Quantum three-dimensional de Sitter space

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We compute the canonical partition function of (2+1)-dimensional de Sitter space using the Euclidean SU(2)×SU(2) Chern-Simons formulation of 3D gravity with a positive cosmological constant. First, we point out that one can work with a Chern-Simons theory with level $k=l/4G$, and its representations are therefore unitary for integer values of k. We then compute explicitly the partition function using the standard character formulas for SU(2) WZW theory and find agreement, in the large k limit, with the semiclassical result. Finally, we note that the de Sitter entropy can also be obtained as the degeneracy of states of representations of a Virasoro algebra with $c=3l/2G$. [S0556-2821(99)05002-X]

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In 2+1 dimensions, general relativity has been shown to be equivalent to Chern-Simons theory with a six dimensional gauge group whose structure depends on the value of the cosmological constant and on the signature of the spacetime metric [1]. In this paper, we shall focus on the case of a positive cosmological constant and Euclidean metrics. This case has the attractive feature that the gauge group is SU(2) ×SU(2), and its representations are therefore well defined. We shall compute the canonical partition function of de Sitter space by reducing the Chern-Simons theory to a boundary Wess-Zumino-Witten (WZW) theory. For related calculations see [2,3]. This calculation is similar to those yielding the Bañados-Teitelboim-Zanelli (BTZ) black hole entropy by considering a boundary conformal field theory either at the black hole horizon [4,5] or at spatial infinity [6,7,5].

The metric for (2+1)-dimensional de Sitter space is

$$ds^2=-\left(1-\frac{r^2}{l^2}\right)dt^2+\left(1-\frac{r^2}{l^2}\right)^{-1}dr^2+r^2d\phi^2 \qquad (1)$$

where $\Lambda=1/l^2$ is the cosmological constant, and the horizon at $r=l$ is that seen by an observer travelling along the timelike geodesic at $r=0$. Performing a Wick rotation yields the Euclidean de Sitter metric

$$ds_E^2=\left(1-\frac{r^2}{l^2}\right)dt_E^2+\left(1-\frac{r^2}{l^2}\right)^{-1}dr^2+r^2d\phi^2 \qquad (2)$$

where the period β of the time coordinate t_E $(0\leq t_E<\beta)$ is fixed as $\beta=2\pi l$ by the requirement that the metric be everywhere regular. Below, we shall allow a conical singularity at $r=0$. This will change the value of β such that the metric is still regular at $r=l$.

A simple coordinate transformation $r=l\cos\varrho$ makes it clear that this is a three sphere, since

$$ds_E^2=\sin^2\varrho\, dt_E^2+l^2d\varrho^2+l^2\cos^2\varrho\, d\phi^2. \qquad (3)$$

Note that $\varrho=0$ describes the horizon and it is the origin of the angular coordinate t_E, while the observer's worldline lies at $\varrho=\pi/2$ and is the origin of ϕ. In the following, it will be convenient to introduce a new time coordinate x^0 with period 1 related to t_E by $x^0=t_E/\beta$.

In computing the entropy of de Sitter space, we shall consider a boundary surface that encloses the Euclidean world line of the timelike geodesic observer. (This surface may be thought of as analogous to a surface at spatial infinity in the black hole case.) The boundary surface results from removing (a thickened version of) the observer's worldline from Euclidean de Sitter space. Since a three sphere can be constructed by gluing together a pair of solid tori, the remaining space is itself a solid torus. The topology of the Euclidean manifold that we consider is thus exactly the same as for the BTZ black hole [5].

A gravity theory of Riemannian metrics with a positive cosmological constant is classically equivalent to a Chern-Simons theory for the group SU(2)×SU(2). The corresponding SU(2) gauge fields are

$${}^{+}A^a=\omega^a+\frac{1}{l}e^a, \qquad {}^{-}A^a=\omega^a-\frac{1}{l}\,e^a, \qquad (4)$$

where w^a and e^a are the spin connection and triad respectively.

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Using the metric (3) we can read off the triad and spin connection. Actually, we shall consider a generalization of the spacetime (3) described by the two-parameter gauge field

$$^{\pm}A^1 = -\gamma \sin \varrho \left(d\phi \mp \frac{\beta}{l} dx^0 \right),$$

$$^{\pm}A^2 = \pm d\varrho,$$

$$^{\pm}A^3 = \pm \gamma \cos \varrho \left(d\phi \mp \frac{\beta}{l} dx^0 \right) \tag{5}$$

from which de Sitter space is obtained for $\gamma=1$. The parameter γ parametrizes the deficit angle of a conical singularity located at $\varrho=\pi/2$ $(r=0)$, or equivalently a holonomy around a non-contractible loop in the solid torus. Indeed, since the origin of ϕ is located at $\varrho=\pi/2$, and $A^1_\phi=-\gamma$ at that point, the gauge field (5) has a holonomy[1] for $\gamma\neq 1$. We shall only consider the case $0<\gamma\leqslant 1$ which, from the metric point of view, represents an angular deficit in ϕ. We shall see below that the condition $\gamma<1$ will be protected quantum mechanically by the bound in the spin $(2s<k)$ arising in affine SU(2) representations.

Since x^0 is also an angular coordinate whose origin is located at $\varrho=0$, the requirement that there be no conical singularities or holonomies at the horizon $(\varrho=0)$ fixes $\gamma\beta/l=2\pi$ and thus

$$\beta = \frac{2\pi l}{\gamma}. \tag{6}$$

For $\gamma=1$ we recover the period of de Sitter space and Eq. (5) yields the Euclidean metric (3).

The conical singularities arising in three dimensional gravity with a non-zero cosmological constant were introduced in [8]. Their statistical mechanical properties have been recently studied in [3]. Note that the presence of this singularity provides another reason to remove the observer's wordline $\varrho=\pi/2$ since the singularity is located at that point. The conformal field theory lives precisely on that surface.

In the following we shall consider only the positive chirality gauge field ^+A and denote it simply by A. All the following equations can be straightforwardly generalized to include the other SU(2) field.

We work here in the canonical ensemble. The first step in defining the canonical partition function is to formulate boundary conditions. Motivated by the gauge field (5), and as in BTZ case [5,9], we fix the boundary conditions at $\varrho=\pi/2$ to be

$$A_0 = -\frac{\beta}{l} A_\phi, \tag{7}$$

where β will be the argument of the canonical partition function. Since β is the period of the Euclidean time coordinate, it can be interpreted as the inverse temperature. As explained before, de Sitter space has $\beta=2\pi l$. Other values of β correspond to holonomies in the gauge field, and, for $\beta>2\pi l$, they can be interpreted as conical singularities.

As in [5] we shall also add a term at the horizon $\varrho=0$ that imposes the constraint

$$A^a_0|_{\varrho=0} = -2\pi\delta^a_3 \tag{8}$$

which can be achieved by using a Wilson line along the horizon [5].

The boundary conditions (7) and (8) are motivated by the on-shell gauge field (5) associated to de Sitter space. However, they give rise to a far bigger space. Indeed, the phase space is infinite dimensional and is described by a chiral WZW model (see below).

The Euclidean action appropriate to our boundary conditions [(8) and (7)] is[2]

$$\begin{aligned} I_E[A,\beta] = {} & \frac{k}{4\pi}\int_M \epsilon^{kl}\, \mathrm{Tr}(iA_k\partial_0 A_l - A_0F_{kl})d^2x dx^0 \\ & - \frac{k\beta}{4l\pi}\int_{\varrho=\pi/2} \mathrm{Tr}(A_\phi)^2 d\phi dx^0 \\ & - \frac{k}{2}\int_{\varrho=0} A^{(3)}_\phi d\phi dx^0. \end{aligned} \tag{9}$$

with $k=l/4G$.

This choice of action requires some explanation. The Chern-Simons action does not depend on the metric and therefore the spacetime signature does not affect its form. In the Chern-Simons formulation of general relativity, the information about the spacetime signature is contained in the local gauge group.

However, let k be a positive integer, $\mathcal{G}$ a compact Lie group, and consider the Chern-Simons action $I[A] = (k/4\pi)\int_M \mathrm{Tr}\ \epsilon^{ij}(-A_i\dot{A}_i + A_0F_{ij})$ (plus boundary terms) on a manifold M with the topology $\Sigma\times S_1$. We take S_1 as the time direction and, as usual in Euclidean field theory, we relate its period with the inverse temperature β. We now ask the question of what is the right measure in the path integral which produces a partition function of the form Tr $e^{-\beta H}$, with H real and positive, and the trace being taken over a well defined Hilbert space. If we integrate $e^{iI[A]}$, one finds a Kac-Moody algebra at level k whose representations are well defined, but the Hamiltonian has an unwanted $i\beta$ coefficient. If, on the contrary, we integrate $e^{-I[A]}$, one finds the right

[1]Given an angular coordinate θ with period 2π whose origin is located at a point $r=0$, in general, there is a holonomy around $r=0$ if the zero mode of A_θ is different from zero at $r=0$. Suppose $A=\gamma d\theta J$ with J any of the Hermitian SU(2) generators. This value of A can be set equal to zero by the gauge transformation $g=e^{-\gamma\theta J}$. However, g is single valued only if γ is an integer. In this case, the above holonomy is trivial. For $\gamma\neq n$, g is multivalued and therefore A does have a non-trivial holonomy.

[2]We use the conventions $T_a=-(i/2)\sigma_a$, $[T_a,T_b]=-\epsilon_{abc}T^c$, $Tr(T_aT_b)=-\frac{1}{2}\delta_{ab}$. The total action including both chiralities is $I=I[^+A,^+\beta]-I[^-A,^-\beta]$.

measure $e^{-\beta H}$, but the representations are not well defined because this yields a Kac-Moody algebra at level ik. The source of this problem is not the choice of the measure but the action itself. Indeed, Chern-Simons theory is of first order (linear in the time derivatives) therefore its correct ''Euclidean'' version is $I_E[A]=(k/4\pi)\int \mathrm{Tr}\, \epsilon^{ij}(iA_i\dot{A}_i+A_0F_{ij})$ (plus boundary terms) and one integrates e^{-I_E}. This is the form of the action (9). We stress that here the terminology ''Euclidean'' refers to properties of the path integral and not to the spacetime signature (which is encoded in the local gauge group).

To further justify Eq. (9) we note, first, that the equations of motion derived from it are $F_{ij}=0$ and $i\partial_0 A_i=D_iA_0$ and they are solved by the de Sitter gauge field (5), and second, the value of I on the solution (5) coincides with that of the Euclidean Hilbert action on the metric (2) with $k=l/4G$ (see below). Finally, and perhaps most importantly in the context of Euclidean integrals in quantum gravity, the action (9) gives rise to a well defined partition function that can be computed exactly and yields sensible answers. In a future publication [10], we shall argue that a similar approach leads to a simplified treatment of Riemannian anti–de Sitter metrics, although in that case the gauge group is necessarily complex.

The partition function associated to A is then equal to

$$Z_A(\beta)=\int D[A]\exp(-I_E[A,\beta]), \tag{10}$$

where the measure $D[A]$ denotes a sum over all gauge fields modulo gauge transformations and $I[A,\beta]$ is given in Eq. (9). Note that this partition function has the right semiclassical value. Indeed, in the saddle point approximation provided by the de Sitter gauge field (5) one finds

$$Z_A(\beta)\sim\sum_\gamma e^{-\beta\gamma^2/(16G)+\pi\gamma l/(4G)}. \tag{11}$$

The sum over γ arises because the only fixed quantity in the canonical calculation is β. A saddle point approximation for that sum yields the relation $\gamma\beta=2\pi l$ which is the classical value of γ that avoids conical singularities at the horizon. From Eq. (11) we can associate to each state labeled by γ an energy $E_\gamma=\gamma^2/(16G)$ and a degeneracy $\rho(\gamma)=\exp[\pi\gamma l/(4G)]$. The de Sitter state corresponds to $\gamma=1$ and has a total entropy [adding the entropy coming from the other SU(2) gauge field] $S=S_l+S_r=\pi l/(2G)$ which is the Gibbons-Hawking value for de Sitter space [11].

Our goal now is to compute the above partition function exactly. In the exact calculation we shall re-encounter the sum (11) but with discretized values of γ. The condition $\gamma<1$, which is necessary for a sensible metric interpretation for the sum over γ as conical singularities, will be protected by the bound on the spin s arising in affine SU(2) representations.

It is well known that Eq. (10) can be reduced to a WZW model at the boundary. The idea is that once the bulk gauge degrees of freedom are eliminated, one is left with an effective theory describing an infinite dimensional residual gauge group at the boundary. This infinite dimensional group is parametrized by a group element g and the action is the chiral WZW model[3]

$$\begin{aligned}I_{CWZW}[g,\beta]=&-\frac{ik}{4\pi}\int \mathrm{Tr}(\partial_\phi g^{-1}\partial_\tau g)d\phi d\tau\\&-\frac{ik}{12\pi}\int_M \mathrm{Tr}(g^{-1}dg)^3 d^2x d\tau\\&-\frac{k\beta}{4l\pi}\int \mathrm{Tr}(g^{-1}\partial_\phi g)^2 d\phi d\tau\\&-\frac{k}{2}\int (g^{-1}\partial_\phi g)^{(3)} d\phi d\tau.\end{aligned} \tag{12}$$

The partition function thus becomes

$$Z(\beta)=\int D[g]\exp(-I_{CWZW}[g,\beta]) \tag{13}$$

and can be evaluated using canonical methods. The canonical Poisson brackets associated with the WZW action are given by the SU(2) Kac-Moody algebra,

$$[T^a_n,T^b_m]=i\epsilon^{ab}{}_c T^c_{n+m}+n\frac{k}{2}\delta^{ab}\delta_{n+m,0}, \tag{14}$$

where the T^a_n are the Fourier components of the gauge field,

$$A_\phi=g^{-1}\partial_\phi g=\frac{2}{k}\sum_{n=-\infty}^{\infty}T^a_n e^{in\phi}. \tag{15}$$

Z can thus be computed as

$$Z_A(\beta)=\sum_{2s=0}^{k}\mathrm{Tr}_s\exp(-\beta L_0/l+2\pi T^3_0), \tag{16}$$

where s labels the spin of the different SU(2) representations and L_0 is a Virasoro generator defined by

$$L_0=\frac{1}{k+Q}\sum_{n=-\infty}^{\infty}:T^a_{-n}T^b_n:\delta_{ab}. \tag{17}$$

The parameter Q is the second Casimir in the adjoint representation. We shall be interested in the $k\gg Q$ limit so this term can be neglected. The computation of the trace in Eq. (16) follows from [12]:

[3]The reduction from Chern-Simons to WZW has been extensively studied in the literature. In our situation, the simplest way to pass from Eq. (10) to Eq. (13) is by solving the constraint $F_{kl}=0$. Alternatively, one can integrate over purely imaginary values of A_0.

$$\mathrm{Tr}_s(q^{L_0}e^{i\theta T_0^3})=\frac{q^{s(s+1)/k}\sum_{n=-\infty}^{\infty}q^{kn^2+(2s+1)n}\{e^{i(s+kn)\theta}-e^{-i(s+1+kn)\theta}\}}{\Pi_{m=1}^{\infty}(1-q^m)(1-q^m e^{i\theta})(1-q^{m-1}e^{-i\theta})}$$
$$=q^{s(s+1)/k}D^{-1}\sum_{n=-\infty}^{\infty}q^{kn^2+(2s+1)n}\frac{\sin[(s+kn+\frac{1}{2})\theta]}{\sin\frac{\theta}{2}} \tag{18}$$

with $q=e^{-\beta/l}$ and $D=\Pi_{m=1}^{\infty}(1-q^m)(1-q^m e^{i\theta})(1-q^m e^{-i\theta})$. Splitting the sine in Eq. (18) according to $\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$, using the fact that in our case $\theta=-2\pi i$, and assuming the main contribution in the trace over the spin representations to come from high values of s, one finally ends up with an effective partition function

$$Z_A(\beta)=\sum_{2s=0}^{k}D^{-1}q^{s(s+1)/k}\frac{\sinh[2\pi(s+\frac{1}{2})]}{\sinh\pi}\times\sum_{n=-\infty}^{\infty}q^{kn^2+(2s+1)n}e^{2\pi kn}. \tag{19}$$

In the semiclassical limit $k\to\infty$ we can concentrate on the numerator since it carries all the k dependence whereas the denominator leads to a subleading contribution.[4]

The sum over n can be calculated by a saddle point approximation obtaining for the saddle point

$$n=\frac{\gamma}{2}-\frac{1}{2k}-\frac{s}{k} \tag{20}$$

where we have replaced $\beta=2\pi l/\gamma$. Since s is bounded from above by $k/2$ and $\gamma<1$, in the limit $k\to\infty$ we have $n<1$ and thus the sum is well approximated by $n=0$.

We thus find an effective quantum mechanical partition function

$$Z_A[\beta]=\sum_{2s=0}^{k}e^{2\pi s}e^{-\beta s(s+1)/lk} \tag{21}$$

which should be compared with the semiclassical sum (11). The energy levels are quantized and given by

$$E_s=\frac{s(s+1)}{lk}, \tag{22}$$

with apparent degeneracies $\rho(s)=e^{2\pi s}$. We calculate the sum (21) by a saddle point approximation. The saddle point occurs when β and s are related as

$$\beta=2\pi l\frac{k}{(2s+1)}. \tag{23}$$

As we mentioned above, the values of β of the form $\beta=2\pi l/\gamma$ with $\gamma<1$ can be interpreted as conical singularities with an angular deficit defined by γ. Thus we find the series of quantized conical singularities with

$$\gamma=\frac{2s+1}{k} \tag{24}$$

which is indeed less than 1 for the allowed states $0\leqslant 2s\leqslant k$.

The state with s taking its maximum value $s=k/2$ has $\beta=2\pi l$ (in the limit $k\to\infty$) and corresponds to de Sitter space. Its energy, according to Eq. (22), is $E=1/16G$ and degeneracy is $e^{\pi k}$. Adding the degeneracy associated to the other SU(2) gauge field (whose de Sitter state has the same degeneracy) yields the total entropy

$$S_{dS}=2\pi k=\frac{2\pi l}{4G} \tag{25}$$

which agrees with the semiclassical approximation for the Gibbons-Hawking entropy of de Sitter space.

Note, that in analogy with the BTZ black hole [5] the density of states arises purely from the presence of the horizon term. This raises the question of whether the density of states could arise dynamically in the context of a theory that does not presuppose the relation (8).

Finally, we point out an interesting relation between this calculation and the use of the twisted Sugawara construction to produce a pair of Virasoro algebras with a classical central charge. This discussion follows the ideas developed in [13,14] but without making a direct connection between the Virasoro algebras and the group of symmetries of de Sitter space at the boundary.

The gauge field (5) obeys the boundary conditions set out in Refs. [5, 14] that allow us to define a pair of Virasoro algebras at the boundary. The central charge of the algebra is

$$c=6k=\frac{3l}{2G} \tag{26}$$

[4]For the denominator $D=\Pi_{n=1}^{\infty}(1-q^n)(1-q^n e^{2\pi})(1-q^n e^{-2\pi})$ we get a subleading contribution to the density of states, but which interestingly is equal to zero for de Sitter space $-\beta/l=\log q=-2\pi$. But this is only the case if we neglect the fact that the temperature gets renormalized in the quantum calculation. This is because the particular form of the quantum Virasoro operator L_0 defined in Eq. (17) forces us to replace β by $\beta'=(1+Q/k)\beta$.

as in the case of negative cosmological constant. This central charge is fixed in the case of negative cosmological constant by the condition that the Virasoro charges leave invariant the anti–de Sitter metric at infinity [15,13,5,16]. In the present case the only justification we have for this choice is by analytic continuation of the result of [5] and that it turns out to give the correct result. Another difference is that in this case our algebra comes from a theory of Euclidean rather than Lorentzian metrics, since this allows us to avoid problems of complex gauge group and complex k.

The Virasoro operators L_0 and $\overline{L_0}$ are defined as

$$L_0=-\frac{k}{4\pi}\int \mathrm{Tr}\left(({}^{+}A_\phi)^2-\frac{1}{2}\right)d\phi=\frac{l}{8G} \tag{27}$$

$$\overline{L_0}=-\frac{k}{4\pi}\int \mathrm{Tr}\left(({}^{-}A_\phi)^2-\frac{1}{2}\right)d\phi=\frac{l}{8G} \tag{28}$$

and from this we see that de Sitter space can be defined by the condition that the asymptotic zero modes satisfy the conditions

$$L_0=c/12, \quad \overline{L_0}=c/12. \tag{29}$$

Note that, just as in the anti–de Sitter case, the Virasoro operators which yield the central charge normalized as $(c/12)n(n^2-1)$ are shifted with respect to the mass by $M=(L_0-c/24)+(\overline{L_0}-c/24)$. The de Sitter energy of each sector is then $lE_{dS}=c/24=l/16G$, just as in the canonical calculation.

It is now interesting to see whether one can extract the density of states from the Cardy formula as in the black hole case [5,6,7]. However, the formula used in those papers is only valid if the eigenvalue N of L_0 is much larger than c. Equation (29) shows that this is not the case. A generalization of the formula is straightforward and can be derived from [17]. The improved formula reads[5]

$$\varrho(l)=\exp\left[2\pi\sqrt{\frac{c}{6}\left(N-\frac{c}{24}\right)}+2\pi\sqrt{\frac{\overline{c}}{6}\left(\overline{N}-\frac{\overline{c}}{24}\right)}\right] \tag{30}$$

which reduces to the formula used in [5] in the case of large N. The entropy following from this formula is

$$S=\frac{\pi l}{2G} \tag{31}$$

in exact agreement with the standard result.

This last result at least suggests that a theory in which the density of states arises dynamically is likely to have an effective central charge of $3l/2G$ as in anti–de Sitter space.

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[5]Note that the use of the formula (30) implies the assumption (that we have as yet been unable to verify) that $Z(-1/\beta)$ is a slowly varying function in the region of the saddle point.

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