Simplifying algebra in Feynman graphs. II. Spinor helicity from the spacecone

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Manifestly Lorentz-covariant Feynman rules are given in terms of a ''scalar'' field for each helicity, dramatically simplifying the calculation of amplitudes with massless particles. The spinor helicity formalism is properly identified as a null complex spacelike (not lightlike) gauge, where two massless external momenta define the reference frame. Usually, this gauge is applied only to external line factors; we extend this method to vertices and propagators by modifying the action itself using light-cone methods. [S0556-2821(98)05324-7]

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I. INTRODUCTION

In other papers in this series [1], we describe a method to simplify Feynman diagram calculations for massive fields, based on the observation that fields with only undotted spinor indices formally have the same structure as nonrelativistic fields, since a Lorentz transformation on such fields is just a complex rotation. Thus spin s is described by a field with $2s+1$ components in a manifestly covariant way: spin 1/2 by a chiral spinor, spin 1 by a self-dual tensor, without their complex conjugates. The result is a unitary gauge where all propagators are simply $1/(p^2+m^2)$. Also, external line factors are unconstrained (i.e., arbitrary); so amplitudes can be completely evaluated covariantly (rather than leaving covariance for the more numerous squares of terms in probabilities). Many supersymmetry relations are already obvious from the graphs, since actions for spins 1/2 and 1 more closely resemble those for spin 0. Much repetitive algebra ordinarily applied in graphs is already done once and for all in the action itself (so the action has more vertices, such as seagulls for spinors, but the amplitude has fewer terms). Since the fields are chiral and self-dual, this method is particularly suited to graphs with maximal helicity violation (MHV), which are those that have the simplest final form.

Although this method can be applied to massless theories by an appropriate limiting procedure (which is trivial for spin 1/2), it is inherently tailored to massive theories. In particular, the concept of using only the physical degrees of freedom suggests the further reduction from $2s+1$ components for the massive case to just 2 for the massless case. In this paper we accomplish this task by combining two well-known methods, the light-cone formalism [2] and twistors [3] (the spinor helicity formalism [4] for external line factors).

The light-cone formalism is defined by choosing a fixed lightlike direction (''$-$''), choosing the gauge where the corresponding lightlike components of gauge fields vanish, and eliminating all auxiliary degrees of freedom (''$+$'') by their nonpropagating (no ''$+$'' derivatives) equations of motion. This last step is the most important one: It reduces all nonzero spins to two degrees of freedom (corresponding to the two helicities); using four-vectors to describe light-cone Yang-Mills theory would miss the whole point of the light-cone approach. This gauge is manifestly unitary, and all propagators are $1/p^2$ because each field has a single component (complex to describe two helicities). The main disadvantage of the light cone is that Lorentz invariance is not manifest: Although the action is Lorentz invariant, Lorentz transformations are nonlinear in the fields (and momenta) and the Feynman rules are asymmetric in the various components of the momenta. An especially simple case is self-dual Yang-Mills theory [5], whose Feynman rules [6] are manifestly covariant under half the Lorentz group [in 2+2 dimensions, one of the two SL(2)'s of SO(2,2)=SL(2)⊗SL(2)]. In particular, the self-dual rules are much simpler than the usual ones for deriving MHV amplitudes.

The spinor helicity method translates all external line factors into twistor notation, essentially using the Penrose transform for free fields. Each massless momentum is expressed as the square of a momentum twistor, while the external line factor (i.e., the external free field) is expressed in terms of that and (for gauge fields) an arbitrary gauge-dependent polarization twistor. If the polarization twistors for all external lines are chosen to point in the same fixed direction, then the gauge is a light-cone gauge, but only for the external fields (and not the internal lines). However, in practice, one chooses to equate the polarization twistors with some of the momentum twistors of the other lines; so the amplitude is expressed completely in terms of momentum twistors (and the probability completely in terms of momenta) [4]. Unlike the light-cone formalism, with this approach not only the vertices but even the external line factors are manifestly covariant.

On the other hand, spinor helicity modifies only the external line factors and does not simplify the vertices. The result is that, although the final expression for MHV amplitudes is extremely simple [7], their calculation does not fully reflect this simplicity. Explicitly, the first two steps in a spinor helicity calculation are the following: (1) Write down all the terms that come from the various graphs, using the usual Feynman rules in color-ordered form [4]. Some simplification comes from using the Gervais-Neveu gauge, which gives only three terms for each three-point vertex and one term for each four-point (for pure Yang-Mills theory). (2) Contract all the vector indices (and spinor ones for

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quarks), coming from vertices, propagator numerators, and external line factors. A remarkable feature of spinor helicity is the large number of terms that vanish in the second step: Many vanish immediately, from the direct effect the external-line factors have on the vertices, but most vanish only after more careful inspection, when polarization vectors are contracted from different parts of the graph via the Kronecker deltas in the propagators. An equally remarkable feature of spinor helicity is that one has to spend a considerable fraction of the calculation writing down these terms that do not contribute. Generally, any method that requires calculation to produce zeros indicates a simpler method is missing for which the zeros would be automatic (e.g., supergraphs for the ''miraculous cancellations'' of supersymmetry).

In this paper we derive new Feynman rules that avoid the second step above altogether by eliminating Lorentz indices on all massless fields, while retaining all the advantages of spinor helicity. This is accomplished by modifying the usual light-cone method to incorporate the principles of the spinor helicity method: (1) Although we choose arbitrary fixed vectors to define the reference frame used to derive the light-cone action, in the amplitudes we choose to identify the two lightlike axes with two of the physical external (''reference'') momenta (as allowed by Lorentz and gauge invariance), which also defines the other two (spacelike) axes, using momentum twistors. The result is then manifestly covariant, since explicit Lorentz components are replaced by Lorentz invariants. On external lines, this is equivalent to spinor helicity, but we have extended the method to internal lines. (2) As a result, the direction chosen to define the gauge is not lightlike, but spacelike and complex, although it is still null. The gauge condition is thus complex, like the Gervais-Neveu gauge, although it is linear. (This distinction does not exist in $2+2$ dimensions, but the Wick rotation to $3+1$ is different.) We thus refer to our formalism as the ''space cone'' (although, as for ''light cone,'' no cone is actually involved; it is an abbreviation for ''complex spacelike null hyperplane''). (3) In the usual light-cone approach, where a fixed component of the gauge field vanishes, one takes care to avoid the vanishing of the corresponding lightlike component of the momentum, which would produce singularities. In our approach its vanishing is *required* by definition of the two reference lines. We show that the external line factors cancel such singularities and find that the resulting vertices for those two lines are much simpler than the rest. This accounts for much of the simplification of the spinor helicity formalism. These external-line factors follow from the covariant ones used in the usual spinor helicity approach and are not the trivial ones normally used in the light-cone approach, although they are still simple. (The field redefinition that relates the two is singular for the reference lines.)

The main advantage of our approach over the usual spinor helicity is that there are no Lorentz indices associated with any lines, although they have an orientation ($+$ helicity at one end, $-$ at the other); all indices show up at the final step, when Lorentz invariants are expressed in terms of momentum twistors. In particular, this means that vanishing graphs and terms in graphs, which in the spinor helicity approach would be seen by expanding the vertices and performing the (vector and spinor) index algebra, are avoided in the space-cone approach without performing any algebra whatsoever.

In the following section we review the basic points of the light-cone formalism, including special features of four dimensions, such as simplification of the interactions and self-duality. In the next section we review twistors and spinor helicity. The space cone is introduced in Sec. IV. Examples are given in Sec. V to illustrate the improvement over previous techniques. Recursion relations are discussed in Sec. VI. The final section contains our conclusions.

II. SHORT REVIEW OF THE LIGHT CONE

We start with the Lagrangian for Yang-Mills theory, in a convenient normalization ($S=\mathrm{tr}L/g^2$):

$$L=\tfrac{1}{8}F^2, \quad F^{ab}=\partial^{[a}A^{b]}+i[A^a,A^b], \tag{1}$$

where $\partial^{[a}A^{b]}=\partial^a A^b-\partial^b A^a$ and the indices a,b are vector indices. (Throughout this paper we will use pure Yang-Mills theory as our standard example, with straightforward generalization to other spins.) We use uppercase letters to denote vectors and lowercase to denote their (contravariant) light-cone components, as

$$A^a=(a,\overline{a},a^+,a^-). \tag{2}$$

In Minkowski space $\overline{a}$ is the complex conjugate of a, while $a^\pm$ are real. Defining our light-cone frame by the Minkowksi-space inner product

$$A\cdot B=a\overline{b}+\overline{a}b-a^+b^--a^-b^+, \tag{3}$$

we choose the light-cone gauge

$$a^-=0. \tag{4}$$

(In the original description of the light-cone gauge, the formalism was described by an ''infinite-momentum frame.'' Of course, Lorentz invariance means that the equations are frame independent, and shortly thereafter it was realized that the infinite-momentum limit was a misconception and could be replaced with a simple change of variables. Even the parton model was originally described in this language, until it was realized that it was only hiding the Lorentz-invariant statement that the one physical assumption of the parton model is a transverse momentum cutoff, which is realized dynamically in QCD via asymptotic freedom.)

The equation of motion for a^+ now contains no ''time'' derivatives ∂^+; so we use this equation to eliminate this ''auxiliary'' component from the Lagrangian: After a little algebra, we find

$$L=\overline{a}\partial^+\partial^-a+\tfrac{1}{4}(F^{-+})^2-\tfrac{1}{4}(F^{t\overline{t}})^2, \tag{5}$$

$$F^{-+}=\partial\overline{a}+\overline{\partial}a+i\,\frac{1}{\partial^-}\left([a,\partial^-\overline{a}]+[\overline{a},\partial^-a]\right),$$
$$F^{t\overline{t}}=\partial\overline{a}-\overline{\partial}a+i[a,\overline{a}]. \tag{6}$$

Only in four dimensions can we simplify the Lagrangian by using the self-dual and anti-self-dual combinations (which would have been indicated if we had used spinor notation):

$$\mathcal{F}=\frac{1}{2}(F^{-+}+\overline{F^{tt}})=\partial\overline{a}+i\frac{1}{\partial^-}[a,\partial^-\overline{a}],$$

$$\overline{\mathcal{F}}=\frac{1}{2}(F^{-+}-\overline{F^{tt}})=\overline{\partial}a+i\frac{1}{\partial^-}[\overline{a},\partial^- a],$$

$$L=\overline{a}\partial^+\partial^- a+\mathcal{F}\overline{\mathcal{F}}$$

$$=-\frac{1}{2}\overline{a}\Box a-i\left(\frac{\overline{\partial}}{\partial^-}a\right)[a,\partial^-\overline{a}]$$

$$-i\left(\frac{\partial}{\partial^-}\overline{a}\right)[\overline{a},\partial^- a]+[a,\partial^-\overline{a}]\frac{1}{(\partial^-)^2}[\overline{a},\partial^- a].$$

$$\frac{1}{2}\Box=\partial\overline{\partial}-\partial^+\partial^-. \tag{7}$$

The Feynman rules may be obtained in a straightforward fashion; for example, the color-ordered three-point vertex contains only two terms and is thus simpler than usual gauges, such as Fermi-Feynman or Gervais-Neveu.

Translation into van der Waerden notation for Weyl spinors is easy: In terms of our light-cone basis, we have the Hermitian 2×2 matrix

$$A^{\alpha\dot{\beta}}=\begin{pmatrix} a^+ & \overline{a} \\ a & a^- \end{pmatrix},\quad A^2=-2\det A. \tag{8}$$

Spinor indices are raised and lowered with the antisymmetric symbol

$$C_{\alpha\beta}=C_{\dot{\alpha}\dot{\beta}}=\text{-}C^{\alpha\beta}=-C^{\dot{\alpha}\dot{\beta}}=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{9}$$

$$\psi^\alpha=C^{\alpha\beta}\psi_\beta,\quad \psi_\alpha=\psi^\beta C_{\beta\alpha},\quad \overline{\psi}^{\dot{\alpha}}=(\psi^\alpha)^\dagger=C^{\dot{\alpha}\dot{\beta}}\overline{\psi}_{\dot{\beta}} \tag{10}$$

$$A\cdot B=A^{\alpha\dot{\beta}}B_{\alpha\dot{\beta}}. \tag{11}$$

Note that each of the four light-cone–spinor-notation components is defined with respect to a null direction: Two are real and lightlike $(\hat{t}\pm\hat{x})$, two are complex and spacelike $(\hat{y}\pm i\hat{z})$. (In $2+2$ dimensions all four are real and lightlike, while in $4+0$ dimensions all are complex and spacelike.)

Spinors can be treated similarly: There is no gauge condition for spin 1/2, but half of the field is auxiliary. Now spinor notation is necessary, and in the light cone a Weyl spinor reduces to a single complex component, as did the vector.

Self-dual Yang-Mills theory also is described more easily in spinor notation, where the field strength naturally separates into self-dual and anti-self-dual parts:

$$[\nabla^{\alpha\dot{\beta}},\nabla^{\gamma\dot{\delta}}]=i(C^{\alpha\gamma}f^{\dot{\beta}\dot{\delta}}+C^{\dot{\beta}\dot{\delta}}f^{\alpha\gamma}). \tag{12}$$

Self-duality $f^{\alpha\beta}=0$ is then partially solved in the light-cone gauge by

$$f^{--}=0\Rightarrow[\nabla^{-\dot{\beta}},\nabla^{-\dot{\delta}}]=0\Rightarrow A^{-\dot{\beta}}=0\,(a^-=a=0), \tag{13}$$

$$f^{-+}=0\Rightarrow\partial^{-[\dot{\beta}}A^{+\dot{\delta}]}=0\Rightarrow A^{+\dot{\beta}}=\partial^{-\dot{\beta}}\phi\,(a^+=\partial\phi,\ \overline{a}=\partial^-\phi). \tag{14}$$

The remaining equation $f^{++}=0$ is identical to that found from the above light-cone action by varying with respect to a and then setting $a=0$: It is the ordinary Yang-Mills field equation restricted to one helicity. If we define self-dual Yang-Mills theory by the Lagrangian $G_{\alpha\beta}f^{\alpha\beta}$, then our light-cone analysis leads directly to the first two terms of the above light-cone action, with $(\partial^-)^{-1}G_{++}$ acting as a replacement for a.

III. TWISTORS AND SPINOR HELICITY

The basic principle of the twistor approach is that a lightlike four-vector can be written as the square of a commuting spinor (‘‘twistor’’):

$$P^2=0\Rightarrow P^{\alpha\dot{\beta}}=\pm p^\alpha\overline{p}^{\dot{\beta}}, \tag{15}$$

where the sign depends on whether the vector is forward or backward (with respect to the time direction): In practice, we ignore this sign, as justified by crossing symmetry or by Wick rotation from $2+2$ dimensions, where p^α and $\overline{p}^{\dot{\alpha}}$ are independent and real (rather than complex conjugates); so the sign can be absorbed into either. This representation is particularly useful for free, massless particles: Using this twistor to describe the momentum, the usual free field equations can be solved for the free field strengths as

$$f^{\alpha_1\cdots\alpha_{2h}}=p^{\alpha_1}\cdots p^{\alpha_{2h}}\varphi_h \tag{16}$$

for a spinor f^α, vector field strength $f^{\alpha\beta}$, Weyl tensor $f^{\alpha\beta\gamma\delta}$, etc. Thus massless degrees of freedom are immediately reduced to (complex) scalars. Actually, Lorentz transformations are reduced to their little group, helicity, which transforms the twistor by a phase (independently of the usual Lorentz) and, thus, the ‘‘scalar’’ by the helicity h times the phase.

An old lesson learned from supersymmetry is that all spinor algebra in four dimensions is performed most conveniently with two-component spinor indices exclusively, and this rule also gives the simplest calculations in the spinor helicity formalism. This is much simpler than algebra with four-component (Dirac) spinors. In particular, Fierz identities are avoided. This also means one should avoid all γ and σ matrices, since they are nothing more than Clebsch-Gordan coefficients. Vectors are described only as objects with 2 two-component indices.

Since almost all spinor algebra for spins $\leqslant 1$ involves objects carrying at most two spinor indices (spinors, vectors, self-dual tensors), for such purposes it is usually convenient to use matrix notation, defined by

$$\langle p| = p^{\alpha}, \quad |p\rangle = p_{\alpha}, \quad [p| = p^{\dot\alpha}, |p] = p_{\dot\alpha} \tag{17}$$

As a result, we also have

$$\langle pq\rangle = p^{\alpha}q_{\alpha}, \quad [pq] = p^{\dot\alpha}q_{\dot\alpha}, \quad \langle pq\rangle^{*} = [pq] = -[pq], \tag{18}$$

$$P = P_{\alpha}{}^{\dot\beta}, \quad P^{*} = -P_{\dot\alpha}{}^{\beta}, \quad \langle k|P|q] = k^{\alpha}P_{\alpha}{}^{\dot\beta}q_{\dot\beta},$$
$$PK^{*} + KP^{*} = (P\cdot K)I, \tag{19}$$

$$f = f_{\alpha}{}^{\beta}, \quad f^{*} = f_{\dot\alpha}{}^{\dot\beta}, \quad \langle p|f|q\rangle = p^{a}f_{\alpha}{}^{\beta}q_{\beta}, \tag{20}$$

$$\langle pq\rangle\langle rs\rangle + \langle qr\rangle\langle ps\rangle + \langle rp\rangle\langle qs\rangle = 0, \tag{21}$$

where the last (''Schouten'') identity is the result of antisymmetrizing three indices that take only two values. (Note that $\langle pq\rangle = -\langle qp\rangle$, $[pq] = -[qp]$ are consequences of using twistors; when using physical, anticommuting spinors, $\langle\psi\chi\rangle = +\langle\chi\psi\rangle$, and $\langle\psi\psi\rangle \neq 0$ occurs in mass terms.)

Although gauge-invariant objects can be written directly in terms of momentum twistors and scalars, the same is not true for gauge fields. Thus for a gauge vector we write (in matrix notation)

$$A = \frac{|\epsilon\rangle[p|}{\langle\epsilon p\rangle}\,\bar\varphi \Rightarrow f^{*} = |p[p|\bar\varphi, \quad f = 0, \tag{22}$$

for + helicity, or

$$A = \frac{|p\rangle[\epsilon|}{[\epsilon p]}\,\varphi \Rightarrow f = |p\langle p|\varphi, \quad f^{*} = 0, \tag{23}$$

for −. (Positive helicity is the same as self-duality; negative is anti-self-dual.) The former agrees with the self-dual Yang-Mills result of the previous section, up to normalization, if we choose

$$\epsilon^{\alpha} \sim \delta^{\alpha}_{+}. \tag{24}$$

These expressions are used for external lines in Feynman diagrams: Setting scalar fields $\varphi = 1$ as usual, the factors multiplying φ and $\bar\varphi$ are identified as the external line factors for a massless vector. Thus the twistor formalism is essentially a covariant way of writing the axial gauge

$$N\cdot A = 0, \quad N = |\epsilon\rangle[\epsilon| \tag{25}$$

in terms of an arbitrary lightlike vector N.

However, a major simplification is achieved in applying spinor helicity methods to explicit evaluation of Feynman diagrams: Instead of choosing the lightlike vectors N to be the same on each external line, as in a light-cone gauge, they are chosen to vary from line to line (i.e., to be momentum dependent). Furthermore, these lightlike vectors, rather than being arbitrary, are identified with some of the external lightlike (massless) momenta (although, of course, no N is identified with the momentum of the same line). One result is that all amplitudes are manifestly Lorentz invariant, since they are expressed completely in terms of the invariant products of momentum twistors (and no other Lorentz structures).

IV. SPACE CONE

Although these principles should be enough to appreciate the advantages of spinor helicity methods, there is one additional rule that, although not required, always gives the simplest results in practice: All the N's are chosen the same for one sign of the helicity and all the same for the other sign. This means that all the e's (+ helicity) are the same undotted spinor $|+\rangle$ and all the $\bar\epsilon$'s (− helicity) are the same dotted spinor $|-]$. However, because no N can be identified with a P of the same line, this means that the ϵ's are identified with a $P = |+\rangle[+|$ of a line with the opposite sign helicity (−) and all $\bar\epsilon$'s with a $P = |-\rangle[-|$ of a line with + helicity. In particular, this means that the one $\bar\epsilon$ used on all such lines is not the complex conjugate of the one ϵ used on all the other lines. The naive way to interpret this would be to say that all the lines of positive helicity are in one light-cone gauge and all the lines of negative helicity are in a different light-cone gauge. However, a much simpler way to interpret this is to say that *all* lines are in the same gauge, defined by the vector

$$N = |+\rangle[-|, \quad N\cdot A = 0. \tag{26}$$

This N is complex; it is also spacelike, since $\langle pp\rangle = 0$. So it is orthogonal to the two lightlike vectors (external momenta) $|\pm\rangle[\pm|$, as well as being null. Thus we have a complex spacelike (axial) gauge.

Repeating the derivation of the light-cone action with the gauge

$$a = 0, \tag{27}$$

eliminating $\bar a$ by its equation of motion, we now have

$$L = \frac{1}{2}\,a^{+}\Box a^{-} - i\left(\frac{\partial^{-}}{\partial}\,a^{+}\right)[a^{+},\partial a^{(-)}]$$
$$-i\left(\frac{\partial^{+}}{\partial}\,a^{-}\right)[a^{-},\partial a^{+}] + [a^{+},\partial a^{-}]\,\frac{1}{\partial^{2}}\,[a^{-},\partial a^{+}]. \tag{28}$$

The fact that the three vertices come in helicities $++-$, $--+$, and $++--$ will prove important later when writing Feynman graphs, since now the $\pm$ index on the field labels its helicity (as seen, e.g., from the external-line factors of the previous section with our present choice of ϵ and $\bar\epsilon$). This action is complex because of the appearance of ∂ without $\bar\partial$. Unlike the Euclidean formulation of light-cone Yang-Mills theory given in Eq. (7), it is not necessary to specify the integration contours in individual Feynman diagrams around the spurious poles introduced by the $1/\partial^{-}$ powers in Eq. (7). The space-cone gauge is complex, and as a result, there are no singularities introduced as a result of the inverse powers of ∂ in the vertices [alternatively, one may simply shift the $i\epsilon$ out of the denominator because in momentum space k and $\bar k$ are complex, as in Eq. (31)].

In spinor notation in matrix form, we use the basis $|+\rangle[+|$, $|-\rangle[-|$, $|-\rangle[+|$, $|+\rangle[-|$:

$$P = p^{+}|+\rangle[+| + p^{-}|-\rangle[-| + p|-\rangle[+| + \overline{p}|+\rangle[-|, \tag{29}$$

with the normalization

$$\langle +-\rangle = [-+] = 1; \tag{30}$$

so we can write, e.g., for massless momentum $P = |p\rangle[p|$,

$$p^{+} = \langle p-\rangle[-p], \quad p^{-} = \langle +p\rangle[p+],$$

$$p = \langle +p\rangle[-p], \quad \overline{p} = \langle p-\rangle[p+]. \tag{31}$$

[We can then identify $\pm$ as labeling rows and columns in 2×2 matrix notation; with our normalization, $\langle +| = \delta_{+}^{\alpha}$, but $\langle -| = -i\delta_{-}^{\alpha}$; so this leads to the harmless redefinition $A^{\alpha\dot{\beta}} = \begin{pmatrix} a^{+} & i\overline{a} \\ -ia & a^{-} \end{pmatrix}$.]

This is as much as we can do in the action; however, when we write down an explicit amplitude, we identify $|+\rangle[+|$ with the momentum of one line with negative helicity (vector or, more generally, spinor) and $|-\rangle[-|$ with that of a line of positive helicity. (Alternatively, we can take one functional derivative each with respect to a^{+} and a^{-} of the *S*-matrix generating functional, to get a propagator in a background, and define the functional integral in terms of the momenta associated with the ends of the propagator. Then the fields in the action in this functional integral become true scalars.) This defines $|\pm\rangle$ up to phases, and thus $|-\rangle[+|$ and $|+\rangle[-|$, on which the phase transformation is a rotation in the plane orthogonal to the two momenta. Thus our choice of the phase $\langle +-\rangle/[-+] = 1$ is a further specification of these phases, while our choice of the magnitude $\langle +-\rangle[-+] = -\langle +|[+|\cdot|-\rangle|-] = 1$ is a choice of (mass) units. In explicit calculations, we restore generality (in particular, to allow momentum integration) by inserting appropriate powers of $\langle +-\rangle$ and $[-+]$ at the end of the calculations, as determined by simple dimensional and helicity analysis. (This avoids a clutter of normalization factors $\sqrt{\langle +-\rangle[-+]}$ at intermediate stages.) For example, looking at the form of the usual spinor helicity external-line factors and counting momenta in the usual Feynman rules, we see that any tree amplitude (or individual graph) in pure Yang-Mills theory must go as

$$\langle\ \rangle^{2-E_{+}}[\ ]^{2-E_{-}}, \tag{32}$$

where $E_{\pm}$ is the number of external lines with helicity $\pm$.

We now return to external-line factors. The naive factors for the above Lagrangian are 1, since the kinetic term resembles that of a scalar. However, this would lead to unusual normalization factors in probabilities, which are not obvious in this complex gauge. Therefore, we determine external-line factors from the earlier spinor helicity expressions for external four-vectors:

$$(\epsilon_{+})^{+} = -\langle -|[-|\cdot\ |+\rangle|p]\langle +p\rangle = \frac{[-p]}{\langle +p\rangle}, \tag{33}$$

$$(\epsilon_{-})^{-} = -\langle +|[+|\cdot\ |p\rangle|-][-p] = \frac{\langle +p\rangle}{[-p]}. \tag{34}$$

Note that these factors are inverses of each other, consistent with leaving invariant (the inner product defined by) the kinetic term.

An exception is the external-line factors for the reference momenta themselves, where $|p\rangle = |\mp\rangle$ for helicity $\pm$ gives vanishing results. However, examination of the Lagrangian shows that this zero can be canceled by a $1/\partial$ in a vertex, since $p = \overline{p} = 0$ for the reference momenta by definition. (Such cancellations occur automatically from field redefinitions in the light-cone formulation of the self-dual theory.) The actual expressions we want to evaluate, before choosing the reference lines, are then

$$\frac{p^{-}}{p}(\epsilon_{+})^{+} = \frac{\langle +p\rangle[p+]}{\langle +p\rangle[-p]}\frac{[-p]}{\langle +p\rangle} = \frac{[p+]}{\langle +p\rangle}, \tag{35}$$

$$\frac{p^{+}}{p}(\epsilon_{-})^{-} = \frac{\langle p-\rangle[-p]}{\langle +p\rangle[-p]}\frac{\langle +p\rangle}{[-p]} = \frac{\langle p-\rangle}{[-p]}. \tag{36}$$

Evaluating the former at $|p\rangle = |-\rangle$ and the latter at $|p\rangle = |+\rangle$, we get 1 in both cases. In summary, for reference lines (1) use only the three-point vertex of the corresponding self-duality ($\pm\ \pm\ \mp$ for helicity $\pm$), and use only the term associating the singular factor with the reference line (the other term and the other vertices give vanishing contributions); (2) including the momentum factors on that line from the vertex, the external line factor is 1.

Obviously, the uniqueness of the vertex term for a reference line is a considerable additional simplification for the rules. As examples, consider the amplitudes in pure Yang-Mills theory that are known to vanish by supersymmetry [8]: By simple counting of $+$'s and $-$'s, we see that the tree graphs with the fewest external $-$'s, those with only self-dual vertices ($++-$), have a single external $-$. Thus the all $+$ amplitude vanishes automatically. Furthermore, the diagrams with a single external $-$ must have that line chosen as one of the reference lines. However, by the above rules that line can carry only the *anti*-self-dual vertex ($--+$); so those amplitudes also vanish.

The vanishing of these amplitudes is also easy to see in the usual spinor helicity formalism: In the space-cone gauge, the only nonvanishing inner products of polarization vectors are between those of opposite helicity, neither of which can be a reference line. Since by momentum counting a tree graph in pure Yang-Mills theory must have at least one such product (as opposed to momentum times polarization), nonvanishing graphs must have at least two external lines of each helicity (including reference lines) [4]. The amazing thing about our space-cone approach is that our corresponding arguments of the previous paragraph make no reference to momenta or inner products; this extends to the avoidance of vanishing terms in nonvanishing amplitudes, where the usual spinor helicity methods require explicit index algebra.

The vanishing amplitudes can also be seen from supersymmetry. On the other hand, supersymmetry is not required on the space cone, because it is already built in: The half of the supersymmetry transformations actually used are trivial on the space cone (or light cone), reflecting the fact that the terms in the Lagrangian are identical for different helicities except for the placement of factors of the transverse momentum component p (or, in the usual light-cone formalism, the longitudinal component p^+). In fact, in the self-dual theory, because of field redefinitions, the terms are identical without exception.

V. MORE EXAMPLES

The simplest nonvanishing amplitude is $++--$. We use color ordering; i.e., we examine only planar diagrams for each permutation of external lines. We consider the case where the helicities are cyclically ordered as $++--$; we label them 1234, and choose 1 and 4 as the reference lines; this amplitude can be denoted as $\oplus+-\ominus$. ($P_4=|+\rangle[+]$, $P_1=|-\rangle[-|$: The positive-helicity reference line gives the reference momentum for negative helicity and vice versa.) There are only three diagrams; however, the $+$ reference line uses only the $++-$ vertex, while the $-$ reference line uses only the $--+$ vertex, and so the four-point-vertex diagram vanishes, as does the diagram with both reference lines at the same vertex. Thus we are left with only one graph. Furthermore, we know that the three-point vertices contribute only one term to the reference line; so this graph has only one term. This means we can immediately write down the answer:

$$\begin{aligned}A_{\mathrm{tree}}(++--)&=\epsilon_{2+}^{+}\epsilon_{3-}^{-}p_2p_3\,\frac{1}{\frac{1}{2}(P_3+P_4)^2}\\&=\frac{[-2]}{\langle+2\rangle}\frac{\langle+3\rangle}{[-3]}\langle+2\rangle[-2]\langle+3\rangle[-3]\\&\quad\times\frac{1}{\langle 34\rangle[34]}\frac{1}{\langle+-\rangle[-+]}\\&=\frac{[12]^2\langle 34\rangle}{[34][41]\langle 14\rangle},\end{aligned}\tag{37}$$

where we have restored helicity and dimensions. Using the identities following from overall momentum conservation,

$$(P_1+P_4)^2=(P_2+P_3)^2\Rightarrow[41]\langle 14\rangle=[23]\langle 32\rangle,\tag{38}$$

$$\sum|p\rangle[p|=0\Rightarrow\langle 34\rangle[14]=-\langle 32\rangle[12],\tag{39}$$

this can be put in the standard form

$$A_{\mathrm{tree}}(++--)=\frac{[12]^4}{[12][23][34][41]}.\tag{40}$$

(Similar manipulations cast it into the form

$$\langle 34\rangle^4/\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle.)$$

By comparison, the Fermi-Feynman gauge would produce 75 terms for this calculation and the Gervais-Neveu gauge 19 terms. The usual spinor helicity methods eventually reduce this to the same one graph and one term, but only after examining the vector products among polarization vectors and momenta. Although relatively simple in this example (the easiest), tracking indices across diagrams (through propagators) can be tedious in general. The space cone eliminates this index algebra, since the fields are effectively scalars. Similar remarks apply to amplitudes involving fermions, where spinor algebra is involved when spinor helicity is applied to the usual Feynman rules, but no such algebra appears on the space cone. Of course, the final result will always contain Lorentz products, and some manipulation may be used for further simplification, but no algebra whatsoever is required on the space cone to eliminate vanishing graphs or terms.

A more complicated example is the $+++--$ amplitude. Again, taking color-ordered (planar) amplitudes, we choose the amplitude cyclically ordered as $+++--$ with lines labeled 12345, picking 1 and 5 as the reference lines, which we denote as $\oplus++-\ominus$. Again dropping all graphs with a reference line at a four-point vertex or two references lines at a three-point vertex, all five graphs with a four-point vertex are destroyed, and only three of the remaining five survive. Since three-point vertices with (without) a reference line have one (two) terms, we are left with only six terms. (We also need to consider various combinations of $+$ and $-$ indices, but only one survives for each graph because of the chirality of three-vertices with reference lines.) The initial result for the amplitude is then

$$\begin{aligned}&A_{\mathrm{tree}}(+++--)\\&=\epsilon_{2+}\epsilon_{3+}\epsilon_{4-}\left[\frac{p_2p_4^2\left(\frac{p_2^-+1}{p_2}-\frac{p_3^-}{p_3}\right)}{(P_1\cdot P_2)(P_4\cdot P_5)}-\frac{p_3^4\left(\frac{p_2^-}{p_2}-\frac{p_3^-}{p_3}\right)}{(P_2\cdot P_3)(P_4\cdot P_5)}\right.\\&\quad\left.-\frac{p_2^2p_4\left(\frac{p_2^-+1}{p_2}-\frac{p_3^-}{p_3}\right)}{(P_1\cdot P_2)(P_3\cdot P_4)}\right],\end{aligned}\tag{41}$$

where we have used the fact that the reference lines have trivial momenta: 1 for the component with $\pm$ index opposite to its helicity, 0 for the remaining components. The two terms for each diagram simplify to 1 using

$$\begin{aligned}\frac{p^-}{p}=\frac{[p+]}{[-p]}\Rightarrow\frac{p_2^-}{p_2}-\frac{p_3^-}{p_3}&=\frac{[2+][-3]-[3+][-2]}{[-2][-3]}\\&=\frac{[23]}{[-2][-3]},\end{aligned}\tag{42}$$

with our normalization. Using this result, we find the similar result

$$\frac{p_2^- + 1}{p_2} - \frac{p_3^-}{p_3} = \frac{\langle +-\rangle[-3] + \langle +2\rangle[23]}{\langle +2\rangle[-2][-3]}$$
$$= \frac{\langle +4\rangle[34]}{\langle +2\rangle[-2][-3]}, \quad (43)$$

applying momentum conservation. We next translate the momentum denominators into twistor notation and also substitute the space-cone expressions for the polarizations and numerators. Canceling identical factors in the numerator and denominator (but no further use of identities), the amplitude becomes $(+=5,\ -=1)$

$$\frac{\langle +4\rangle^3}{\langle +2\rangle\langle +3\rangle}\left(\frac{[-4][34]}{\langle 2-\rangle[4+]} + \frac{[-4]^2}{\langle 23\rangle[4+]} + \frac{\langle +2\rangle[-2]}{\langle -2\rangle\langle 34\rangle}\right)$$
$$= \frac{\langle +4\rangle^3}{\langle +2\rangle\langle +3\rangle}\left(-\frac{\langle +2\rangle[-4]}{\langle 2-\rangle\langle 23\rangle} + \frac{\langle +2\rangle[-2]}{\langle 2-\rangle\langle 34\rangle}\right)$$
$$= -\frac{\langle +4\rangle^3}{\langle 2-\rangle\langle 23\rangle\langle 34\rangle}$$
$$= -\frac{\langle 45\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 51\rangle}, \quad (44)$$

applying momentum conservation twice, restoring normalization, and replacing the numerals for $\pm$.

By comparison, in the usual spinor helicity formalism we find that only the same three diagrams contribute: In this formalism, also, the two reference momenta cannot be on the same three-point vertex. Unlike the space cone, a reference momentum can attach to a four-point vertex as long as the opposite line is off shell; fortunately, that is not possible in this example. In general, a three-point vertex with an attached reference line and one other external line can now have two terms instead of three; the same is true for a three-point vertex with two external lines of the same helicity. This reduces the naive 81 terms to 42. More detailed algebra eventually reduces this to eight terms. The algebra involved in enumerating these terms is comparable in difficulty to that of all the rest of the calculation. (In practice, this amplitude is evaluated by applying supersymmetry to the three-gluon–two-quark amplitude.)

Furthermore, the algebra of the nonvanishing terms is more complicated with the usual spinor helicity, because vector products have to be evaluated in terms of spinor products, after which terms can be collected. For example, the inner product of two polarization vectors in the usual spinor helicity formalism is

$$\epsilon_{i+}\cdot\epsilon_{j-} = \frac{\langle +|[i|}{\langle +i\rangle}\cdot\frac{|j|-}{[-j]} = -\frac{[-i]}{\langle +i\rangle}\frac{\langle +j\rangle}{[-j]}$$
$$= -(\epsilon_{i+})^+(\epsilon_{j-})^- \quad (45)$$

in space-cone language; the inner product of two vectors has been replaced by the product of two scalars. Similar remarks apply to

$$\epsilon_{i\pm,\mathrm{ref}}\cdot P_j = \langle +|[-|\cdot|j\rangle|j] = \langle +j\rangle[-j] = p_j, \quad (46)$$

which is just the transverse momentum component, appearing in the space-cone three-point vertex. With the space-cone method, this algebra has already been done, which allows some collection of terms and common factors even before substituting explicit twistor expressions. In particular, the polarizations now appear as an overall scalar factor for the whole amplitude; the rest is all momenta, and so identities involving momentum conservation can be applied more generally between graphs.

VI. RECURSION RELATIONS

Another method used to derive higher-point amplitudes is the classical Schwinger-Dyson equations, i.e., the classical field equations with perturbative (multiparticle) boundary conditions at infinite times. (In the literature, the field has often been mistaken for the current; as usual, these are distinguished by the fact the field always has an external propagator, while the current has it amputated, since $\Box\phi+\cdots=J$.) The steps are the following: (1) Calculate the first few terms in the series (enumerated by the number of external lines). (2) Guess the general result. (3) Prove that it is correct by induction, using the Schwinger-Dyson equations. Of course, the second part is the hardest in general (at least when one simplifies the third step by using space-cone methods), and has been possible for just a couple of cases, only because the results for those cases are so simple.

The solution to the classical field equations is given by tree graphs with all external lines but one (the field itself) amputated and put on shell [9]. (The usual external-line wave functions describe the asymptotic field, which is free.) This method has been used to find several tree amplitudes at finite point order as well as the two well-known cases with all the on-shell lines possessing the same helicity or one different [4,7]. Note that the field $a^\pm$ has a $\mp$ associated with the opposite end of its external propagator. We then see in the former case, with all $+$'s on on-shell lines, that a^- vanishes because there are no fully amputated diagrams, even off shell, with only $+$'s externally (again counting $+$'s and $-$'s on vertices). Similarly, for the latter case, with only one $-$ on an on-shell line, we see that a^- has only $++-$ vertices, but setting one on-shell $-$ to be a reference line (which by definition must be on shell), such a vertex is not allowed; so a^- vanishes also in this case. This contrasts with the usual spinor helicity method, where the analogue of the vanishing of a^- requires an inductive proof. By similar reasoning, we see that a^+ in the former case consists entirely of $++-$ vertices and in the latter case consists of all $++-$ except for one $--+$ (no $++--$), which must have the $-$ reference line directly attached. The appearance of only the self-dual field (a^+) and almost only the self-dual vertex ($++-$) means that in both cases one is essentially solving the field equations in the self-dual theory [6].

We now consider in more detail the simpler (former) example (the one which does not directly give a nontrivial scattering amplitude). As a slight simplification, we look at the recursion relation for the field as defined in the self-dual theory: From the results at the end of Sec. II, we write

$$a^+ = p\phi. \quad (47)$$

(Implicitly, we also have $a^{-}=p^{-1}\hat{\phi}$. These redefinitions make the $++-$ vertex local.) The recursion relation is now

$$\phi(1,n)=\frac{1}{\frac{1}{2}P^2(1,n)}\sum_{i=1}^{n-1}\phi(1,i)\phi(i+1,n)\times[p^-(1,i)p(i+1,n)-p(1,i)p^-(i+1,n)], \quad (48)$$

$$P(j,k)\equiv\sum_{m=j}^{k}P_m, \quad (49)$$

where we again use color ordering, number the external lines cyclically, and $\phi(j,k)$ denotes the field with on-shell lines with momenta P_j through P_k. (Thus on the left-hand side of the equation the field has n on-shell lines, while on the right-hand side the two fields have i and $n-i$.) Plugging in the twistor expressions for the vertex momenta, we find

$$p^-(1,i)p(i+1,n)-p(1,i)p^-(i+1,n)=\sum_{j=1}^{i}\sum_{k=i+1}^{n}\langle +j\rangle[jk]\langle +k\rangle. \quad (50)$$

If we are clever, we can guess the general result from explicit evaluation of the lower-order graphs; instead, we find in the literature [7], after the above redefinition,

$$\phi(i,j)=\frac{1}{\langle +i\rangle\langle i,i+1\rangle\cdots\langle j-1,j\rangle\langle +j\rangle}. \quad (51)$$

For the initial-condition case $n=1$, this is simply the statement that the external-line factor for ϕ is now

$$\epsilon\phi=\frac{(\epsilon_+)^+}{p}=\frac{1}{\langle +p\rangle^2}. \quad (52)$$

The induction hypothesis is also easy to check: The product of the two ϕ's from the induction hypothesis gives the desired result by itself up to a simple factor:

$$\phi(1,i)\phi(i+1,n)=\phi(1,n)\frac{\langle i,i+1\rangle}{\langle +i\rangle\langle +,i+1\rangle}. \quad (53)$$

(The algebra of the color indices works as usual.) We then perform the sum over i before that over j and k (the complete sum is over all i,j,k with $1\leqslant j\leqslant i\leqslant k\leqslant n$), making use of one of the identities used in the usual spinor helicity evaluation:

$$\frac{\langle ab\rangle}{\langle +a\rangle\langle +b\rangle}+\frac{\langle bc\rangle}{\langle +b\rangle\langle +c\rangle}=\frac{\langle ac\rangle}{\langle +a\rangle\langle +c\rangle}\Rightarrow\sum_{i=j}^{k-1}\frac{\langle i,i+1\rangle}{\langle +i\rangle\langle +,i+1\rangle}=\frac{\langle jk\rangle}{\langle +j\rangle\langle +k\rangle}. \quad (54)$$

Multiplying this by the vertex momentum factor gives a sum over $j<k$ of $\langle jk\rangle[jk]=P_j\cdot P_k$, canceling the external propagator, yielding the desired result.

VII. CONCLUSIONS

We have introduced a complex gauge formalism useful for reducing further the algebraic complexity associated with vector amplitude calculations. The results here generalize naturally to theories containing additional matter. We have seen that the space-cone formalism simplifies the algebra in massless Feynman diagrams to such an extent that supersymmetry identities are no longer needed. The index algebra at intermediate stages of the calculation is eliminated; the only remnant of spin indices on fields is the $\pm$ label for helicity of the two components of the space-cone Yang-Mills field. Our complex spacelike gauge, together with the use of reference momenta, incorporates all the ideas of spinor helicity, and extends them to internal lines and vertices through light-cone-type techniques.

Although we have considered only tree graphs in this paper, the method applies straightforwardly to loops. Alternatively, since auxiliary fields are often useful in the analysis of effective actions (including renormalization), one might use a background field formalism, where a Fermi-Feynman or Gervais-Neveu gauge would be used to calculate the gauge-invariant effective action first, and then the effective action would be evaluated in the space-cone gauge to derive *S*-matrix elements.

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