Phi = m^2\Lambda^2. \tag{58}$$

The highest weight states $\mathcal{D}^{(+)}$ and $\mathcal{D}$ are simplest to describe. We require that $L_+\Phi_H = \bar{L}_+\Phi_H = 0$, and this imposes $h = \bar{h}$. Using this, the d'Alembertian in Eq. (58) acting on highest weight states reduces to

$$h(h-1) = \frac{m^2\Lambda^2}{4} \Rightarrow h_\pm = \frac{1}{2}(1 \pm \sqrt{1+m^2\Lambda^2}). \tag{59}$$

Explicit solutions of the equation $L_+\Phi_H = \bar{L}_+\Phi_H = 0$ give [31]

$$\Phi_H^{(\pm)} = e^{-ih_\pm w - ih_\pm \bar{w}}\frac{1}{(\cosh\mu)^{2h_\pm}} = e^{-i(2h_\pm)t}\frac{1}{(\cosh\mu)^{2h_\pm}} \tag{60}$$

so that h in this section and in the previous section are the same.

For $\nu = h_+ - h_-$ non-integral, $\Phi_H^{(+)}$ are precisely the minimum-energy normalizable modes found in Sec. III B (here $\tan\rho = \sinh\mu$ as in the Appendix, subsection 1). $\Phi_H^{(-)}$ are the non-normalizable modes of lowest energy in the spectrum (39). Other non-fluctuating modes will reside in non-highest-weight representations. Descendant states are constructed on the primary $\Phi_H^{(\pm)}$ by the action of $(L_-)^p(\bar{L}_-)^q$ and have weights $h = h_\pm + p$ and $\bar{h} = h_\pm + q$. Examining the differential operator L_- shows that all of these solutions have the same boundary behavior as the primary states and therefore share their normalizability properties. Finally, $L_0 + \bar{L}_0$ is the generator of time translations, and $L_0 - \bar{L}_0$ is the generator of rotations, so that the frequency and angular momentum are given by $\omega = h + \bar{h}$ and $l = h - \bar{h}$. The spectra of the two towers of states are given by

$$\omega_\pm = 2n + 2h_\pm + l, \quad n = 0,1,2,\ldots. \tag{61}$$

This matches the spectra for the $\nu \notin \mathbb{Z}$ solutions found in Sec. III B.

For integral ν, the situation is the same for representations built on h_+; the modes residing in this representation are again the normalizable solutions $\Phi^{(+)}$. The non-fluctuating states will fall into two types. If ω is not quantized according to Eq. (46), the solutions will reside in the representations $\mathcal{D}(L^2,\omega)$. If the energies are quantized according to Eq. (46), then the solutions $\Phi^{(-)}$ will reside in nondecomposable representations containing the finite-dimensional representations $\mathcal{R} = \mathcal{D}(h_-) \times \mathcal{D}(h_-)$. The lowest energy (highest weight) state in $\mathcal{R}$ has $\omega = 2h_-$ and is given by

$$e^{-iwh_- - i\bar{w}h_-}(\cosh\mu)^{-2h_-}. \tag{62}$$

The highest energy (lowest weight) state in $\mathcal{R}$ has $\omega = -2h_- = 2h_+ - 2$ and is given by

$$e^{iwh_-+i\bar{w}\bar{h}_-}(\cosh\mu)^{-2h_-} \tag{63}$$

which is annihilated by L_- and $\bar{L}_-$. One may also find these modes in the manner described in Sec. III B 2.

The case $\nu=1$ is quite straightforward and interesting. In addition to the (unitary) identity representation at $l=0$ there are solutions with $(h\in\mathbb{Z}^+,\ \bar{h}=0)$:

$$\Phi=e^{-ilw}\tanh^l\mu. \tag{64}$$

One may apply $L_\pm$ to reach other such solutions. For general l, applying $L_\pm$ to $(h,\bar{h})=(l,0)$ gives us the state $(h,\bar{h})=(l\mp1,0)$. Applying L_+ to $(h=1,\ \bar{h}=0)$ gives the irreducible identity representation $\mathcal{D}(0)\times\mathcal{D}(0)$. Applying $\bar{L}_+$ annihilates all states $(h,\bar{h})=(l,0)$. Applying $\bar{L}_-$ to such states brings one to the normalizable set of solutions forming the irrep $\mathcal{D}^+(1,0)\times\mathcal{D}^+(1,0)$. There are similar solutions for $(h=0,\ \bar{h}\in\mathbb{Z}^+)$ and states generated from these by applying the raising and lowering operators. The full representation in this example is a nondecomposable representation and was discussed in Ref. [35]. Such representations were discussed for $\nu=0,1$ in AdS_{d+1} in Refs. [36,37] from a somewhat different point of view.

In summary, we have found that every solution to the wave equation which is well-behaved in the bulk lies in a representation of the bulk-isometry–boundary-conformal group $G=\mathrm{SL}(2,\mathbb{R})_L\times\mathrm{SL}(2,\mathbb{R})_R$. The energy is simply the L_0 eigenvalue, and the singularity or zero of the wave function at infinity is related to the invariant h of the representation. The map is quite natural. Normalizable modes (plus the singleton) correspond to unitary representations of G; it makes sense to quantize these modes. Non-normalizable modes correspond to non-unitary representations of G. These are the modes we wish to keep as non-fluctuating backgrounds.

Relation to conformal field theory on the cylinder. As $\mu\to\infty$, the generators $\mathrm{SL}(2,\mathbb{R})_L$ acting on surfaces of fixed μ become

$$L_0=i\partial_w,\quad L_-=ie^{-iw}\partial_w,\quad L_+=ie^{iw}\partial_w. \tag{65}$$

These generators and their $\mathrm{SL}(2,\mathbb{R})_R$ companions which replace w by $\bar{w}$ are the standard conformal symmetries of the cylinder. Acting on descendants of a primary state Φ_H with weight h, the generators at the boundary become

$$L_0=i\partial_w,\quad L_-=ie^{-iw}(\partial_w-ih),\quad L_+=ie^{iw}(\partial_w+ih). \tag{66}$$

The shift of $L_\pm$ by $\pm ih$ arises from the radial derivatives in the bulk generators.

C. Poincaré coordinates

In Poincaré coordinates the AdS_3 metric and d'Alembertian are

$$ds^2=\left(\frac{\Lambda^2}{r^2}\right)(-dt^2+dx^2+dr^2) \tag{67}$$

$$\Box=r^2\partial_r-r\partial_r+r^2(\partial_x^2-\partial_t^2). \tag{68}$$

As discussed in the Appendix, subsection 1, these coordinates only cover a region of AdS_3 and there is a horizon at $r=\infty$. A convenient basis for $\mathrm{SL}(2,\mathbb{R})_L$ in these coordinates is

$$L_0=-L_2,\quad L_+=i(L_1+L_3),\quad L_-=i(L_1-L_3). \tag{69}$$

Starting with the explicit generators in Eq. (47) and implementing the Poincaré coordinates in the Appendix, subsection 1, yields the generators

$$L_0=\frac{-r}{2}\partial_r-z\partial_z \tag{70}$$

$$L_-=i\Lambda\partial_z \tag{71}$$

$$L_+=-\frac{i}{\Lambda}[zr\partial_r+z^2\partial_z+r^2\partial_{\bar{z}}]. \tag{72}$$

Here $z=t+x$ and $\bar{z}=t-x$. The generators $\bar{L}$ of $\mathrm{SL}(2,\mathbb{R})_R$ simply exchange z and $\bar{z}$.

Translations and CFT on the plane. It is instructive to examine the action of the generators (70)–(72) on surfaces of constant r at the boundary $r=0$:

$$L_0=-z\partial_z,\quad L_-=i\Lambda\partial_z,\quad L_+=\frac{-iz^2}{\Lambda}\partial_z. \tag{73}$$

These are the standard generators of conformal transformations on the plane. (The factors of i and Λ arise because we are in Minkowski space and z is a dimensionful coordinate.) In the case of the CFT on the cylinder, L_0 generated translations. L_- generates translations on the plane, while L_0 generates dilatations. Furthermore, the basis (69) for $\mathrm{SL}(2,\mathbb{R})$ is different from the basis (54) we used for global coordinates. As discussed in the Appendix, the boundary of the patch of spacetime covered by Poincaré coordinates is conformal to Minkowski space and we have chosen the corresponding natural basis for L_0 and $L_\pm$. It is important to emphasize that we are *not* dealing with the standard bijective map between CFTs on the cylinder and the plane. The plane that appears at the Poincaré boundary is merely a patch of the cylinder and can only be expected to describe the theory on the cylinder in a thermal sense, after tracing over some degrees of freedom.

The Poincaré mode solutions of Sec. III A with fixed frequency ω and momentum k are eigenstates of L_- and $\bar{L}_-$ with eigenvalues $\omega\pm k$ (see also [31]). L_- generates a noncompact subgroup of $\mathrm{SL}(2,\mathbb{R})$ [30]; thus the wave functions we get by diagonalizing this operator should be continuously moded, as we have found.

Representing the isometries. We can also construct a set of scalar fields in Poincaré coordinates that carry well-defined weights under L_0 and $\bar{L}_0$. Again, the d'Alembertian is the sum of left and right Casimir operators:

$$\Box\Phi=-2(L_L^2+\bar{L}_R^2)\Phi=m^2\Lambda^2 \tag{74}$$

where we have used the definition in Eq. (50) of SL(2,$\mathbb{R}$) Casimirs operators. States of weight $(h,\bar{h})$ under L_0 and $\bar{L}_0$ must satisfy the equations

$$-\left(\frac{r}{2}\partial_r+z\partial_z\right)\Phi=h\Phi,\quad -\left(\frac{r}{2}\partial_r+\bar{z}\partial_{\bar{z}}\right)\Phi=\bar{h}\Phi. \quad (75)$$

Since L_0 and $\bar{L}_0$ are linear in derivatives, a product of eigenstates of these operators is still an eigenfunction. A general class of eigenstates is given by

$$\Phi=r^a z^b \bar{z}^c (r+\sqrt{z\bar{z}})^d (r-\sqrt{z\bar{z}})^e. \quad (76)$$

The left and right weights are then

$$(h,\bar{h})=(-[b+(a+d+e)/2],\quad -[c+(a+d+e)/2]). \quad (77)$$

In global coordinates the eigenstates of L_0 essentially provided a Fourier basis for mode expansions. Here the L_0 eigenstates provide an expansion in a power series of functions.

To find highest weight (primary) states we want to additionally solve the equations $L_+\Phi_H=\bar{L}_+\Phi_H=0$. Simultaneous solution of these conditions requires that $z\partial_z\Phi=\bar{z}\partial_{\bar{z}}\Phi$ implying that $h=\bar{h}$ and imposing symmetry between z and $\bar{z}$ in primary states. Using $z\partial_z\Phi=\bar{z}\partial_{\bar{z}}\Phi$ and the equation for $L_0\Phi=h\Phi$ in $L_+\Phi_H=0$ gives

$$\partial_r\Phi_H=\frac{-2h}{r}\left(\frac{r^2+z\bar{z}}{r^2-z\bar{z}}\right)\Phi_H \quad (78)$$

which is easily solved to give

$$\Phi_H=r^{2h}(r^2-z\bar{z})^{-2h}. \quad (79)$$

It can be explicitly checked that this is a primary state with weights (h,h). Requiring that the d'Alembertian acting on this solution have eigenvalue $m^2\Lambda^2$ yields the condition

$$h(h-1)=\frac{m^2\Lambda^2}{4}\Rightarrow h_\pm=\frac{1}{2}(1\pm\sqrt{1+m^2\Lambda^2}) \quad (80)$$

exactly as in the case of global coordinates.

For $m^2>0$ and $z\bar{z}<0$ the solutions with $h=h_+$ vanish at the boundary $(r=0)$ and at the horizon $(r=\infty)$ while the the h_- solutions diverge in both locations. This agrees with the claim in Sec. III that the h_- representation alters the boundary conditions but does not fluctuate. Furthermore, it suggests that in Poincaré coordinates the horizon should also be treated as a boundary with which flux can be exchanged. Indeed, it is well known that quantum field theory in a spacetime with horizon requires the specification of horizon boundary conditions. For $z\bar{z}\geqslant 0$ the situation is more disturbing—the h_+ solution is singular in the bulk of spacetime at $r^2=z\bar{z}$. This appears to be a pathology that arises because the surface $(r^2-z\bar{z})=0$ is a fixed point of L_0, the generator of dilatations. The eigenstates of L_0 are accordingly singular on this surface.

V. DISCUSSION AND CONCLUSIONS

A. Understanding the bulk from the boundary

The original motivation for this work was the desire to study quantum gravity in the bulk spacetime from the perspective of the boundary gauge theory. In particular, we would like to study the appearance of spacetime singularities and horizons (see [10,38], for example) which we expect to be related to nonperturbative issues in the gauge theory. A preliminary step is to determine whether we can say anything about local spacetime physics from the boundary perspective.

The existence of the normalizable modes displayed in this article implies that there is a natural Hilbert space of small fluctuations around the bulk AdS background. These fluctuating modes are the probes of bulk causal structure. Since we can map such states into the boundary Hilbert space, we might expect that there is an analysis of local bulk processes from the boundary point of view.

On the other hand it seems that we cannot reconstruct position space correlation functions in terms of the correspondence as written in Eq. (1). To see this let us expand both sides of Eq. (1) in a formal series in the boundary field Φ_b. For the moment we will work in Euclidean space where this is a well-defined procedure since the bulk field Φ is uniquely determined by Φ_b. (We will return to the Lorentzian version below.) As pointed out in [4], Φ can be expressed in terms of Φ_b via the equation (here, in Poincaré coordinates)

$$\Phi(x^0,\vec{x})=\int_{\mathcal{B}}d\vec{x}'K(x^0,\vec{x};\vec{x}')\Phi_b(\vec{x}'). \quad (81)$$

where K is a solution to the wave equation behaving as a delta function times a given power of x^0 on the boundary. The quadratic piece of the action can be written as

$$S_{\rm eff}=\int dz dx dz' dx' \Phi(z,x)\mathcal{F}(z,x;z',x')\Phi(z',x')+\cdots. \quad (82)$$

$\mathcal{F}$ is the inverse spacetime propagator for Φ, which we would like to extract from the boundary theory. This piece can be written as a quadratic expression in Φ_b via Eq. (81). Expanding (formally) Eq. (1) to quadratic order, we find

$$\langle\mathcal{O}(t_1,\vec{x}_1)\mathcal{O}(t_2,\vec{x}_2)\rangle =-2i\int dt'd\vec{x}'dr'dt''d\vec{x}''dr''K(r',t',\vec{x}';t_1,\vec{x}_1) \quad (83)$$

$$\times\mathcal{F}(r',t',\vec{x}';r'',t'',\vec{x}'')K(r'',t'',\vec{x}'';t_2,\vec{x}_2). \quad (84)$$

Extracting $\mathcal{F}$ from this would require ''inverting'' K, which seems impossible. The integration over r',r'' washes out any information about localization in the direction perpendicular to the boundary. Essentially, this point was made in Ref. [16]—the correspondence (1) relates off-shell operators in the boundary to on-shell fields in the bulk. Since the r dependence of the latter depends on the $(t,\vec{x})$ dependence, full localization is not possible.

The point is that instead of being able to reconstruct arbitrary off-shell correlators, we must be content with a description of on-shell quantities in the bulk. Indeed, our knowledge of the mapping between the bulk and boundary Hilbert spaces and their Hamiltonians implies that transition amplitudes between arbitrary *physical* states are computable in either of the dual descriptions. Off-shell physics can be probed to the extent that on-shell correlators receive contributions from off-shell modes in intermediate states. This situation will force the boundary analysis of the bulk geometry to be somewhat subtle and indirect, but still possible in principle.

In addition to correlation functions, we want to be able to ask how the vacuum in the bulk string theory relates to the boundary field theory. This question really concerns the global causal structure—the existence of different natural vacua in a spacetime often reflects the presence of horizons, for example. Understanding this is also relevant to the study of possible singularities in black hole backgrounds from the dual gauge theory point of view. It may be that although local bulk physics is quite difficult to examine via the boundary field theory, the global causal structure may somewhat easier to address.

B. Apparent ambiguities in the effective action

The presence of normalizable modes in Lorentzian AdS might appear to render the correspondence (1) ambiguous, since there is no preferred solution corresponding to a given boundary value. However, there is a natural prescription—we must sum the effective action over all spacetime backgrounds with the same boundary behavior, with a weighting which depends upon the state of the system. We can generalize the calculations in [4] that we outlined above, by appropriately modifying Eq. (81):

$$\Phi=\Phi_n+\int_{\mathcal{B}}K\Phi_b=\Phi_n+\Phi_{nn}, \tag{85}$$

where Φ_n is an arbitrary normalizable solution of the field equations. Here K is a particular solution we have chosen which solves the wave equation and has the same behavior at infinity as in the Euclidean case. $S_{\rm eff}$ can be written as

$$S_{\rm eff}=\int [\partial(\Phi_n+\Phi_{nn})]^2+m^2\Phi^2. \tag{86}$$

Upon integrating by parts, the bulk term vanishes. It is easy to see that upon including the correct measure factors, the non-vanishing boundary terms are

$$S_{\mathcal{B}}=\int_{\mathcal{B}}dS^i[2\Phi_n\partial_i\Phi_{nn}+\Phi_{nn}\partial_i\Phi_{nn}]. \tag{87}$$

When summing over field histories in the Lorentzian path integral, one must also specify a state at early and late times in the form of wave functionals $\Psi_{i,f}[\Phi]$. A path integral obtained by continuation from Euclidean signature will pick out vacuum wave functionals, since the $i\epsilon$ prescription damps out all excited states. In such a case, the normalizable mode Φ_n is set to zero and the boundary contribution to the action in Eq. (86) vanishes. More generally, one could choose to study excited states at early and late times by explicitly including the appropriate wave functionals; in this case the boundary terms become physically relevant and give contributions to correlation functions evaluated in excited states.

For calculations in interacting theories, the effects of the ambiguity can be more pronounced. In Euclidean AdS, a dramatic example of the effect of multiple saddlepoints with same boundary behavior is the high-temperature transition between AdS space and the AdS black hole discussed in Sec. 3.2 of [4]. Our normalizable modes represent a large class of saddlepoints of the bulk action in Lorentzian AdS and, in an interacting theory, they encode the non-trivial dynamics of the bulk.

C. Conclusions

In this work we have developed a Lorentzian signature version of the bulk-boundary correspondence. This required understanding the respective roles played by normalizable and non-normalizable modes. The two sets of modes emerge naturally, either from direct solution of the field equations or from the field representations of the AdS isometry group. The non-normalizable modes act as backgrounds and couple to local operators in the boundary description, while normalizable modes describe fluctuations in the bulk.

The picture we have presented suggests several avenues for the study of black hole spacetimes from the boundary perspective. Black holes can be constructed in AdS spaces [39] by making discrete identifications of the geometry. Unlike pure AdS spacetime, the resulting bulk spacetime has global horizons and singularities in the classical supergravity approximation. The question is whether and how the boundary theory describes the interior of the black hole. One would hope that the Hilbert space of states within the black hole is identified with a sector of states in the boundary theory; this would realize a form of the black hole complementarity advocated by 't Hooft and Susskind. In our picture, this issue would be studied by considering the roles of the normalizable and non-normalizable solutions to the wave equation in the black hole spacetime.

ACKNOWLEDGMENTS

We have enjoyed discussions with E. Gimon, S. Kachru, E. Keski-Vakkuri, R. Leigh, E. Martinec, S. Mathur, J. Schwarz, E. Silverstein, A. Strominger and C. Vafa. We are also grateful to O. Aharony, J. Maldacena, R. Helling, R. Myers, and S. Ross for useful comments on the original version of this work. P.K. thanks Stanford University for their hospitality during the course of this work. V.B. is supported by the Harvard Society of Fellows and by the NSF under grant NSF-Phy-92-18167. P.K. is supported by in part by DOE grant DE-FG03-92-ER40701 and by the DuBridge foundation. A.L. is supported by NSF grant NSF-Phy-92-18167.

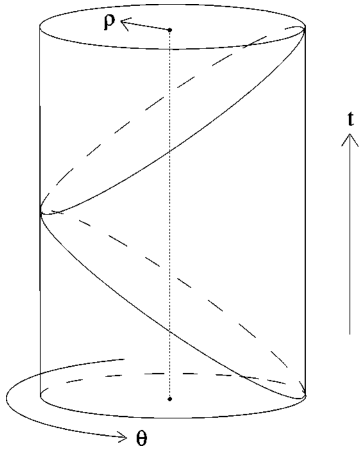

FIG. 1. Anti–de Sitter spacetime displayed as the interior of a cylinder. For the single cover of AdS spacetime the top and bottom boundaries should be identified, whereas for its universal covering space (CAdS) an infinite number of copies should be attached above and below the displayed region. The boundary of AdS spacetime is identified with the boundary of the cylinder. The coordinates indicated correspond to those in Eq. (A2). Horizons in AdS spacetime are obtained by making two diagonal cuts through the cylinder, as shown. The cuts divide AdS spacetime into two regions, each of which is covered by a set of Poincare coordinates. The boundary divides into two diamond shaped regions, which are each conformal to copies of flat Minkowski space.

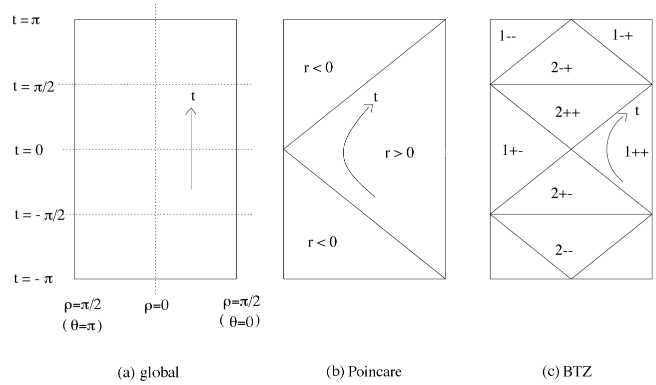

FIG. 2. Penrose diagrams for anti–de Sitter space. Displayed are vertical cross sections which cut through the center of the AdS cylinder. In each figure, regions demarcated by solid lines identify the portion of the spacetime covered by a single coordinate patch. (a) Global coordinates. The boundary of the region is the surface of a cylinder. (b) Poincaré coordinates. Here AdS space is divided into two patches, with the two boundaries at $r=0$ being conformal to flat Minkowski space. At the horizons, $r=\pm\infty$. (c) BTZ coordinates. AdS space is divided into 12 patches, 8 of which appear in the 2 dimensional slice shown. The 8 boundaries are each conformal to flat Minkowski space.

APPENDIX: COORDINATE SYSTEMS ON AdS_{d+1}

1. AdS_3

AdS_3 is defined as the hyperboloid $-U^2-V^2+X^2+Y^2=-\Lambda^2$ embedded in a space with metric $ds^2=-dU^2-dV^2+dX^2+dY^2$.

Global coordinates. Global coordinates for AdS_3 are defined by

$$U=\Lambda\cosh\mu\sin t,\quad V=\Lambda\cosh\mu\cos t,$$

$$X=\Lambda\sinh\mu\cos\theta,\quad Y=\Lambda\sinh\mu\sin\theta.$$

These yield the metric

$$ds^2=\Lambda^2[-\cosh^2\mu dt^2+d\mu^2+\sinh^2\mu d\theta^2]. \quad (A1)$$

Here $0\leqslant\mu\leqslant\infty$, $0\leqslant\theta\leqslant 2\pi$ and $0\leqslant t\leqslant 2\pi$. We unwrap t to have range $-\infty$ to ∞ in order to work on $CAdS_3$, the universal cover of AdS_3.

It is often convenient to make the coordinate transformation $\sinh\mu=\tan\rho$ with $0\leqslant\rho\leqslant\pi/2$. The metric then becomes

$$ds^2=\Lambda^2[-\sec^2\rho dt^2+\sec^2\rho d\rho^2+\tan^2\rho d\theta^2]. \quad (A2)$$

From the above we see that AdS_3 has the topology of a disk times a line. The boundary of spacetime at $\rho=\pi/2$ is a cylinder $S^1\times R$ (see Fig. 1). The bulk-boundary correspondence asserts that a conformal field theory on this cylinder is dual to quantum gravity in the bulk.

A Penrose diagram illustrating the causal structure can be drawn by considering a two dimensional cross section of the hyperboloid. We choose to display a $X=0$ slice, and obtain the diagram in Fig. 2(a).

Poincaré coordinates. Poincaré coordinates are defined by

$$U=\frac{1}{2r}(\Lambda^2+x^2+r^2-t^2),\quad V=\Lambda\frac{t}{r},$$

$$Y=\frac{-1}{2r}(-\Lambda^2+x^2+r^2-t^2),\quad X=\Lambda\frac{x}{r}, \quad (A3)$$

giving the metric

$$ds^2=(\Lambda^2/r^2)(-dt^2+dx^2+dr^2). \quad (A4)$$

Here t and x range between $-\infty$ and ∞, and $0\leqslant r\leqslant\infty$. Poincaré coordinates only cover one half of AdS_3, as shown in Fig. 1. There is a horizon in these coordinates at $r=\infty$. The boundary at $r=0$ is clearly conformal to flat Minkowski space $R^{1,1}$. The bulk-boundary correspondence asserts that a conformal field theory on this boundary plane is dual to quantum gravity in the bulk. Note however, that the plane only covers half of the cylindrical boundary of global AdS_3.

The second half of AdS_3 displayed in Fig. 1 can be covered by labelling the hyperboloid as in Eq. (A3) but now letting $-\infty\leqslant r\leqslant 0$. These two patches together cover AdS_3, while to cover $CAdS_3$ one assembles a vertical tower of such patches. The Penrose diagram is displayed in Fig. 2(b).

BTZ coordinates. For completeness, we consider a third coordinate system which is useful for constructing the BTZ black hole. Divide the hyperboloid into three regions:

$$\text{region 1:}\quad -U^2+X^2\leqslant 0,\quad -V^2+Y^2\geqslant 0,$$
$$\text{region 2:}\quad -U^2+X^2\leqslant 0,\quad -V^2+Y^2\leqslant 0,$$
$$\text{region 3:}\quad -U^2+X^2\geqslant 0,\quad -V^2+Y^2\leqslant 0. \tag{A5}$$

Note that one cannot take both $-U^2+X^2$ and $-V^2+Y^2$ to be positive. We then cover each region with four coordinate patches.

$$U=\pm\hat{r}\cosh\hat{\phi},\quad V=\sqrt{\hat{r}^2-\Lambda^2}\sinh\hat{t},$$
$$X=\hat{r}\sinh\hat{\phi},\quad Y=\pm\sqrt{\hat{r}^2-\Lambda^2}\cosh\hat{t},$$
$$ds^2=-(\hat{r}^2-\Lambda^2)d\hat{t}^2+\Lambda^2(\hat{r}^2-\Lambda^2)^{-1}d\hat{r}^2+\hat{r}^2d\hat{\phi}^2, \tag{A6}$$
region 1,

$$U=\pm\hat{r}\cosh\hat{\phi},\quad V=\pm\sqrt{\Lambda^2-\hat{r}^2}\cosh\hat{t},$$
$$X=\hat{r}\sinh\hat{\phi},\quad Y=\sqrt{\Lambda^2-\hat{r}^2}\sinh\hat{t},$$
$$ds^2=-(\hat{r}^2-\Lambda^2)d\hat{t}^2+\Lambda^2(\hat{r}^2-\Lambda^2)^{-1}d\hat{r}^2+\hat{r}^2d\hat{\phi}^2, \tag{A7}$$
region 2,

$$U=\hat{r}\sinh\hat{\phi},\quad V=\pm\sqrt{\hat{r}^2+\Lambda^2}\cosh\hat{t},$$
$$X=\pm\hat{r}\cosh\hat{\phi},\quad Y=\sqrt{\hat{r}^2+\Lambda^2}\sinh\hat{t},$$
$$ds^2=(\hat{r}^2+\Lambda^2)d\hat{t}^2+\Lambda^2(\hat{r}^2+\Lambda^2)^{-1}d\hat{r}^2-\hat{r}^2d\hat{\phi}^2, \tag{A8}$$
region 3.

In all three regions, $\hat{t}$ and $\hat{\phi}$ range between $-\infty$ and ∞. Here $\hat{r}$ has range $\Lambda\leqslant\hat{r}\leqslant\infty$ in region 1, $0\leqslant\hat{r}\leqslant\Lambda$ in region 2, and $0\leqslant\hat{r}\leqslant\infty$ in region 3. So altogether there are 12 patches covering AdS_3. To draw a Penrose diagram we again consider the slice $X=0$. Note that on this slice region 3 is a dimension one submanifold, while regions 1 and 2 are dimension 2 submanifolds. Thus only regions 1 and 2 will appear in the Penrose diagram, and we label the various patches as $1\pm\pm$, $2\pm\pm$ in an obvious notation. The Penrose diagram then appears as in Fig. 2(c).

To make a BTZ black hole [39] of mass M from the above coordinates one simply makes $\hat{\phi}$ periodic with period $2\pi\sqrt{M}$.

2. AdS_{d+1}

The various coordinate systems defined above generalize straightforwardly to arbitrary dimensions. Global coordinates for AdS_{d+1} give the metric

$$ds^2=\Lambda^2[-\sec^2\rho dt^2+\sec^2\rho d\rho^2+\tan^2\rho d\Omega_{d-1}^2] \tag{A9}$$

with $0\leqslant\rho\leqslant\pi/2$, $-\infty\leqslant t\leqslant\infty$. Thus AdS_{d+1} is globally a d-dimensional disk times a line and the boundary at $\rho=\pi/2$ is a cylinder $S^{d-1}\times R$. The d'Alembertian operator in global coordinates is

$$\Lambda^2\Box=-\cos^2\rho\partial_t^2+\cos^2\rho\partial_\rho^2+(d-1)\cot\rho\partial_\rho+\cot^2\rho\nabla_{S^{d-1}}^2. \tag{A10}$$

Poincaré coordinates yield a metric

$$ds^2=(\Lambda^2/r^2)(-dt^2+d\vec{x}^2+dr^2). \tag{A11}$$

Here $d\vec{x}^2$ is the flat metric on R^{d-1} and $0\leqslant r\leqslant\infty$. There is a horizon at $r=\infty$ and the boundary at $r=0$ is the plane $R^{d-1,1}$. The d'Alembertian operator in Poincaré coordinates is

$$\Lambda^2\Box=-r^2\partial_t^2+r^2\partial_r^2-(d-1)r\partial_r+r^2\nabla_{R^{d-1}}^2. \tag{A12}$$

[1] S. R. Das and S. D. Mathur, Nucl. Phys. **B478**, 561 (1996); J. Maldacena and A. Strominger, Phys. Rev. D **56**, 4975 (1997); I. R. Klebanov, Nucl. Phys. **B496**, 231 (1997); M. R. Douglas, J. Polchinski and A. Strominger, J. High Energy Phys. **B12**, 003 (1997); S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. **B499**, 217 (1997); M. Cvetic and F. Larsen, Nucl. Phys. B (Proc. Suppl.) **62**, 443 (1998).

[2] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).

[4] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).

[5] S. J. Avis, C. J. Isham and D. Storey, Phys. Rev. D **18**, 3565 (1978).

[6] P. Breitenlohner and D. Z. Freedman, Phys. Lett. **115B**, 197 (1982); Ann. Phys. (N.Y.) **144**, 249 (1982).

[7] C. G. Callan and F. Wilczek, Nucl. Phys. **B340**, 366 (1990).

[8] N. Seiberg, Prog. Theor. Phys. Suppl. **102**, 319 (1990); presented at the 1990 Cargese meeting on Random Surfaces, Quantum Gravity, and Strings; presented at the Yukawa Institute Seminar on Common Trends in Mathematics and Quantum Field Theories.

[9] N. Seiberg and S. Shenker, Phys. Rev. D **45**, 4581 (1992).

[10] G. T. Horowitz and H. Ooguri, Phys. Rev. Lett. **80**, 4116 (1998).

[11] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, ''Correlation functions in the CFT_d/AdS_{d+1} correspondence,'' MIT Report No. MIT-CTP-2727, hep-th/9804058.

[12] C. Fronsdal, Phys. Rev. D **12**, 3819 (1975).

[13] E. J. Martinec, ''Matrix models of AdS gravity,'' Chicago Report No. EFI-98-01, hep-th/9804111.

[14] O. Coussaert, M. Henneaux, and P. van Driel, Class. Quantum Grav. **12**, 2961 (1995).

[15] M. Banados, Phys. Rev. D **52**, 5816 (1996).
[16] T. Banks and M. B. Green, J. High Energy Phys. **05**, 002 (1998).
[17] L. Mezinescu and P. K. Townsend, Ann. Phys. (N.Y.) **160**, 406 (1985).
[18] S. W. Hawking, Phys. Lett. **126B**, 175 (1983).
[19] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).
[20] B. Biran, A. Casher, F. Englert, M. Rooman and P. Spindel, Phys. Lett. **134B**, 179 (1984).
[21] L. Castellani, R. D'Auria, P. Fre, K. Pilch and P. van Nieuwenhuizen, Class. Quantum Grav. **1**, 339 (1984).
[22] P. van Nieuwenhuizen, Class. Quantum Grav. **2**, 1 (1985).
[23] H. J. Kim, L. J. Romans and P. van Nieuwenhuizen, Phys. Rev. D **32**, 389 (1985).
[24] S. Deger, A. Kaya, E. Sezgin, and P. Sundell, ''Spectrum of $D=6$, $N=4b$ supergravity on $AdS_3\times S^3$,'' Texas A&M Report No. CTP-TAMU-15/98, hep-th/9804166.
[25] O. Aharony, Y. Oz and Z. Yin, Phys. Lett. B **430**, 87 (1998).
[26] S. Minwalla, J. High Energy Phys. **10**, 002 (1998).
[27] S. Minwalla, ''Restrictions imposed by superconformal invariance on quantum field theories,'' Princeton Report No. PUTP-1748, hep-th/9712074.
[28] *Higher Transcendental Functions*, edited by A. Erdélyi (Kreiger, Malabar, FL, 1981), Vols. 1–3.
[29] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, San Diego, CA, 1980).
[30] A. O. Barut and C. Fronsdal, Proc. R. Soc. London **A287**, 532 (1965).
[31] J. Maldacena and A. Strominger, ''AdS_3 black holes and a stringy exclusion principle,'' Harvard Report No. HUTP-98/A016, hep-th/9804085.
[32] M. Cvetic and F. Larsen, Nucl. Phys. B (Proc. Suppl.) **62**, 443 (1998).
[33] V. Bargmann, Ann. Math. **48**, 568 (1947).
[34] P. A. M. Dirac, J. Math. Phys. **4**, 901 (1963).
[35] M. Flato and C. Fronsdal, Lett. Math. Phys. **20**, 65 (1990).
[36] S. Ferrara and C. Fronsdal, Class. Quantum Grav. **15**, 2153 (1998).
[37] S. Ferrara and C. Fronsdal, Phys. Lett. B **433**, 19 (1998).
[38] G. T. Horowitz and S. F. Ross, J. High Energy Phys. **04**, 015 (1998).
[39] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).