Hidden symmetries, $AdS_D \times S^n$, and the lifting of one-time physics to two-time physics

Itzhak Bars
Department of Physics and Astronomy, University of Southern California, Los Angeles, California 90089-0484

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The massive non-relativistic free particle in $d-1$ space dimensions, with a Lagrangian $L=(m/2)\dot{\mathbf{r}}^2$, has an action with a surprising non-linearly realized SO(d,2) symmetry. This is the simplest example of a host of diverse one-time-physics systems with hidden SO(d,2) symmetric actions. By the addition of gauge degrees of freedom, they can all be lifted to the *same* SO(d,2) covariant unified theory that includes an extra spacelike and an extra timelike dimension. The resulting action in $d+2$ dimensions has manifest SO(d,2) Lorentz symmetry and a gauge symmetry Sp(2,R). The symmetric action defines two-time physics. Conversely, the two-time action can be gauge fixed to diverse one-time physical systems. In this paper three new gauge fixed forms that correspond to the non-relativistic particle, the massive relativistic particle, and the particle in $AdS_{d-n}\times S^n$ curved spacetime will be discussed at the classical level. The last case is discussed at the first quantized and field theory levels as well. For the last case the popularly known symmetry is SO($d-n-1$,2) $\times$SO($n+1$), but yet we show that the classical or quantum versions are symmetric under the larger SO(d,2). In the field theory version the action is symmetric under the full SO(d,2) provided it is improved with a quantized mass term that arises as an anomaly from operator ordering ambiguities. The anomalous mass term vanishes for $AdS_2\times S^0$ and $AdS_n\times S^n$ (i.e., $d=2n$). A quantum test for the presence of two-time-physics in a one-time physics system is that the SO(d,2) Casimir operators have fixed eigenvalues independent of the system. It is shown that this test is successful for the particle in $AdS_{d-n}\times S^n$ by computing the Casimir operators and showing explicitly that they are independent of n. The strikingly larger symmetry could be significant in the context of the proposed AdS/CFT duality. [S0556-2821(99)02104-9]

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I. HIDDEN SO(d,2) IN ONE-TIME PHYSICS

In this section we will begin by showing some examples of surprising non-linearly realized hidden SO(d,2) symmetry in simple one-time-physics systems. We will then explain the true and systematic origin of these symmetries, not only in these examples but also in a host of many others, as being a simple and direct consequence of two time physics. Two-time physics has been defined and explained in [1–4] and it will be briefly outlined below, but the reader can understand the symmetries discussed here from the traditional one-time physics point of view. The main point of the examples is that the hidden symmetry allows us to embed standard one-time physics in a larger spacetime with one more spacelike and one more timelike dimensions as compared to standard one-time physics. The lifting to higher dimensions is done with the addition of gauge degrees of freedom such that diverse actions for one-time-physics systems converge to the *same* unified action in two-time physics that also has an Sp(2,R) gauge symmetry. The Sp(2,R) acts on position and momentum (X^M,P^M) as a doublet. This establishes an Sp(2,R) duality symmetry among the diverse one-time-physics systems. There are consequences and some tests of two-time-physics as will be illustrated in Sec. II.

A. Non-relativistic particle

1. Hidden symmetry

Consider the free massive non-relativistic particle in $d-1$ space dimensions with the action

$$S=\int d\tau \frac{1}{2} m\dot{\mathbf{r}}^2. \tag{1}$$

We will discuss this simple example from different angles because it serves as a prototype for understanding the more complicated cases. The case of the massless relativistic particle (with and without spin) discussed in [1,3] can also serve as a prototype, but it is perhaps not sufficiently complicated to illustrate some of the issues.

As is well known, the obvious symmetry of this system is described by the Galilean group consisting of rotations SO($d-1$) and translations T_{d-1} in ($d-1$) dimensions. The Hamiltonian $H=\mathbf{p}^2/2m$ commutes with the generators of these symmetries. Until now there has not been any clue that this system has a higher symmetry structure. However, it can be checked that the action (not the Hamiltonian) is symmetric under the larger symmetry SO(d,2) as follows.

Define a basis for an SO(d,2) vector with an index $M=(+',-',0,i)$, with $i=1,2,\ldots,(d-1)$ denoting the space coordinates as in $\mathbf{r}^i$. The parameters of SO(d,2) form an antisymmetric matrix ε_{MN} with independent components $\varepsilon_{+'-'},\varepsilon_{+'0},\varepsilon_{-'0},\varepsilon_{+'i},\varepsilon_{-'i},\varepsilon_{0i},\varepsilon_{ij}$, where the last ε_{ij} are the parameters for rotations for the linearly realized rotations SO($d-1$). The hidden SO(d,2) symmetry of the action above is obtained by the following *off-shell* linear and non-linear transformations of $\mathbf{r}(\tau)$

$$\delta \mathbf{r}^i(\tau)=\varepsilon_{ij}\mathbf{r}^j+\varepsilon_{+'-'}(\mathbf{r}^i-2\tau\dot{\mathbf{r}}^i)+\varepsilon_{+'0}\frac{-\tau(\mathbf{r}^i-\dot{\mathbf{r}}^i\tau)}{\sqrt{(\mathbf{r}-\tau\dot{\mathbf{r}})^2}}+\varepsilon_{-'0}\left[\dot{\mathbf{r}}^i\sqrt{(\mathbf{r}-\tau\dot{\mathbf{r}})^2}-\frac{\tau\dot{\mathbf{r}}^2}{2}\frac{(\mathbf{r}^i-\dot{\mathbf{r}}^i\tau)}{\sqrt{(\mathbf{r}-\tau\dot{\mathbf{r}})^2}}\right]-\varepsilon_{+'i}\tau+\varepsilon_{-'j}\left[-\mathbf{r}^i\dot{\mathbf{r}}^j+\mathbf{r}^j\dot{\mathbf{r}}^i-\mathbf{r}\cdot\dot{\mathbf{r}}\delta^{ij}+\tau\dot{\mathbf{r}}^i\dot{\mathbf{r}}^j+\tau\frac{\dot{\mathbf{r}}^2}{2}\delta^{ij}\right]+\varepsilon_{0j}\left[-\delta^{ij}\sqrt{(\mathbf{r}-\tau\dot{\mathbf{r}})^2}+\frac{\tau\dot{\mathbf{r}}^j(\mathbf{r}^i-\dot{\mathbf{r}}^i\tau)}{\sqrt{(\mathbf{r}-\tau\dot{\mathbf{r}})^2}}\right]. \tag{2}$$

Note the explicit τ, in addition to the implicit τ in $\mathbf{r}^i(\tau)$, which will be related below to a gauge transformation. The Lagrangian transforms into a total derivative $\delta(\dot{\mathbf{r}}^2/2)=\partial_\tau\Lambda(\tau,\varepsilon_{MN})$, with $\Lambda(\tau,\varepsilon_{MN})$ given by

$$\Lambda(\tau,\varepsilon_{MN})=-\varepsilon_{+'-'}\tau\dot{\mathbf{r}}^2-\varepsilon_{+'0}\left[\tau\frac{\dot{\mathbf{r}}\cdot(\mathbf{r}-\tau\dot{\mathbf{r}})}{\sqrt{(\mathbf{r}-\tau\dot{\mathbf{r}})^2}}+\frac{1}{2}\sqrt{(\mathbf{r}-\tau\dot{\mathbf{r}})^2}\right]+\varepsilon_{-'0}\left[\frac{\dot{\mathbf{r}}^2}{2}\sqrt{(\mathbf{r}-\tau\dot{\mathbf{r}})^2}-\frac{\tau\dot{\mathbf{r}}^2}{2}\frac{\dot{\mathbf{r}}\cdot(\mathbf{r}-\tau\dot{\mathbf{r}})}{\sqrt{(\mathbf{r}-\tau\dot{\mathbf{r}})^2}}\right]-\varepsilon_{+'i}\mathbf{r}^i+\varepsilon_{-'j}\left[-\dot{\mathbf{r}}^j\dot{\mathbf{r}}\cdot\mathbf{r}+\mathbf{r}^j\frac{\dot{\mathbf{r}}^2}{2}+\tau\dot{\mathbf{r}}^j\dot{\mathbf{r}}^2\right]+\varepsilon_{0j}\left[\tau\dot{\mathbf{r}}^j\frac{\dot{\mathbf{r}}\cdot(\mathbf{r}-\tau\dot{\mathbf{r}})}{\sqrt{(\mathbf{r}-\tau\dot{\mathbf{r}})^2}}\right]. \tag{3}$$

Hence the action is symmetric under $SO(d,2)$.

2. Generators

The generators of this $SO(d,2)$ symmetry can be derived by using a generalized Noether theorem. Using canonical variables $\mathbf{r}(\tau),\mathbf{p}(\tau)=m\dot{\mathbf{r}}(\tau)$ they are given at any τ by

$$SO(d-1): L^{ij}=\mathbf{r}^i\mathbf{p}^j-\mathbf{r}^j\mathbf{p}^i \tag{4}$$

$SO(1,2)$:

$$\begin{cases} L^{+'-'}=-\left(\mathbf{r}-\tau\frac{\mathbf{p}}{m}\right)\cdot\mathbf{p}, \quad L^{+'0}=-m\sqrt{\left(\mathbf{r}-\tau\frac{\mathbf{p}}{m}\right)^2}, \\ L^{-'0}=-\frac{\mathbf{p}^2}{2m}\sqrt{\left(\mathbf{r}-\tau\frac{\mathbf{p}}{m}\right)^2}, \end{cases} \tag{5}$$

$$L^{+'i}=-m\left(\mathbf{r}^i-\tau\frac{\mathbf{p}^i}{m}\right), \quad L^{0i}=\mathbf{p}^i\sqrt{\left(\mathbf{r}-\tau\frac{\mathbf{p}}{m}\right)^2} \tag{6}$$

$$L^{-'i}=-\frac{\mathbf{p}^2}{2m}\left(\mathbf{r}^i-\tau\frac{\mathbf{p}^i}{m}\right)+\mathbf{p}\cdot\left(\mathbf{r}-\tau\frac{\mathbf{p}}{m}\right)\frac{\mathbf{p}^i}{m}. \tag{7}$$

The Poisson brackets of these $L^{MN}(\tau)$ form the $SO(d,2)$ algebra at every τ (which is treated as a parameter)

$$\{L^{MN},L^{RS}\}=\eta^{MR}L^{NS}+\eta^{NS}L^{MR}-\eta^{NR}L^{MS}-\eta^{MS}L^{NR}, \tag{8}$$

including the $SO(1,2)$ and $SO(d-1)$ subalgebras as indicated. Furthermore, the L^{ij} together with $\mathbf{p}^i\sim L^{0i}/L^{+'0}$ form the Galilean subalgebra, which is the familiar symmetry of the non-relativistic particle. The Galilean generators are the only ones that do not have explicit τ dependence. The general τ dependent L^{MN} generate the new hidden $SO(d,2)$ symmetries of the action (1). The τ dependent terms may be regarded as generating τ-dependent local transformation on the independent *off-shell* dynamical variables $\mathbf{r}(\tau),\mathbf{p}(\tau)$.

The $SO(d,2)$ transformations of the independent canonical degrees of freedom $\mathbf{r},\mathbf{p}$ are obtained at any τ by evaluating the Poisson brackets while treating τ as a parameter

$$\delta\mathbf{r}^i(\tau)=\frac{1}{2}\varepsilon_{MN}\{L^{MN}(\tau),\mathbf{r}^i(\tau)\}, \quad \delta\mathbf{p}^i(\tau)=\frac{1}{2}\varepsilon_{MN}\{L^{MN}(\tau),\mathbf{p}^i(\tau)\}. \tag{9}$$

Under these transformations the first order form of the action

$$S=\int_0^T d\tau\left(\dot{\mathbf{r}}\cdot\mathbf{p}-\frac{\mathbf{p}^2}{2m}\right), \tag{10}$$

is invariant under $SO(d,2)$. Here $\mathbf{r}(\tau),\mathbf{p}(\tau)$ are treated as independent *off-shell* fields whose τ dependence are unrelated to each other. However, if they are related to each other by using the equation of motion for momentum $\mathbf{p}=m\dot{\mathbf{r}}$, then the $\delta\mathbf{r}^i$ of Eq. (9) reduces to the $\delta\mathbf{r}^i$ in Eq. (2) which corresponds to the transformation law for the invariance of the action (1) in the second order form.

It can be checked that the $SO(d,2)$ generators can be rewritten formally as the antisymmetric product of two $(d+2)$-dimensional vectors in the form

$$L^{MN}=X_0^MP_0^N-X_0^NP_0^M \tag{11}$$

with

$$M=(+',-',0,\quad i) \tag{12}$$

$$X_0^M=\left(\tau,\quad \frac{\mathbf{r}\cdot\mathbf{p}}{m}-\frac{\tau\mathbf{p}^2}{2m^2},\ \sqrt{\left(\mathbf{r}-\tau\frac{\mathbf{p}}{m}\right)^2},\mathbf{r}^i\right) \tag{13}$$

$$P_0^M=\left(m,\frac{\mathbf{p}^2}{2m},0,\mathbf{p}^i\right). \tag{14}$$

These satisfy $X_0^2=P_0^2=X_0\cdot P_0=0$ with a metric η_{MN}, such that $\eta_{+'-'}=-1,\eta_{00}=-1,\ \eta_{ij}=\delta_{ij}$. This is the metric invariant under $SO(d,2)$ with two timelike dimensions.

3. Lifting to two-time physics

The SO(d,2) symmetry with this structure implies that the non-relativistic particle action can be lifted to a manifestly SO(d,2) symmetric form by the addition of gauge degrees of freedom. From the form of Eq. (11) we can deduce that the manifestly symmetric form of the symmetry is the Lorentz symmetry SO(d,2) realized linearly on a vector $X^M(\tau)$ and its canonical conjugate $P^M(\tau)$. These describe a particle (0-brane) in a spacetime with d spacelike and 2 timelike dimensions (X^M,P^M are lifted forms of X_0^M,P_0^M including gauge degrees of freedom). This shows that the non-relativistic particle is connected to the realm of two-time physics, a formulation that also has a sufficiently large gauge symmetry Sp($2,R$) to kill all ghosts and connect back to one-time physics as discussed in recent papers [1–4].

The Sp($2,R$) gauge theory for zero branes takes the form [1]

$$S_0=\frac{1}{2}\int_0^T d\tau (D_\tau X_i^M)\varepsilon^{ij}X_j^N\eta_{MN}$$

$$=\int_0^T d\tau\left(\partial_\tau X_1^M X_2^N-\frac{1}{2}A^{ij}X_i^M X_j^N\right)\eta_{MN}. \tag{15}$$

The canonical conjugates are $X_1^M=X^M$ and $\partial S/\partial \dot{X}_1^M=X_2^M=P^M$. They are consistent with the idea that (X_1^M,X_2^M) is the Sp($2,R$) doublet (X^M,P^M). The symmetric A^{ij} are the 3 gauge potentials of Sp($2,R$). The equations of motion for A^{ij} give the first class constraints

$$X\cdot X=X\cdot P=P\cdot P=0 \tag{16}$$

that form the Sp($2,R$) Lie algebra. The action is evidently symmetric under SO(d,2). The generators are gauge invariant

$$L^{MN}=X_i^M X_j^N\varepsilon^{ij}=X^M P^N-X^N P^M. \tag{17}$$

In this form all components of X^M and P^M are canonical and $\delta X^M,\delta P^M$ are obtained by using the basic Poisson brackets $\delta X^M=\frac{1}{2}\varepsilon_{RS}\{L^{RS},X^M\}$, etc. In this fully covariant approach the constraints are applied on the states, as discussed in [1–4].

The three gauge choices that reduce the general system to the non-relativistic particle are

$$X^{+'}(\tau)=\tau,\quad P^{+'}(\tau)=m,\quad P^0(\tau)=0. \tag{18}$$

After solving the three constraints (16) explicitly in this gauge, $X^M(\tau)$ and $P^M(\tau)$ take the form given in Eqs. (13),(14) . Note that the non-relativistic particle action (10) can then be written as

$$S=\int d\tau\,\partial_\tau X_0\cdot P_0=\int_0^T d\tau\left(\dot{\mathbf{r}}\cdot\mathbf{p}-\frac{\mathbf{p}^2}{2m}\right). \tag{19}$$

This follows from the fully gauge invariant and SO(d,2) invariant two-time-physics action (15) after the gauge (13),(14) has been inserted.

4. An intermediate gauge

It is also interesting to consider an intermediate gauge. For example, if we choose only two gauges $P^{+'}(\tau)=m$, $P^0(\tau)=0$ and solve two constraints $X^2=X\cdot P=0$, there remains one gauge freedom and one constraint. Then X^M,P^M are parametrized in terms of the d canonical degrees of freedom $(t(\tau),\mathbf{r}^i(\tau))$ and their canonical conjugates $(H(\tau),\mathbf{p}^i(\tau))$ as follows:

$$M=(+',-',0,i)$$

$$X^M=\left(t,\quad \frac{\mathbf{r}\cdot\mathbf{p}}{m}-t\frac{H}{m},\sqrt{\mathbf{r}^2-2\frac{t\mathbf{r}\cdot\mathbf{p}}{m}+2\frac{H}{m}t^2},\quad \mathbf{r}^i\right) \tag{20}$$

$$P^M=(m,H,0,\mathbf{p}^i). \tag{21}$$

We derive the dynamics for the remaining degrees of freedom $t,\mathbf{r},H,\mathbf{p}$ by inserting this gauge fixed form in the original action (15). The result is a one-time action given by

$$S=\int_0^T d\tau\left(\partial_\tau X^M P^N-\frac{1}{2}A^{22}(-2mH+\mathbf{p}^2)-0-0\right) \tag{22}$$

$$=\int_0^T d\tau\left[-H\partial_\tau t+\mathbf{p}^i\partial_\tau\mathbf{r}^i-\frac{1}{2}A^{22}(-2mH+\mathbf{p}^2)\right]. \tag{23}$$

We have dropped a total derivative term $\partial_\tau(\mathbf{r}\cdot\mathbf{p})$ that does not contribute to the dynamics. The last form of the action confirms that (t,H) and $(\mathbf{r},\mathbf{p})$ are canonical conjugates with Poisson brackets

$$\{t,H\}=-1,\quad \{\mathbf{r}^i,\mathbf{p}^j\}=\delta^{ij}. \tag{24}$$

The A^{22} equation of motion gives the constraint $H=\mathbf{p}^2/2m$. This is the same as the $P^2=0$ constraint. The remaining local symmetry corresponds to τ reparametrizations. In the gauge $t(\tau)=\tau$ the dynamics describes the free nonrelativistic massive particle. In fact, if this additional gauge is chosen the action reduces to Eq. (10).

We expect that this form of one-time-physics action (23) is also symmetric under SO(d,2). To construct the generators we insert the gauge choice of Eqs. (20),(21) in the gauge invariant L^{MN} of Eq. (17). At the classical level, without watching orders of operators, they are given by (now there is no explicit τ dependence)

$$\mathrm{SO}(d-1): \quad L^{ij}=\mathbf{r}^i\mathbf{p}^j-\mathbf{r}^j\mathbf{p}^i \tag{25}$$

$$\mathrm{SO}(1,2): \quad \begin{cases} L^{+'-'}=2tH-\mathbf{r}\cdot\mathbf{p}, \quad L^{+'0}=-m\sqrt{\mathbf{r}^2-2t\dfrac{\mathbf{r}\cdot\mathbf{p}}{m}+2\dfrac{H}{m}t^2}, \\ L^{-'0}=-H\sqrt{\mathbf{r}^2-2t\dfrac{\mathbf{r}\cdot\mathbf{p}}{m}+2\dfrac{H}{m}t^2}, \end{cases} \tag{26}$$

$$L^{+'i}=t\mathbf{p}^i-m\mathbf{r}^i, \quad L^{-'i}=-t\frac{H}{m}\mathbf{p}^i+\frac{\mathbf{r}\cdot\mathbf{p}}{m}\mathbf{p}^i-H\mathbf{r}^i, \tag{27}$$

$$L^{0i}=\mathbf{p}^i\sqrt{\mathbf{r}^2-2\frac{t\mathbf{r}\cdot\mathbf{p}}{m}+2\frac{H}{m}t^2}. \tag{28}$$

Using the basic Poisson brackets (24) it can be shown that these L^{MN} form the SO(d,2) algebra. They also generate the transformation rules for $t,H,\mathbf{r}^i,\mathbf{p}^i$ by evaluating the Poisson brackets $\delta t=\frac{1}{2}\varepsilon_{MN}\{L^{MN},t\}$, etc. The action is not invariant under these transformations alone; for invariance one must also transform A^{22}. The reason is that the constraint $(-2mH+\mathbf{p}^2)$ that multiplies A^{22} in the action is not invariant, but transforms into itself with an overall factor

$$\delta(-2mH+\mathbf{p}^2)=\gamma(\varepsilon_{MN},\tau)(-2mH+\mathbf{p}^2), \tag{29}$$

where

$$\gamma(\varepsilon_{MN},\tau)=\left(\frac{\mathbf{p}^j(\tau)}{m}\varepsilon_{-'j}-\varepsilon_{+'-'} + \frac{t(\tau)\left(\varepsilon_{+'0}+\dfrac{H(\tau)}{m}\varepsilon_{-'0}-\varepsilon_{0j}\dfrac{\mathbf{p}^j(\tau)}{m}\right)}{\sqrt{\mathbf{r}^2(\tau)-2t(\tau)\dfrac{\mathbf{r}(\tau)\cdot\mathbf{p}(\tau)}{m}+2\dfrac{H(\tau)}{m}t^2(\tau)}}\right). \tag{30}$$

This term is cancelled by taking $\delta A^{22}=-2A^{22}\gamma(\varepsilon_{MN},\tau)$. This factor can be understood as follows. Recall that when a gauge is fixed the new generators L^{MN} perform a naive SO(d,2) transformation (that disturbs the gauge) followed by an Sp(2,R) gauge transformation (that restores the gauge). The constraints (16) transform as a triplet under the restoring gauge transformation. Since two of the constraints are already zero explicitly, the third one transforms into itself with an overall factor $\delta(P^2)=\gamma(\varepsilon_{MN})\times P^2$, and this must be compensated by the transformation of the gauge field δA^{22} as given above.

5. Field theory

When we do not make the last gauge choice the remaining constraint must be applied on the states. A complete Hilbert space for the quantum theory is given in configuration space as $|t,\mathbf{r}\rangle$. The physical subset of states $|\psi\rangle$ are those that satisfy the constraint

$$\left(H-\frac{\mathbf{p}^2}{2m}\right)|\psi\rangle=0. \tag{31}$$

In terms of the wave function in configuration space $\psi(t,\mathbf{r})=\langle t,\mathbf{r}|\psi\rangle$ the physical state condition takes the form of the non-relativistic Schrödinger equation

$$i\partial_t\psi(t,\mathbf{r})=-\frac{\nabla^2}{2m}\psi(t,\mathbf{r}). \tag{32}$$

The effective field theory that reproduces this equation is

$$S_{eff}=\int dt\,d\mathbf{r}\left[i\psi^*\partial_t\psi-\frac{1}{2m}\nabla\psi^*\cdot\nabla\psi\right]. \tag{33}$$

The norm of the physical state is then given by integrating the time component of the probability current at fixed time

$$\langle\psi|\psi\rangle=\int d^{d-1}\mathbf{r}\,\psi^*(t,\mathbf{r})\psi(t,\mathbf{r}). \tag{34}$$

This norm is independent of t due to the conservation of the probability current $(\psi^*\psi,(\psi^*\nabla\psi-\nabla\psi^*\psi)/2im)$ as a result of the physical state condition (32).

Now we ask the question: is the field theoretic version of the theory also SO (d,2) invariant under the transformation

$$\delta\psi(t,\mathbf{r})=\frac{i}{2}\varepsilon_{MN}\langle t,\mathbf{r}|L^{MN}|\psi\rangle=\frac{1}{2}\varepsilon_{MN}(\hat{L}^{MN}\psi)(t,\mathbf{r}) \tag{35}$$

where $\hat{L}^{MN}$ are differential operators obtained from the operators L^{MN} in Eqs. (25)–(28) by replacing $H=i\hbar\partial_t$, and $\mathbf{p}=-i\hbar\nabla$ as applied on $\psi(t,\mathbf{r})$. The correct quantum operators to all orders of $\hbar$ must correspond to a particular order of the canonical operators $t,H,\mathbf{r},\mathbf{p}$, but we have not attempted to find the order. Here we face a difficult problem with the non-linear form of the L^{MN} since an infinite number of possibilities of ordering of an infinite series is possible. Therefore we have not been able to give a definitive answer to this question.[1] It would be amazing if one can find an ordering of operators that would give SO(d,2) invariance for the non-relativistic Schrödinger field theory action (33). If there is no such order, it would imply that the quantum theory in the form (33) produces anomalies that break the SO(d,2) symmetry. If this is the case one may ask if there is an anomalous term that can be added to the field theory to yield the correct quantum version with an SO(d,2) symmetry. This question remains open for now.

B. Massive relativistic particle

1. Lifting to the intermediate SO($d-1$,1) covariant gauge

To understand better the hidden symmetries and their origins it is useful to start with the fully gauge fixed form of the relativistic massive particle action and first lift it to the intermediate gauge which is manifestly SO($d-1$,1) Lorentz covariant. The answer is well known, but by using similar steps as the previous section it may be helpful to make analogies to the non-relativistic case, thus clarifying some concepts that may have remained hazy to the reader. Consider the action for the massive relativistic particle

$$S=m\int_0^T d\tau\sqrt{1-\dot{\mathbf{r}}^2}, \tag{36}$$

which as Eq. (1) is also symmetric under rotations and translations. This action has a "hidden" off-shell symmetry under $\delta\mathbf{r}(\tau)=\beta^i\tau-\beta\cdot r(\tau)\dot{\mathbf{r}}^i(\tau)$, where β^i are constant parameters, since the Lagrangian transforms into a total derivative $\delta\sqrt{1-\dot{\mathbf{r}}^2}=\partial_\tau[\mathbf{r}\cdot\beta\sqrt{1-\dot{\mathbf{r}}^2}]$. Using a generalized Noether's theorem one can derive the generator of this transformation, and by writing it in terms of the canonical variables $\mathbf{r}(\tau),\mathbf{p}(\tau)=m\dot{\mathbf{r}}/\sqrt{1-\dot{\mathbf{r}}^2}$ in the form

$$\mathbf{K}^i(\tau)=\tau\mathbf{p}^i(\tau)-\mathbf{r}^i(\tau)\sqrt{\mathbf{p}^2(\tau)+m^2}, \tag{37}$$

one can recognize that it is the generator of relativistic boosts. The $\delta\mathbf{r}(\tau)$ used above can be written as the Poisson bracket $\delta\mathbf{r}(\tau)=\{-\beta\cdot\mathbf{K}(\tau),\ \mathbf{r}^i(\tau)\}$. Note the explicit τ dependence in $\mathbf{K}^i(\tau)$ and in $\delta\mathbf{r}^i(\tau)$ which is analogous to the explicit τ that appeared in the previous non-relativistic case. Although the action is symmetric under the boosts, the Hamiltonian $H=\sqrt{\mathbf{p}^2+m^2}$ is not symmetric, but transforms under them in a well defined manner. We can compare the "hidden" boost symmetry of Eq. (36) to a subset of the hidden symmetries SO(d,2) of Eq. (1).

[1]The analogous question for the $\mathrm{AdS}_{d-n}\times S^n$ will be answered in the affirmative in the last section.

The "hidden" boost symmetry can be made manifest by lifting the action (36) to its well known Lorentz symmetric form

$$S=m\int d\tau\sqrt{-(\dot{x}^\mu)^2}. \tag{38}$$

To do this lifting we must add gauge degrees of freedom and then the action is gauge invariant under τ-reparametrizations. As is well known this action can be rewritten in the first order form by introducing the canonical momentum $p^\mu(\tau)$ and an einbein $A^{22}(\tau)$ that plays the role of a Lagrange multiplier to implement the constraint on this momentum

$$S=\int_0^T d\tau\left[\dot{x}^\mu p_\mu-\frac{1}{2}A^{22}(p_\mu^2+m^2)\right]. \tag{39}$$

Integrating out p^μ and A^{22} gives back Eq. (38). This form should be compared to the non-relativistic case in Eq. (23). The generators of the Lorentz symmetry are now given in terms of canonical variables $L^{\mu\nu}=x^\mu p^\nu-x^\nu p^\mu$ while the constraint is applied on the physical states. Fixing the gauge $x^0(\tau)=\tau$ reduces the action (38) back to Eq. (36) while L^{0i} becomes the $\mathbf{K}^i(\tau)$ of Eq. (37).

2. Lifting to two-time physics

We now note the surprising SO(d,2) symmetry of the action (39) as follows. Using the basis for a $(d+2)$-dimensional vector with index $M=(0',1',\mu)$ the parameters of SO(d,2) are given as an antisymmetric tensor with components $\varepsilon_{0'1'},\varepsilon_{0'\mu},\varepsilon_{1'\mu},\varepsilon_{\mu\nu}$. The last $\varepsilon_{\mu\nu}$ correspond to the linearly realized Lorentz symmetry. The full linearly and non-linearly realized *off-shell* SO(d,2) transformation is

$$\begin{aligned}\delta x^\mu=&\varepsilon^{\mu\nu}x_\nu-\varepsilon_{0'1'}\frac{x^\mu x\cdot p}{\sqrt{m^2x^2+(x\cdot p)^2}}\\&+\varepsilon_{1'\nu}\left[\eta^{\nu\mu}\frac{x\cdot p}{m}+\frac{p^\nu}{m}x^\mu\right]\\&-\varepsilon_{0'\nu}\left[\frac{p^\nu}{m}\frac{x^\mu x\cdot p}{\sqrt{m^2x^2+(x\cdot p)^2}}+\eta^{\nu\mu}\frac{\sqrt{m^2x^2+(x\cdot p)^2}}{m}\right]\end{aligned} \tag{40}$$

and

$$\begin{aligned}\delta p^\mu=&\varepsilon^{\mu\nu}p_\nu+\varepsilon_{0'1'}\frac{m^2x^\mu+x\cdot p\,p^\mu}{\sqrt{m^2x^2+(x\cdot p)^2}}-\varepsilon_{1'\nu}\left[\eta^{\nu\mu}m+\frac{p^\nu}{m}p^\mu\right]\\&-\varepsilon_{0'\nu}p^\nu\frac{m^2x^\mu+x\cdot p\,p^\mu}{\sqrt{m^2x^2+(x\cdot p)^2}}\end{aligned} \tag{41}$$

and

$$\delta A^{22}=A^{22}\left[\frac{\left(\varepsilon_{0'1'}+\varepsilon_{0'\nu}\frac{p^{\nu}}{m}\right)x\cdot p}{\sqrt{m^2x^2+(x\cdot p)^2}}+2\varepsilon_{1'\nu}\frac{p^{\nu}}{m}\right]+\varepsilon_{1'\nu}\frac{\dot{x}^{\nu}}{m}. \tag{42}$$

This transformation gives a total derivative $\delta L=\partial_\tau\Lambda(\varepsilon_{MN},\tau)$ with

$$\Lambda(\varepsilon_{MN},\tau)=\varepsilon_{0'1'}\sqrt{m^2x^2+(x\cdot p)^2}+\varepsilon_{1'\nu}p^{\nu}x\cdot p-m\varepsilon_{1'\nu}x^{\nu}+\varepsilon_{0'\nu}\frac{p^{\nu}}{m}\frac{(x\cdot p)^2}{\sqrt{m^2x^2+(x\cdot p)^2}}. \tag{43}$$

Hence the action (39) is invariant.

The generators of this transformation are

$$L^{0'1'}=\sqrt{m^2x^2+(x\cdot p)^2},\quad L^{0'\mu}=p^{\mu}\sqrt{m^2x^2+(x\cdot p)^2} \tag{44}$$

$$L^{1'\mu}=-\frac{x\cdot p}{m}p^{\mu}-mx^{\mu},\quad L^{\mu\nu}=x^{\mu}p^{\nu}-x^{\nu}p^{\mu}. \tag{45}$$

They close under Poisson brackets to form the SO$(d,2)$ Lie algebra.

These generators are written in the form of cross products

$$L^{MN}=X_0^MP_0^N-X_0^NP_0^M \tag{46}$$

with

$$M=(0',1',\mu)$$

$$X_0^M=\left(\sqrt{x^2+\left(\frac{x\cdot p}{m}\right)^2},-\frac{x\cdot p}{m},x^{\mu}\right) \tag{47}$$

$$P_0^M=(0,m,p^{\mu}). \tag{48}$$

These satisfy $X_0^2=X_0\cdot P_0=0$ while $P_0^2=p^2+m^2$, with the metric $-\eta^{0'0'}=\eta^{1'1'}=-\eta^{0'0'}=1$ and $\eta^{\mu\nu}=$Minkowski. This form suggests that we may lift the system to two-time physics.

Therefore we may start from the Sp$(2,R)$ gauge symmetric two-time physics action (15), choose the two gauges

$$P^{0'}(\tau)=0,\quad P^{1'}(\tau)=m, \tag{49}$$

and solve the two constraints $X^2=X\cdot P=0$. The result is the gauge fixed form (47),(48). The dynamics of the remaining degrees of freedom (x^{μ},p^{μ}) is obtained by inserting the gauge fixed form (47),(48) into the two-time physics action (15). The result is the one-time-physics action (39) for the relativistic particle. This action has one remaining gauge symmetry (τ reparametrization) and imposes the remaining constraint $P^2=p^2+m^2=0$ as the equation of motion for A^{22}.

This shows that both the relativistic and the nonrelativistic particle are lifted to the *same* two-time-physics action. Hence to an observer in two-time physics these two systems are the same, since they are just gauge fixed versions of the same theory.

C. Particle on $\mathrm{AdS}_{d-n}\times \mathrm{S}^n$

A particle moving in the curved background $\mathrm{AdS}_{d-n}\times\mathrm{S}^n$ is described by the action

$$S=\int_0^T d\tau(G_{\mu\nu}^{AdS}(x)\dot{x}^{\mu}\dot{x}^{\nu}+G_{ab}^{S^n}(y)\dot{y}^a\dot{y}^b), \tag{50}$$

where $m=0,1,\ldots,d-n-1$ and $a=1,2,\ldots,n$. There are many ways of parametrizing the AdS metric. The particular parametrization used below is convenient for discussing and resolving the quantum ordering problem which will be dealt with in the next section. The point that we will make is that for $\mathrm{AdS}_D\times\mathrm{S}^n$ the full symmetry of the action is SO$(D+n,2)$. Furthermore, as long as $D+n=d$ is a constant the various models distinguished by n are Sp$(2,R)$ dual to each other because they are obtained from the same Sp$(2,R)$ gauge symmetric two-time-physics action by gauge fixing.

As in the previous cases, the larger SO$(d,2)$ symmetry comes as a surprise since the popularly known symmetry in this background is SO$(d-n-1,2)\times$SO$(n+1)$ which is smaller than SO$(d,2)$. For example, we claim that the action for AdS_3 alone has SO(3,2) symmetry which is larger than the popularly known SO(2,2). Similarly the action for $\mathrm{AdS}_5\times\mathrm{S}^5$ has SO(10,2) symmetry which is larger than the popularly known SO(4,2)$\times$SO(6); and the action for $\mathrm{AdS}_4\times\mathrm{S}^7$ or $\mathrm{AdS}_7\times\mathrm{S}^4$ has SO(11,2) symmetry.

Instead of lifting the $\mathrm{AdS}_{d-n}\times\mathrm{S}^n$ action (50) to the two-time-physics action (15), we will construct Eq. (50) as a gauge fixed form of Eq. (15). Lifting would correspond to the reverse process.

Consider the $(d+2)$-dimensional vectors X^M,P^M in the basis $M=(+',-',\mu,i)$ for $\mu=0,1,\ldots,d-n-1$ and $i=1,2,\ldots,n+1$. The metric is $\eta^{+'-'}=-1,\eta^{ij}=\delta^{ij}$ and $\eta^{\mu\nu}$ = Minkowski. We choose two gauges by demanding $|X^i|=1$ and $P^{+'}=0$. Then the unit vector $X^i=\mathbf{u}^i/|\mathbf{u}|\equiv\mathbf{\Omega}^i$ describes a sphere S^n as the boundary of a ball in $n+1$ dimensions. The radius of the ball $|X^i|$ is one of the coordinates that has been gauge fixed to 1. The constraints $X^2=X\cdot P=0$ are solved by the following parametrization:

$$M=(+',-',\mu,i)$$

$$X^M=\left(|\mathbf{u}|,\frac{1+\mathbf{u}^2x^2}{2|\mathbf{u}|},|\mathbf{u}|x^{\mu},\frac{\mathbf{u}^i}{|\mathbf{u}|}\right) \tag{51}$$

$$P^M=\left(0,-\frac{\mathbf{u}\cdot\mathbf{k}}{|\mathbf{u}|}+\frac{x\cdot p}{|\mathbf{u}|},\frac{p^{\mu}}{|\mathbf{u}|},\left(|\mathbf{u}|\mathbf{k}^i-2\mathbf{k}\cdot\mathbf{u}\frac{\mathbf{u}^i}{|\mathbf{u}|}\right)\right). \tag{52}$$

The bold vectors $\mathbf{u}^i,\mathbf{k}^i$ are in $n+1$ dimensions and x^{μ},p^{μ} are in $d-n-1$ dimensions. For $n=0$ we replace $\mathbf{u}^i/|\mathbf{u}|$ by 1. Inserting this gauge fixed form into the original two-time physics action (15) gives an action that determines the dynamics of $x^{\mu}(\tau),p^{\mu}(\tau),\mathbf{u}^i(\tau),\mathbf{k}^i(\tau)$

$$S=\int d\tau\left(p\cdot\dot{x}+\mathbf{k}\cdot\dot{\mathbf{u}}-\frac{1}{2}A^{22}\left(\frac{p^2}{\mathbf{u}^2}+\mathbf{u}^2\mathbf{k}^2\right)\right) \tag{53}$$

$$\to\int d\tau\frac{1}{2A^{22}}\left(\frac{\dot{\mathbf{u}}^2}{\mathbf{u}^2}+\mathbf{u}^2\dot{x}^2\right) \tag{54}$$

$$=\int d\tau\frac{1}{2A^{22}}\left(\frac{\dot{u}^2}{u^2}+u^2\dot{x}^2+\dot{\mathbf{\Omega}}^2\right). \tag{55}$$

The second form of the action is obtained by integrating out the momenta. From the first line we see that the vectors $p^\mu,\mathbf{k}^i$ are indeed the canonical conjugates to $x^\mu,\mathbf{u}^i$ respectively. The last line is obtained by making a transformation from Cartesian coordinates to spherical coordinates $\mathbf{u}^i=u\mathbf{\Omega}^i$. This action describes the particle in the curved background $\mathrm{AdS}_{d-n}\times S^n$ with metric

$$ds^2=u^2(dx^\mu)^2+\frac{(du)^2}{u^2}+(d\mathbf{\Omega})^2$$

where the $d-n$ coordinates of AdS_{d-n} are (x^μ,u) and the n coordinates of S^n are those that parametrize the unit vector $\mathbf{\Omega}^i$ embedded in $n+1$ dimensions. This form of the metric has been used in recent discussions of the proposed AdS conformal field theory (CFT) duality [6], and we find it useful for the discussion of operator ordering that will be dealt with in the next section.[2] There are many other possible parametrizations of the AdS metric. Each one of them will correspond to some form of gauge choice in our formalism. For such other gauge choices for AdS see [4] and [3].

The point here is that our construction shows that the symmetry of the action is SO$(d,2)$ which is larger than the popularly known SO$(d-n-1,2)\times$SO$(n+1)$. In our approach the SO$(d,2)$ generators are obtained by inserting the gauge fixed forms of X_0^M and P_0^M given in Eqs. (51),(52) into the gauge invariant L^{MN} of Eq. (17). At the classical level (operator ordering ignored) we obtain $L^{MN}=X_0^MP_0^N-X_0^NP_0^M$ in the form

$$L^{+'-'}=-\mathbf{u}\cdot\mathbf{k}+x\cdot p,\quad L^{+'\mu}=p^\mu,\quad L^{+'i}=\mathbf{u}^2\mathbf{k}^i-2\mathbf{k}\cdot\mathbf{u}\mathbf{u}^i \tag{56}$$

$$L^{-'\mu}=\frac{p^\mu}{2\mathbf{u}^2}+\mathbf{u}\cdot\mathbf{k}x^\mu+\frac{1}{2}x^2p^\mu-x\cdot p x^\mu \tag{57}$$

$$L^{-'i}=\frac{1}{2}\mathbf{k}^i+\frac{x^2}{2}(\mathbf{u}^2\mathbf{k}^i-2\mathbf{k}\cdot\mathbf{u}\mathbf{u}^i)-x\cdot p\frac{\mathbf{u}^i}{\mathbf{u}^2} \tag{58}$$

$$L^{\mu\nu}=x^\mu p^\nu-x^\nu p^\mu,\quad L^{ij}=\mathbf{u}^i\mathbf{k}^j-\mathbf{u}^j\mathbf{k}^i \tag{59}$$

$$L^{\mu i}=x^\mu(\mathbf{u}^2\mathbf{k}^i-2\mathbf{k}\cdot\mathbf{u}\mathbf{u}^i)-p^\mu\frac{\mathbf{u}^i}{\mathbf{u}^2}. \tag{60}$$

By using the basic Poisson brackets among $(\mathbf{u},\mathbf{k}),(x^\mu,p^\mu)$ it is easily seen that these form the SO$(d,2)$ algebra

$$\{L^{MN},L^{RS}\}=\eta^{MR}L^{NS}+\eta^{NS}L^{MR}-\eta^{NR}L^{MS}-\eta^{MS}L^{NR}. \tag{61}$$

The generators for the subgroup SO$(n+1)\times$SO$(d-n-1,2)$ are L^{ij} and $(L^{\mu\nu},L^{+'-'},L^{+'\mu},L^{-'\mu})$ respectively. The additional symmetry generators that complete to SO$(d,2)$ are $L^{+'i},L^{-'i},L^{\mu i}$. It is well known that the action (54) is symmetric under SO$(n+1)\times$SO$(d-n-1,2)$. To show that it is also symmetric under the full SO$(d,2)$ it is sufficient to show that it is symmetric under the $L^{\mu i}$ since the remaining $L^{\pm' i}$ are obtained from these by SO$(d-n-1,2)$ rotations. The transformations generated by $L^{\mu i}$ are given by evaluating the Poisson brackets $\delta\mathbf{u}^i=\{\varepsilon_{\nu j}L^{\nu j},\mathbf{u}^i\}$, $\delta x^\mu=\{\varepsilon_{\nu j}L^{\nu j},x^\mu\}$:

$$\delta\mathbf{u}^i=2\varepsilon^{\nu j}x_\nu\mathbf{u}_j\mathbf{u}^i-x_\nu\varepsilon^{\nu i}\mathbf{u}^2,\quad \delta x^\mu=\varepsilon^{\mu j}\frac{\mathbf{u}_j}{\mathbf{u}^2}. \tag{62}$$

The Lagrangian transforms as follows:

$$\delta\left(\frac{\dot{\mathbf{u}}^2}{\mathbf{u}^2}+\mathbf{u}^2\dot{x}^2\right)=(2\varepsilon_{\mu i}x^\mu\mathbf{u}^i)\left(\frac{\dot{\mathbf{u}}^2}{\mathbf{u}^2}+\mathbf{u}^2\dot{x}^2\right).$$

This is equivalent to a conformal rescaling of the metric which can be cancelled by a transformation of the einbein

$$\delta A^{22}=(2\varepsilon^{\nu j}x_\nu\mathbf{u}_j)A^{22}. \tag{63}$$

Therefore the action for a particle on $\mathrm{AdS}_{d-n}\times S^n$ is invariant under SO$(d,2)$ for all n.

II. SO$(d,2)$ GENERATORS IN FIRST QUANTIZATION

Since the $\mathrm{AdS}_{d-n}\times S^n$ case is of current interest due to the proposed AdS-CFT duality [6], we will also discuss the first quantized theory in that gauge. We will resolve quantum ordering ambiguities in the generators of SO$(d,2)$, and then compute the quadratic Casimir eigenvalue of SO$(d,2)$ for all values of n at fixed d, to show that these gauge invariant quantities are independent of n and are the same as those computed in other gauges, namely $C_2(\mathrm{SO}(d,2))=1-d^2/4$. This confirms the gauge invariant prediction of two-time physics, thus verifying its presence.

The full *physical information of the theory is contained in the gauge invariant* L^{MN}. Using the constraints $X^2=P^2=X\cdot P=0$ it is straightforward to show that all the Casimir operators of SO$(d,2)$ vanish at the classical level

$$\text{classical:}\quad C_n(\mathrm{SO}(d,2))=\frac{1}{n!}\mathrm{Tr}(iL)^n=0. \tag{64}$$

[2] There is a similarity between our parametrization and one used in [5], however our's treats the radius of AdS or the sphere (here scaled to $R=1$) as an additional coordinate that has been gauge fixed [i.e., $|X^i(\tau)|=R=1$]. This additional coordinate together with the global and gauge symmetries of the action (15) is what permits us to have the larger symmetry SO$(d,2)\supset$SO$(d-n-1,2)\times$SO$(n+1)$.

In the first quantized theory the $C_n(\mathrm{SO}(d,2))$ are not zero after taking quantum ordering into account. Since the L^{MN} are gauge invariant we must find the same eigenvalues in any gauge. First consider the SO$(d,2)$ covariant quantization without choosing any gauges, as treated in [1]. In this case all components of (X^M,P^M) are independent canonical degrees of freedom and the first class constraints are applied on the states. The constraints form the Sp$(2,R)$ algebra. The states are labelled simultaneously by the Casimir operators of Sp$(2,R)$ as well as the Casimir operators of SO$(d,2)$ since these groups commute $|C_2(\mathrm{Sp}(2,R)),C_n(\mathrm{SO}(d,2))\rangle$. We need to find their eigenvalues for physical states. The following relations are proven by writing out all the Casimir operators in terms of X,P. First, all Casimir eigenvalues $C_n(\mathrm{SO}(d,2))$ are rewritten in terms of $C_2(\mathrm{SO}(d,2))$ and d. For example $C_3(\mathrm{SO}(d,2))=(d/3!)C_2(\mathrm{SO}(d,2))$. Second, the quadratic Casimir operator of Sp$(2,R)$ is related to the quadratic Casimir operator of SO$(d,2)$. Third, since physical states are gauge invariant, the quadratic Casimir operator of Sp$(2,R)$ must vanish in the physical sector. The last condition fixes all the Casimir eigenvalues for SO$(d,2)$ to unique non-zero values in terms of d. Therefore the quantum system can exist only in a unique unitary representation of SO$(d,2)$ characterized by

$$\text{quantum:}\quad C_2(\mathrm{Sp}(2))=0,\qquad \begin{cases} C_2(\mathrm{SO}(d,2))=1-\dfrac{d^2}{4}, \\ C_3(\mathrm{SO}(d,2))=\dfrac{d}{3!}\left(1-\dfrac{d^2}{4}\right), \\ \cdots. \end{cases} \tag{65}$$

The first quantization of the theory in several one-time physics gauges (massless relativistic particle, H-atom, harmonic oscillator, and all of these with spin) has already been performed elsewhere [1–3], and the correct value of the Casimir operators (which change with spin) have been obtained, in agreement with the prediction. Hence for these diverse systems the Hilbert space corresponds to the same unique representation of SO$(d,2)$. This establishes a Sp$(2,R)$ duality among these systems at the quantum level. This may be considered a successful test of the unification in the form of two-time physics at the quantum level.

Now we consider the first quantized theory in the $\mathrm{AdS}_{d-n}\times \mathrm{S}^n$ gauge. We want to find the correct operator ordering of the generators in the quantum theory and then compute the quadratic Casimir eigenvalue. We must find that $C_2(\mathrm{SO}(d,2))=1-d^2/4$ since this is the prediction of the gauge invariant two-time physics. Confirming this result is tantamount to the presence of two time physics in the one-time quantum theory of the $\mathrm{AdS}_{d-n}\times \mathrm{S}^n$ particle, and to establishing that the Hilbert space is the same as the other cases already mentioned.

With operator ordering taken into account the quantum generators have the form

$$L^{MN}=|\mathbf{u}|^{-d/2+n+2}L_0^{MN}|\mathbf{u}|^{d/2-n-2}. \tag{66}$$

Evidently the factors of $|\mathbf{u}|^{\pm(d/2-n-2)}$ drop out in the classical theory, but they are essential for the correct symmetry generators in the quantum theory as proved in the last section [see Eq. (97)]. The (non-unitary looking) similarity transformation with the $|\mathbf{u}|^{d/2-n-2}$ will be explained in the next section. This transformation is required for Hermiticity of the generators according to a scalar product defined in Eq. (90) that is appropriate for an AdS covariant quantization scheme. The L^{MN} are Hermitian in the physical Hilbert space provided the L_0^{MN} are the following operator ordered versions of the classical generators (56)–(60):

$$L_0^{+'-'}=\frac{1}{2}(x\cdot p+p\cdot x-\mathbf{u}\cdot\mathbf{k}-\mathbf{k}\cdot\mathbf{u})+i \tag{67}$$

$$L_0^{+'\mu}=p^\mu,\quad L_0^{+'i}=\mathbf{V}^i(\mathbf{u},\mathbf{k}) \tag{68}$$

$$L_0^{-'\mu}=\frac{1}{2}x^\nu p^\mu x_\nu-\frac{1}{2}x\cdot px^\mu-\frac{1}{2}x^\mu p\cdot x-ix^\mu \tag{69}$$

$$+\frac{p^\mu}{2\mathbf{u}^2}+\frac{1}{2}(\mathbf{u}\cdot\mathbf{k}+\mathbf{k}\cdot\mathbf{u})x^\mu \tag{70}$$

$$L_0^{-'i}=\frac{1}{2}\mathbf{k}^i+\frac{x^2}{2}\mathbf{V}^i(\mathbf{u},\mathbf{k})-\frac{1}{2}(x\cdot p+p\cdot x+2i)\frac{\mathbf{u}^i}{\mathbf{u}^2} \tag{71}$$

$$L_0^{\mu\nu}=x^\mu p^\nu-x^\nu p^\mu,\quad L_0^{ij}=\mathbf{u}^i\mathbf{k}^j-\mathbf{u}^j\mathbf{k}^i \tag{72}$$

$$L_0^{\mu i}=x^\mu\mathbf{V}^i(\mathbf{u},\mathbf{k})-p^\mu\frac{\mathbf{u}^i}{\mathbf{u}^2}. \tag{73}$$

$\mathbf{V}^i(\mathbf{u},\mathbf{k})$ is the operator ordered version of the classical expression $\mathbf{u}^2\mathbf{k}^i-2\mathbf{k}\cdot\mathbf{u}\mathbf{u}^i$

$$\mathbf{V}^i(\mathbf{u},\mathbf{k})=\mathbf{u}^k\mathbf{k}^i\mathbf{u}^k-\mathbf{u}^i\mathbf{k}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{k}\mathbf{u}^i \tag{74}$$

$$=\mathbf{u}^2\mathbf{k}^i-\mathbf{u}^i(\mathbf{u}\cdot\mathbf{k}+\mathbf{k}\cdot\mathbf{u}) \tag{75}$$

$$=\mathbf{k}^i\mathbf{u}^2-(\mathbf{u}\cdot\mathbf{k}+\mathbf{k}\cdot\mathbf{u})\mathbf{u}^i. \tag{76}$$

Some of its interesting properties are

$$\left[\frac{\mathbf{u}^i}{\mathbf{u}^2},\mathbf{V}^j\right]=i\delta^{ij},\quad [\mathbf{V}^i,\mathbf{V}^j]=0,\quad \mathbf{V}^2=\mathbf{u}^2\mathbf{k}^2\mathbf{u}^2 \tag{77}$$

$$[\mathbf{k}^i,\mathbf{V}^j]=-2iL^{ij}+i\delta^{ij}(\mathbf{u}\cdot\mathbf{k}+\mathbf{k}\cdot\mathbf{u}) \tag{78}$$

$$[\mathbf{u}^i,\mathbf{V}^j]=i\delta^{ij}(\mathbf{u}^2-2\mathbf{u}^i\mathbf{u}^j),\quad [|\mathbf{u}|,\mathbf{V}^j]=-i|\mathbf{u}|\mathbf{u}^j. \tag{79}$$

These may be used to verify the closure of the algebra at the quantum level.[3] Since $|\mathbf{u}|^{\pm(d/2-n-2)}$ is a similarity transformation, the commutation relations are the same for L^{MN} or L_0^{MN} at the quantum level.

The quadratic Casimir operator may now be computed for L_0^{MN} or L^{MN}. All operators cancel and it reduces to a pure number independent of n for all $\mathrm{AdS}_{d-n}\times \mathrm{S}^n$

$$C_2(SO(d,2))=1-d^2/4. \tag{80}$$

This is the correct value imposed by the overall structure of two-time physics as given in Eq. (65). The similarity transformation of Eq. (66) cannot change the Casimir operators

$$C_n(L^{MN})=|\mathbf{u}|^{-d/2+n+2}C_n(L_0^{MN})|\mathbf{u}|^{d/2-n-2}=C_n, \tag{81}$$

since they are pure numbers.

III. FIELD THEORY IN $\mathrm{AdS}_{d-n}\times \mathrm{S}^n$ BACKGROUND

As seen from the action (53) the equation of motion for A^{22} generates the classical constraint

$$P^2=\left(\frac{p^2}{\mathbf{u}^2}+\mathbf{u}^2\mathbf{k}^2\right)=G^{mn}p_mp_n=0,\quad p_m\equiv(p_\mu,\mathbf{k}_i). \tag{82}$$

In the quantum theory the constraint is applied on the Hilbert space to find the physical states which are annihilated by it

$$(:G^{mn}p_mp_n:)|\phi\rangle=0. \tag{83}$$

The columns (:) indicate that the operator form of the constraint must first be defined by resolving ambiguities in the ordering of the operators. This must be done in such a way as to preserve the SO(d,2) symmetry of the system.

One possible ordering follows from the definition of Laplacian in general relativity. This is guaranteed to preserve the symmetries $\mathrm{SO}(d-n-1,2)\times \mathrm{S}(n+1)$ of the background $\mathrm{AdS}_{d-n}\times \mathrm{S}^n$, so it is a good starting point. In configuration space the constraint on the wave function $\phi(x^\mu,\mathbf{u}^i)=\langle x^\mu,\mathbf{u}^i|\phi\rangle$ takes the form of the Laplace equation

$$\partial_m(\sqrt{-G}G^{mn}\partial_n\phi(x^\mu,\mathbf{u}^i))=0. \tag{84}$$

This follows from the effective action $S_{eff}=\frac{1}{2}\int d^dX\sqrt{-G}(G^{mn}\partial_m\phi\partial_n\phi)$. Using

$$G_{mn}=\begin{pmatrix} u^2\eta_{\mu\nu} & 0\\ 0 & \dfrac{1}{u^2}\delta_{ij}\end{pmatrix},\quad \sqrt{-G}=u^{d-2n-2}, \tag{85}$$

$$G^{mn}=\begin{pmatrix} \dfrac{1}{u^2}\eta^{\mu\nu} & 0\\ 0 & u^2\delta^{ij}\end{pmatrix}, \tag{86}$$

we find the effective action

$$S^0_{eff}(\phi)=\frac{1}{2}\int d^{d-n-1}xd^{n+1}u(u^{d-2n-4}\partial_\mu\phi^*\partial^\mu\phi +u^{d-2n}\partial_i\phi^*\partial_i\phi) \tag{87}$$

$$=\frac{1}{2}\langle\phi|(|\mathbf{u}|^{d-2n-4}p^2+\mathbf{k}^i|\mathbf{u}|^{d-2n}\mathbf{k}^i)|\phi\rangle. \tag{88}$$

The norm of the state is not $\langle\phi|\phi\rangle=\int\phi^*\phi$, but rather it is defined by the scalar product $(\phi,\phi)=\int d\Sigma_mJ^m$, using the conserved probability current $J^m=\sqrt{-G}G^{mn}(\phi^*i\partial_n\phi-i\partial_n\phi^*\phi)$, by integrating over a spacelike surface, such as fixed time

$$(\phi,\phi)=\int_{x^0=fixed}\sqrt{-G}G^{0n}(\phi^*i\partial_n\phi-i\partial_n\phi^*\phi) \tag{89}$$

$$=\int_{x^0=fixed}(d^{d-n-2}x)(d^{n+1}\mathbf{u})|\mathbf{u}|^{d-2n-4}\times(\phi^*i\partial_0\phi-i\partial_0\phi^*\phi). \tag{90}$$

The adjoint of an operator and its Hermiticity in the physical Hilbert space must be defined relative to this scalar product. This approach defines the *AdS-covariant quantization scheme* consistent with field theory. The operators L^{MN} defined in the previous section are Hermitian according to the scalar product in this quantization scheme. This explains the reason for the similarity transformation $|\mathbf{u}|^{\pm(d/2-n-2)}$ and the other insertions of i in Eqs. (66),(67)–(73). The analog of this approach for Lorentz covariant quantization of the relativistic particle in flat space, with and without spin, was discussed in [1,3].

The Laplace equation may be written in operator form $\hat{S}_0|\phi\rangle=0$, where

$$\hat{S}_0=|\mathbf{u}|^{d-2n-4}p^2+\mathbf{k}^i|\mathbf{u}|^{d-2n}\mathbf{k}^i. \tag{91}$$

This is just the constraint condition with a particular order of operators. Thus, general covariance imposes a particular order. To check the symmetries of the effective field theory action $S_{eff}(\phi)$ for this order of operators we transform the wave function

$$\delta|\phi\rangle=-\frac{i}{2}\varepsilon_{MN}L^{MN}|\phi\rangle,\quad \delta\langle\phi|=\frac{i}{2}\varepsilon_{MN}\langle\phi|(L^{MN})^\dagger. \tag{92}$$

[3] The following change of variables $\mathbf{r}^i=\mathbf{u}^i/\mathbf{u}^2,\mathbf{p}^i=\mathbf{V}^i(\mathbf{u},\mathbf{k})$ is a canonical transformation at the quantum level. With this substitution one may notice that the generators of SO(d,2) take the same form we found in [1] for the free massless relativistic particle. Hence the computation of the Casimir operator is easily explained. This may also shed light on the overall structure of the generators, and helps explain the anomaly terms i in $L^{+'-'}$, $-ix^\mu$ in $L^{-'\mu}$, and $i\mathbf{u}^i/\mathbf{u}^2$ in $L_0^{-'i}$, as due to Hermiticity in Lorentz covariant quantization in flat space. The AdS covariant quantization introduces the further anomaly terms generated by the similarity transformation $|\mathbf{u}|^{\pm(d/2-n-2)}$ given in Eq. (66).

Then the transformation of the action $\delta S^0_{eff}(\phi)$ can be written in the form

$$\delta S^0_{eff}(\phi)=\frac{i}{2}\varepsilon_{MN}\langle\phi|[(L^{MN})^\dagger\hat{S}_0-\hat{S}_0L^{MN}]|\phi\rangle. \quad (93)$$

Note that $(L^{MN})^\dagger$ is the naive Hermitian conjugation[4] (using $x,p,\mathbf{u},\mathbf{k}$ that are naively Hermitian). Now we can verify that the generators L^{ij} and $(L^{+'-'},L^{+'\mu},L^{-'\mu},L^{\mu\nu})$ are indeed symmetries of the effective action as expected in the General Relativity formalism. Indeed we find explicitly $\delta S^0_{eff}(\phi)=0$, confirming the $\mathrm{SO}(d-n-1,2)\times\mathrm{SO}(n+1)$ invariance of the action and of the quantization procedure.

Next we turn to the remaining generators of $\mathrm{SO}(d,2)$, $L^{+'i},L^{-'i},L^{\mu i}$. We find that the ordering of operators given by $\hat{S}_0$ introduces quantum anomalies that break the bigger symmetry $\mathrm{SO}(d,2)$ for the generic $\mathrm{AdS}_{d-n}\times \mathrm{S}^n$ background in field theory. There are exceptions for the cases $\mathrm{AdS}_2\times\mathrm{S}^0$ and $\mathrm{AdS}_n\times\mathrm{S}^n$ (i.e., $d=2n$) for which the anomaly is zero and the full symmetry is active. On the other hand it is also possible to improve the effective action by adding the following anomaly term to the action in such a way as to preserve the full $\mathrm{SO}(d,2)$ for all d,n

$$\hat{S}=\hat{S}_0-\frac{1}{4}(d-2)(d-2n)|\mathbf{u}|^{d-2n-2}, \quad (94)$$

$$S_{eff}(\phi)=\frac{1}{2}\langle\phi|\hat{S}|\phi\rangle=S^0_{eff}(\phi)+S^1_{eff}(\phi). \quad (95)$$

The anomaly $S^1_{eff}(\phi)$ results from a different ordering of the operators and may be seen as a potential term (no momenta) added on to the kinetic term defined by general relativity. Actually it is just a mass term in the field theory formalism, $S^1_{eff}(\phi)=-\frac{1}{2}m^2\int\sqrt{-G}\,\phi^*\phi$, with the quantized mass (in units of the S^n radius^{-2} that was set to $R=1$)

$$m^2=\frac{1}{4}(d-2)(d-2n). \quad (96)$$

Of course, the mass term is invariant separately under the subgroup $\mathrm{SO}(n+1)\times\mathrm{SO}(d-n-1,2)$. On the other hand, the total action is invariant under the full $\mathrm{SO}(d,2)$ thanks to the relations

$$(L^{MN})^\dagger\hat{S}-\hat{S}L^{MN}=0 \quad (97)$$

that are satisfied just for the special value of the mass, and the precise ordering of operators in L^{MN} as given in Eqs. (66),(67)–(73). For the special cases $\mathrm{AdS}_2\times\mathrm{S}^0$ and $\mathrm{AdS}_n\times\mathrm{S}^n$ (i.e., $d-n=n$) the mass vanishes.

In recent literature the cases of $\mathrm{AdS}_2,\mathrm{AdS}_3\times\mathrm{S}^3$ and $\mathrm{AdS}_5\times\mathrm{S}^5$ have been investigated in the context of the AdS-CFT correspondance [6–9]. These are among the cases that, according to our results, have higher symmetries SO(2,2), SO(6,2) and SO(10,2) respectively, with vanishing mass term. The higher symmetry may be of interest in future investigations.

We have shown that in order to be consistent with two-time physics the quantum theory must be carefully constructed. The formalism sets constraints that are non-trivial. One of the signatures of two-time physics is the $\mathrm{SO}(d,2)$ symmetry realized in a unique unitary representation with special values of the Casimir operators. This is a unifying aspect since it connects diverse one-time physics systems in the same quantum representation of $\mathrm{SO}(d,2)$. Furthermore, our work establishes a quantum duality for $\mathrm{AdS}_{d-n}\times\mathrm{S}^n$ for all n among themselves, as well as with all other one-time physics systems that we derived before [1–4] from the same action.

[4]Because of the i insertions and the factors of $|\mathbf{u}|^{\pm(d/2-n-2)}$ in Eqs. (66),(67)–(73) $(L^{MN})^\dagger$ may not be equal to L^{MN}. This is of no concern since the L^{MN} are Hermitian in the correct sense defined above, not in the naive sense.

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