Simplifying algebra in Feynman graphs. I. Spinors

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We present a general formalism for simplifying manipulations of spin indices of massless and massive spinors and vectors in Feynman diagrams. The formalism is based on covariantly reducing the number of field components in the action in favor of chiral or self-dual fields. In this paper we concentrate on calculational simplifications involving fermions in gauge theories by eliminating half of the components of Dirac spinors. Some results are (1) we find reference momenta for massless fermions analogous to those used for external gauge bosons, (2) many of the known supersymmetry identities (tree and one loop) are seen in a simple manner from the graphs, (3) manipulations with external line factors for massive fermions are unnecessary, and (4) some of the simplifications for nearly maximally helicity violating gluonic amplitudes are built into the Feynman rules. [S0556-2821(98)04924-8]

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I. INTRODUCTION

In the past decade there have been many advances in the art of doing perturbative gauge theory calculations. (In [1] reviews of these methods are presented for massless QCD.) Color ordering, new Feynman rules and methods inspired directly from string theory, and techniques based on unitarity and analyticity requirements on *S*-matrix elements have been important developments [2,3,4]. Notable among the new techniques was the introduction of reference momenta for external polarization tensors. However, the simplifications for doing these calculations when external fermions are present have not been as dramatic, and no analogous simplifications for massive particles have been presented. In this work we fill this gap by introducing reference momenta for massless external fermion lines analogous to those used for gluons and, similarly, covariantly reduce the components of massive external line factors for spinors and vectors. Many of the simplifications are introduced directly into the action by covariantly reducing the components of the fields themselves. This leads to further simplifications for propagators and vertices.

Twistors [5], also known in gauge theory calculations as ‘‘spinor helicity’’ [6], are formulated by writing all massless (on-shell) momenta in terms of commuting spinors,

$$k^2=0 \Rightarrow k_{\alpha\dot{\alpha}}=\pm k_\alpha k_{\dot{\alpha}}, \tag{1}$$

where k_α and $k_{\dot{\alpha}}$ are two-component Weyl spinors of left or right chirality. In matrix notation these spinors may also be represented as

$$\langle k|=k^\alpha, \quad |k\rangle=k_\alpha, \quad [k|=k^{\dot{\alpha}}, \quad |k]=k_{\dot{\alpha}}, \tag{2}$$

and we may write

$$k=\pm|k\rangle[k|. \tag{3}$$

The indices are raised and lowered through the metrics $C^{\alpha\beta}$ and $C^{\dot{\alpha}\dot{\beta}}$ on the SL(2,*C*) covering group of the Lorentz group,

$$k^\alpha=C^{\alpha\beta}k_\beta, \quad k^{\dot{\alpha}}=C^{\dot{\alpha}\dot{\beta}}k_{\dot{\beta}}, \tag{4}$$

where

$$C^{\alpha\beta}=C^{\dot{\alpha}\dot{\beta}}=-C_{\alpha\beta}=-C_{\dot{\alpha}\dot{\beta}}=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{5}$$

As usual, symmetrization and antisymmetrization of the two indices α,β are denoted through $(\alpha\beta)$ and $[\alpha\beta]$. We shall also use the conventions with the inner product $V\cdot W=\frac{1}{2}V^{\alpha\dot{\alpha}}W_{\alpha\dot{\alpha}}$ and the Abelian (self-dual) field strength is defined as $F_{\alpha\beta}=\frac{1}{2}\partial_{(\alpha}{}^{\dot{\alpha}}A_{\beta)\dot{\alpha}}$.

Furthermore, the polarization vectors $\epsilon^{\pm}_{\alpha\dot{\alpha}}$ satisfying the required normalization conditions may also be represented in terms of commuting spinors as

$$\epsilon^{+}_{\alpha\dot{\alpha}}(k)=-i\,\frac{k_{\dot{\alpha}}p_\alpha}{k^\beta p_\beta}, \quad \epsilon^{-}_{\alpha\dot{\alpha}}(k)=i\,\frac{k_\alpha p_{\dot{\alpha}}}{k^{\dot{\beta}}p_{\dot{\beta}}} \tag{6}$$

or, in matrix notation,

$$\epsilon^{+}=-i\,\frac{|p\rangle[k|}{\langle kp\rangle}, \quad \epsilon^{-}=i\,\frac{|k\rangle[p|}{[kp]}. \tag{7}$$

Our conventions are such that $(\psi^\alpha)^*=\psi^{\dot{\alpha}}$ and $(\psi_{\dot{\alpha}})^*=-\psi_\alpha$. Because of gauge invariance, the polarization spinor p_α is arbitrary (but $k^\alpha p_\alpha\neq 0$):

$$\begin{aligned}\delta A_{\alpha\dot{\alpha}}=\partial_{\alpha\dot{\alpha}}\lambda &\Rightarrow \delta\epsilon_{\alpha\dot{\alpha}}(k)=ik_{\alpha\dot{\alpha}}\lambda(k)\\ &\Rightarrow \delta p_\alpha=\zeta(k)k_\alpha+\xi(k)p_\alpha,\end{aligned} \tag{8}$$

where $\zeta(k)$ comes from the gauge transformation parameter $\lambda(k)$, while the scale parameter $\xi(k)$ comes from the normalization of the polarization vectors. The perturbative cal-

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culations can then be simplified by identifying the null vector $p_{\alpha\dot{\alpha}}=p_{\alpha}p_{\dot{\alpha}}$ with any other massless (''reference'') momentum in the amplitude.

This approach has been applied both to eliminate much algebra in intermediate stages of calculations and to reduce the expressions for the final result of the amplitude. In some cases the resulting amplitudes are extremely simple, such as the case of pure gluon amplitudes where all external legs except one or two are similarly polarized [4,7]. The simplicity of these expressions is now understood to arise from the relation of these particular amplitudes to self-dual field theories [8]. The simplifications we present in this paper are based on actions that take advantage of chirality and self-duality.

In our formulation reference momenta may also be given to external *fermion* lines. This is interesting in that, unlike external vector bosons, there is no apparent local symmetry incorporating the fermion field. In our approach we shall first integrate out half of the massive spin-1/2 fields (e.g., $\psi^{\dot{\alpha}}$) coupled to gauge fields. In such theories with a labeling of the fields according to an SL$(2,C)$ (van der Waerden) Weyl-spinor notation, the massive spinor fields with dotted indices may be eliminated in favor of those with undotted ones; we treat the former fields as Lagrange multipliers. Taking the massless limit of the resulting Lagrangian defines our massless theory; the definition of the in and out states then allows us to present the reference momenta for the external fermion lines.

In Sec. II we present the reformulation of the actions for massive and massless spin-1/2 fields. We give the improved Feynman rules in Sec. III, including the reference momenta for external fermions. In Sec. IV we make some comparisons of our approach to the usual formulation. In Sec. V we reproduce a well-known result: We illustrate our technique with a sample calculation of the high-energy limit $e^{+}e^{-}\to\gamma\gamma$ tree-level scattering process. The known supersymmetry identities and their implication for the vanishing of certain tree and one-loop amplitudes are immediately obvious from the action and are discussed in Sec. VI. Section VII contains our final comments. In the sequel we will apply similar methods to vectors (Abelian and non-Abelian).

II. ACTIONS

The action for a massive particle in an external massless vector field (Yang-Mills or electromagnetism) can be written as

$$\mathcal{L}=\bar{\psi}^{\dot{\alpha}}i\nabla_{\alpha\dot{\alpha}}\psi^{\alpha}+\frac{m}{2}\,(\psi^{\alpha}\psi_{\alpha}+\bar{\psi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}), \tag{9}$$

where $\nabla_{\alpha\dot{\alpha}}=\partial_{\alpha\dot{\alpha}}+iA_{\alpha\dot{\alpha}}$ is our convention for the covariant derivative. We may treat the dotted field as an auxiliary one and eliminate it using the field equations ($i\nabla_{\alpha\dot{\alpha}}\psi^{\alpha}=-m\bar{\psi}_{\dot{\alpha}}$); alternatively, one may functionally integrate out the barred fermion field after completing the square. We then have [9]

$$\mathcal{L}=\frac{1}{2m}\,(\nabla^{\dot{\alpha}}{}_{\alpha}\psi^{\alpha})(\nabla_{\beta\dot{\alpha}}\psi^{\beta})+\frac{m}{2}\,\psi^{\alpha}\psi_{\alpha} \tag{10}$$

or

$$\mathcal{L}=-\frac{1}{2m}\,\psi^{\alpha}(\Box-m^{2})\psi_{\alpha}-\frac{i}{2m}\,\psi^{\alpha}F_{\alpha}{}^{\beta}\psi_{\beta}$$

$$\equiv-\frac{1}{2m}\,\psi^{\alpha}\left(\Box+\frac{i}{2}\,F_{\beta}{}^{\gamma}S_{\gamma}{}^{\beta}-m^{2}\right)\psi_{\alpha} \tag{11}$$

(which defines the spin operator $S_{\alpha\beta}=S_{\beta\alpha}$) for real representations of the gauge group. In the obtaining the result in Eq. (9), we have integrated by parts in the covariant derivative. When the real representation is complex plus complex conjugate (e.g., QED or QCD), we can write $\psi^{\alpha}=(\chi^{\alpha},\xi^{\alpha})$, where χ and ξ are in complex conjugate representations:

$$\mathcal{L}=-\frac{1}{m}\,\chi^{\alpha}(\Box-m^{2})\xi_{\alpha}-\frac{i}{m}\,\chi^{\alpha}F_{\alpha}{}^{\beta}\xi_{\beta}$$

$$\equiv-\frac{1}{m}\,\chi^{\alpha}\left(\Box+\frac{i}{2}\,F_{\beta}{}^{\gamma}S_{\gamma}{}^{\beta}-m^{2}\right)\xi_{\alpha}. \tag{12}$$

In addition, the same procedure can be applied for complex representations, i.e., cases with parity violation, but the result is more complicated.

Turning out attention back to the modified theory in Eq. (11), because the original action possessed a quadratic term $\bar{\psi}^{2}$, the elimination of $\bar{\psi}^{\dot{\alpha}}$ produced only a trivial functional determinant factor in the path integral. The overall $1/m$ can also be removed from the kinetic term by scaling the fermionic field:

$$\psi\to\sqrt{m}\,\psi. \tag{13}$$

It is interesting that upon setting the anti-self-dual part $F_{\alpha\beta}$ of the field strength to zero the action (11) loses its spin dependence and the couplings of the fermions to the gauge fields are exactly what one obtains for scalars. For this reason there is much improvement in using the action (11) to compute S-matrix elements involving external fermions; this will be demonstrated in later sections.

The idea of using fields with only undotted spinor indices has been used previously for spin-1/2 fields in [9], but advantage was not taken of the simplification of external-line factors; the fermion reference momentum described in the following section was not introduced. More importantly, the γ matrices were replaced by the four-vector σ matrices; in [9], the σ matrices and their transposes were effectively treated as independent. The result of these manipulations does not change the amount of γ-matrix algebra in actual calculations.

There is also a major simplification in using the Lagrangian in Eq. (11) for the case in which the massless Yang-Mills fields that couple to the fermion field are (almost) all of the same helicity: Only the self-dual part $S_{\alpha\beta}$ of the spin operator appears in the action (9) in the magnetic-moment coupling. The coupling of anti-self-dual Yang-Mills fields does not enter through the $\psi^\alpha F_\alpha{}^\beta \psi_\beta$ term and, as a result, is spin independent. It only couples through the covariantized box, as a scalar coupling would enter. In calculating amplitudes closer to self-dual ones [i.e., maximum helicity violation (MHV)], the fermions resemble scalars in actual calculations. Examples of this will be discussed in Sec. V on supersymmetry identities.

Because our method treats dotted and undotted indices in an asymmetric fashion, the action (11) is complex and unitarity is not readily apparent. However, because unitarity is present within the original action (9), it must persist after integrating out the auxiliary fermionic fields $\overline{\psi^\alpha}$. [Even in non-Abelian gauge theories used with a complex gauge-fixing term, for example the Gervais-Neveu gauge $L_{gf} = (1/\lambda)\mathrm{Tr}(\partial\cdot A + iA^2)^2$, unitarity is not immediately obvious from the reality properties of the action.]

As another example of this asymmetry between the dotted and undotted spinor treatments, we analyze the classical magnetic moment; because only the self-dual part of the classical magnetic-moment coupling appears in the action (11), it is not immediately obvious that a complete contribution arises. The effect of this coupling may be found from the covariantized Pauli-Lubanski vector $W_{\alpha\dot\beta} = S_{\alpha\beta}\nabla^\beta{}_{\dot\beta}$. Commuting this operator with the ''Hamiltonian''

$$H = \Box - \frac{i}{2} F^{\alpha\beta} S_{\beta\alpha} \tag{14}$$

gives the usual precessions

$$-i\,\frac{d}{d\tau} W_{\alpha\dot\beta} = [H, W_{\alpha\dot\beta}] = i W_\alpha{}^{\dot\delta} F_{\dot\delta\dot\beta} + i W^\rho{}_{\dot\beta} F_{\rho\alpha}, \tag{15}$$

as follows from truncation of the usual expressions that include $S_{\dot\alpha\dot\beta}$ terms for W and H. We see that the precession of the spin as described by the covariantized Pauli-Lubanski vector has contributions from both self-dual and anti-self-dual fields.

We also note that the same method used to obtain Eq. (11) has also been applied to the classical mechanics and classical field theory of the massive superparticle (describing spins 0 and 1/2) and the subcritical (Liouville) superstring [10]. There, the undotted indices are carried by the anticommuting coordinates. Such ''chiral superspaces'' are natural for self-dual supersymmetric theories of any spins [11].

III. FEYNMAN RULES

The rules for the propagators and vertices can be read from the action (11) as usual. The vector propagator is the usual one, the same as the scalar propagator up to index factors:

$$\langle A_{\beta\dot\epsilon}(k) A_{\gamma\dot\zeta}(-k)\rangle = \frac{1}{k^2} C_{\beta\gamma} C_{\dot\epsilon\dot\zeta}. \tag{16}$$

The fermion propagator is now also like a scalar propagator:

$$\langle \psi^\alpha(k)\psi^\beta(-k)\rangle = \frac{1}{k^2+m^2} C^{\alpha\beta}. \tag{17}$$

Correlations between external F type couplings are simple; for example, we have

$$\begin{aligned}\langle F_{\alpha\beta}(k) F_{\gamma\delta}(-k)\rangle &= \tfrac{1}{4} k_{(\alpha}{}^{\dot\epsilon} \langle A_{\beta)\dot\epsilon}(k) A_{\dot\zeta(\gamma}(-k)\rangle k_{\delta)}{}^{\dot\zeta} \\ &= \tfrac{1}{4} C_{\gamma(\alpha} C_{\beta)\delta}.\end{aligned} \tag{18}$$

The coupling of a gauge field to the charged spinors may be found from the Lagrangian (11) and appears in three types. The fermion line either emits a gauge boson in a spin-independent fashion through the expansion of the gauge-covariant box or through the self-dual field strength. We first expand the box as

$$\nabla^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha} = \partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha} + 2iA^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha} + i\partial^{\alpha\dot\alpha}A_{\alpha\dot\alpha} - A^{\alpha\dot\alpha}A_{\alpha\dot\alpha}, \tag{19}$$

so that we may clearly read off the contributions to the Feynman rules. From the spin-independent couplings, i.e., from a $\psi^\alpha \Box \psi_\alpha$, we have the $e^+e^-\gamma$ coupling:

$$V^{\mu\nu}_{\alpha\dot\alpha} = \tfrac{1}{2} e (k_1 - k_2)_{\alpha\dot\alpha} C^{\mu\nu}. \tag{20a}$$

This Feynman rule represents the emission a gauge vector $A_{\alpha\dot\alpha}$ from two fermions ψ^μ and ψ^ν with momenta k_1 and k_2. The $e^+e^-\gamma$ vertex coming from the $\psi^\alpha F_\alpha{}^\beta\psi_\beta$ term in the Lagrangian is

$$V^{\mu\nu}_{\alpha\dot\alpha} = \tfrac{1}{2} e (k_1 - k_2)^{(\mu}{}_{\dot\alpha}\delta^{\nu)}_{\delta}, \tag{20b}$$

with the same quantum number assignments. The external vector emitted through the chiral $F_{\alpha\beta}(k)$-type coupling is a ''$-$'' helicity state. This is clearly seen in the light-cone gauge where the gauge degree of freedom contained in $F_{\alpha\beta}$ is only the ''$-$'' helicity scalar state (and $F_{\dot\alpha\dot\beta}$ is the ''$+$'' helicity state); alternatively, contracting an on-shell ''$+$'' helicity state with the vertex gives zero. The final vertex is the four-point $e^+e^-\gamma\gamma$ coupling:

$$V^{\mu\nu}_{\alpha\dot\alpha,\beta\dot\beta} = e^2 C_{\alpha\beta} C_{\dot\alpha\dot\beta} C^{\mu\nu}, \tag{21}$$

which does not contain any momentum dependence and where the gauge fields and spinors have the assignments $A_{\alpha\dot\alpha}, A_{\beta\dot\beta}$ and ψ^μ, ψ^ν. The couplings (20a) and (21) are of the same form as the scalar ones, but with additional indices $(\mu\nu)$ of the fermions contracted with the external fermion lines.

It might appear that the Feynman rules for vertices are more complicated than in the usual formalism, because there are more (two three-point vertices and a four-point one). However, all such terms arise in either method: here, directly in the action; by the usual method, after performing the Dirac algebra. The advantage in the formulation presented here is that the γ-matrix algebra has been performed once and for all in the action itself, whereas in the usual method one must reshuffle the terms individually in each Feynman diagram at each vertex and propagator. Also, our form allows for a more convenient comparison of the contributions to scattering amplitudes arising from particles of other spins (e.g., for supersymmetry).

In deriving the *S*-matrix elements in terms of the reduced action (11), we need to specify the ingoing and outgoing line factors for the external fermions. In the massive case external line factors are arbitrary; their choice corresponds to one of the spin axes. Unlike Dirac spinors, in our case massive external line factors are not needed to determine the four components in terms of two polarizations, since we have already reduced them explicitly to only two. With the usual methods, external line factors are needed to reduce the 4^n matrix elements for n external fermion lines to 2^n appropriate to the two polarizations of spin 1/2. The alternative is to square the amplitude before performing Dirac algebra. However, this can be cumbersome, especially when numerical evaluation of momentum integrals is involved, since N diagrams produce N^2 terms in the cross section.

In the massless case, the line factors are determined by helicity: The two helicity states for ψ_α are, with the normalization $\epsilon^{+,\alpha}\epsilon^-_\alpha=1$,

$$\epsilon^+_\alpha=p_\alpha, \quad \epsilon^-_\alpha=\frac{q_\alpha}{p^\beta q_\beta} \tag{22}$$

or, in matrix notation,

$$\epsilon^+=|p\rangle, \quad \epsilon^-=\frac{|q\rangle}{\langle pq\rangle}, \tag{23}$$

in terms of an arbitrary twistor q_α. These states correspond to solutions to the field equations in the original action (9), $\psi_\alpha=\epsilon^+_\alpha$ and $\overline{\psi}_{\dot\alpha}=p_{\dot\alpha}=-i\partial^\alpha{}_{\dot\alpha}\epsilon^-_\alpha$. The relation to Eq. (11) can be seen, e.g., by introducing a four-component background spinor for Eq. (9) before integrating out the barred fields. The ambiguity in choosing q_α is analogous to choosing reference momenta for gauge field polarization vectors in the spinor helicity formalism (an independent derivation of a fermionic reference momenta was given in [12]). Note that the vector polarizations are products of the spinor polarizations and their complex conjugates (including normalization). In the remaining part of this section, we illustrate the simplifications obtained by using these Feynman rules.

The rules are simplest when the amplitudes possess external vector lines mostly of the same helicity (i.e., of the ''+'' type). If an external gauge boson of helicity ''−'' is emitted through an $F_{\alpha\beta}$ coupling, we can immediately apply twistor techniques to write the corresponding external-line factor with momentum $k^2=0$ in momentum space:

$$F_{\alpha\beta}(k)=\frac{i}{2}\,k_{(\alpha}{}^{\dot\alpha}A_{\beta)\dot\alpha}(k)=k_\alpha k_\beta \tag{24}$$

or

$$F=|k\rangle\langle k|. \tag{25}$$

Because of gauge invariance, it is independent of the reference momentum chosen for the vector boson.

As an example of the reduced spinor algebra associated with these couplings, we consider the case of a single fermion line with a number of attached external gauge bosons. All the algebra associated with the indices of the external fermions comes from the F coupling. We immediately obtain with the propagator (17) the contracted vertex algebra

$$F_\alpha{}^\beta(k_1)F_\beta{}^\sigma(k_2)F_\sigma{}^\rho(k_3)\cdots=|1\rangle\langle 12\rangle\langle 23\rangle\cdots, \qquad \langle i|=k_i{}^\alpha, \quad |j\rangle=k_{j\alpha}, \tag{26}$$

from the couplings. The numbers 1,2, . . . label the consecutive F's that appear in the product of external F fields attached to the fermion line; the remaining spin-independent couplings appear as scalars and do not effect the matrix algebra associated with the SU(2) indices. If the fermion line is a closed loop, then the n F's contract to a cyclic product

$$P=\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle; \tag{27}$$

if the line is open, then the product is terminated on either end with the polarization spinors $\langle f|$ and $|i\rangle$ of the corresponding external fermion fields,

$$P=\langle f1\rangle\cdots\langle ni\rangle, \tag{28}$$

with the helicity states (22) in the massless case.

Last, we comment on the use of the Lagrangian (11) for fermions in massive QED calculations. Consider, for example, the scattering of a fermion-antifermion pair going into $n-2$ vectors of the same helicity. Because the $F_\alpha{}^\beta$ vertex does not enter into the calculations (it generates the ''−'' helicity states), we find that the expressions for the graphs are the same as the ones obtained for $n-2$ photons emitted along a scalar line. The Feynman rules in this particular example completely avoid the usual Dirac matrix algebra, external line factors, and field equations usually used to simplify the algebra. Similar simplifications occur for amplitudes with a general set of polarizations for the external vectors or virtual photons connecting to other fermion lines.

IV. COMPARISONS TO THE USUAL FORMALISM

In this section we make several remarks comparing the formalism discussed in this work with the usual Feynman diagrammatic techniques. First, it is not equivalent to other known tricks known to simplify the γ-matrix algebra. Our reformulation of the original vertex is analogous to applying the Gordon identity,

$$\overline{u}(p)\gamma^a u(q)=\frac{1}{2m}\,\overline{u}[(p+q)^a+2S^{ab}(q-p)_b]u(q), \tag{29}$$

where $S^{ab}=(i/4)[\gamma^a,\gamma^b]$ is the spin operator. The rescaling of the fermion by the mass factor $\sqrt{m}$ occurs naturally in the above when taking the massless limit. The ''squared-propagator'' trick, originally introduced by Feynman and Schwinger, uses the fact that $(\nabla\!\!\!/-m)(\nabla\!\!\!/+m)=\Box+\cdots$ is more analogous to the scalar theory. Although these manipulations give a convenient rearrangement of γ matrices along fermion lines, the number of γ matrices involved is the same.

Second, a major advantage of our method is that the σ matrices are treated as self-dual tensors instead of vectors. With the squared-propagator trick, only even numbers of γ matrices appear; so the matrix algebra consists of only 8 of the usual 16, namely, I, γ_5, and the spin operators S_{ab}. When we restrict ourselves to chiral spinors, a further reduction to just 4 matrices (σ and I) is achieved, since then γ_5 is identified with the identity and only the self-dual part of the spin survives. In notation where representations are written as (m,n), where m and n are either integral or half-integral (corresponding to $2m$ undotted indices and $2n$ dotted indices), these σ matrices appear as (1,0), not as $(\frac{1}{2},\frac{1}{2})$. Thus the σ matrices we use are treated as spin, as in nonrelativistic quantum mechanics, and not as part of γ_a. (I.e., they are treated as bosonic, not fermionic.) The action, however, is still manifestly Lorentz covariant: SL(2,C), when restricted to just undotted indices, is indistinguishable from SU(2). More generally, our massive fields are all $(m,0)$ representations. We emphasize that the resulting matrices carry *only* an effective SU(2) algebra, and not the usual SU(2)$\otimes$SU(2) associated with the covering group SL(2,C), since the fields carry only undotted indices. Also, instead of using explicit Pauli σ matrices (which are the Clebsch-Gordan coefficients for $\frac{1}{2}\otimes\frac{1}{2}=1\oplus 0$), we use SU(2) spinor notation, which is simpler even in nonrelativistic quantum mechanics.

A fermion line in the old formalism gives a string of the form

$$\overline{u_f}\not{V}_1\not{P}_1\not{V}_2\cdots\not{P}_{n-1}\not{V}_n u_i \tag{30}$$

for n vertices $\not{V}_j$ and $n-1$ propagators $\not{P}_k$; the new way gives

$$V_1V_2\cdots V_n. \tag{31}$$

Matrix multiplication is now trivial, of the form

$$AB=(A\cdot B)I+A\times B, \tag{32}$$

rather than

$$\gamma^{(m)}\gamma^{(n)}=\gamma^{(m+n)}+\gamma^{(m+n-1)}+\cdots+I \tag{33}$$

for the antisymmetrized product $\gamma^{(n)}$ of n γ matrices, since any 2×2 matrix contains only the singlet or (1,0) representations. Similarly, for two fermion lines the old method required Fierz identities, which are not needed in our formalism. In the next section an explicit example will illustrate our approach.

V. MASSLESS EXAMPLE

In this section we reproduce two very simple QED (and QCD) scattering processes using these rules (in the massless or high-energy limit).

A classic scattering example is $e^+e^-\to\gamma^-\gamma^+$. A similar analysis will provide the massless QCD amplitude for $q\overline{q}\to g^+g^-$ after including the color factors and the additional four-point contact term within the non-Abelian F_α^β spin-dependent interaction. We label the incoming fermion line with momentum k_1 and the outgoing one with k_2; the photons with outgoing momentum k_3 and k_4 have helicity $-$ and $+$, respectively. In a color-ordered format, there are three diagrams—one with an ordered labeling 1234 of the external legs (containing an s_{14} channel), one with the crossed legs 1243 (containing a s_{24} channel), and an additional one from the $e^+e^-\gamma\gamma$ four-point contact term within the covariant box.

Because the outgoing photons have opposite helicities, there is at most one emission of a vector through the spin-dependent F-type coupling (which generates a minus helicity vector state). The first ordered diagram receives contributions from the emission of a "$-$" state through either a $\psi^\alpha F_\alpha{}^\beta\psi_\beta$ coupling or the $A\cdot\partial$ within the covariantized box; we label the contributions as T_0 and T_1 with the index labeling the number of F couplings. The Feynman rules give the expressions for the diagrams:

$$T_0^{(1234)}=e^2\,\frac{\langle 2q\rangle\langle 32\rangle[2k]\langle 1p\rangle}{\langle q1\rangle\langle 14\rangle[k3]\langle p4\rangle},$$

$$T_0^{(1243)}=e^2\,\frac{\langle 2q\rangle\langle 42\rangle[2p]\langle 1k\rangle}{\langle q1\rangle\langle 13\rangle[p4]\langle k3\rangle} \tag{34}$$

and

$$T_1^{(1234)}=e^2\,\frac{\langle 23\rangle\langle 3q\rangle\langle 1p\rangle}{\langle q1\rangle\langle 14\rangle\langle p4\rangle},\quad T_1^{(1243)}=-e^2\,\frac{\langle 23\rangle\langle 3q\rangle\langle 2p\rangle}{\langle q1\rangle\langle 24\rangle\langle p4\rangle}, \tag{35}$$

where k and p are the reference momenta for the photons with momentum k_3 and k_4, respectively, and q is that for the fermion with momentum k_1. The next contribution arises from the four-point contact term and is

$$C=e^2\,\frac{\langle 2q\rangle}{\langle q1\rangle}\,\frac{\langle 3p\rangle[4k]}{[3k]\langle 4p\rangle}. \tag{36}$$

The final result after adding up all the contributions is guaranteed to be independent of the reference momentum. However, we should choose them to simplify the intermediate steps in the calculation.

For example, upon taking $q=k_2$ we eliminate $T_0^{(1234)}$, $T_0^{(1243)}$, and C. Choosing the reference momentum $p=k_2$ for the "$-$" helicity outgoing photon eliminates $T_1^{(1243)}$. The entire result for the amplitude then arises from the $T_1^{(1234)}$ contribution; it is

$$A(e^+,e^-;k_3^-,k_4^+)=e^2\,\frac{\langle 23\rangle^2}{\langle 14\rangle\langle 24\rangle}=-e^2\,\frac{[14]\langle 23\rangle}{\langle 42\rangle[23]}, \tag{37}$$

where we have multiplied by $[23]/[23]$ and used $k_2\cdot k_3=k_1\cdot k_4$ to show agreement with the result obtained by conventional techniques. In practice, the reference momenta are chosen at the beginning of the calculation; so only the single graph $T_1^{(1234)}$ is actually calculated. Although the amplitude here is a relatively easy one to evaluate with more conventional techniques, our evaluation did not involve any γ-matrix algebra; for higher-point diagrams, this is a significant advantage. Furthermore, we expect that in one-loop amplitudes the fermionic reference momenta will lead to significant calculational advantages.

VI. SUPERSYMMETRY IDENTITIES

In this section we rederive several of the known supersymmetry identities relevant to maximally helicity-violating amplitudes [13,1] and their relation to self-dual Yang-Mills theory in $2+2$ dimensions. The reformulation of the gauge theory we present is naturally suited to deriving these identities.

First, consider the scattering at the tree level of a $q\bar{q}$ pair into a series of $+$ helicity outgoing non-Abelian vectors, $A(q,\bar{q},g^+,\ldots,g^+)$. This example is particularly simple to describe with our rules because an off-shell field $\langle A_{\alpha\dot{\alpha}}\rangle$ evaluated at the tree level between on-shell self-dual states is itself self-dual [14]; i.e., any tree amplitude with a number of external legs of the same helicity attached to the fermion line vanishes if it is coupled through an $F_{\alpha\beta}=0$ term in the Lagrangian (9). Along the fermion line the vectors are then emitted through the (scalar-type) coupling found from expanding the covariantized box $\psi^\alpha\Box\psi_\alpha$. Because the gluons are emitted through a scalar-type coupling, the incoming and outgoing fermions contract immediately to give $\langle fi\rangle=\langle 1q\rangle/\langle 2q\rangle$. (The $q,\bar{q}$ lines possess momenta k_1, k_2.) The entire contribution is immediately seen to vanish upon taking $q=k_1$. A similar analysis shows that amplitudes with any number of fermions and vectors containing a maximally helicity violating assignment of polarizations vanish.

Next, the well-known relation for the partial amplitude, where ϕ and $\bar{\phi}$ are complex scalars,

$$A_{\text{tree}}(e^+,1^+,\ldots,j^-,\ldots,(n-1)^+,e^-)=\frac{\langle j1\rangle}{\langle jn\rangle}A_{\text{tree}}(\bar{\phi}_1,1^+,\ldots,j^-,\ldots,(n-1)^+,\phi_n), \quad (38)$$

may also be easily found with the use of our new Feynman rules. In this example, for simplicity we consider Abelian gauge fields only. The coupling of scalars to the gauge field is similar to that of the fermions; however, there is no spin-dependent F term:

$$\mathcal{L}=-\bar{\phi}(\Box-m^2)\phi. \quad (39)$$

In deriving the e^+e^- amplitude, the ''$-$'' helicity jth vector along the fermion line may be emitted through a $\psi^\alpha F_\alpha^{\ \beta}\psi^\beta$ vertex or through the fermion-vector coupling found in the expansion of the covariant box. In the former case, within each diagram the external fermion line factors do not contract with any of the algebra associated with the vector emission except for the single $F_{\alpha\beta}(k_j)$ field to give, as described in Eq. (26),

$$\frac{\langle gj\rangle\langle jn\rangle}{\langle q1\rangle}. \quad (40)$$

With the simple choice of $q=k_j$, all of these diagrams are set to zero and do not contribute to the amplitude. The external fermion line factors in the remaining $e^+e^-\rightarrow$vector diagrams, i.e., those without any explicit $F_{\alpha\beta}$ couplings, contract directly to give

$$\left.\frac{\langle qn\rangle}{\langle q1\rangle}\right|_{q=k_j}=\frac{\langle jn\rangle}{\langle j1\rangle}. \quad (41)$$

In comparing the diagrams contributing to the $e^+e^-\gamma^+\cdots\gamma^+$ and $\phi\phi\gamma^+\cdots\gamma^+$ amplitudes, we find the relation in Eq. (38).

The last identity on *S*-matrix elements we discuss is one which relates different MHV amplitudes at one loop. These *S*-matrix elements are known to satisfy, for any number of external legs, the identity

$$A_{n;1}^{[1]}=A_{n;1}^{[0]}=-A_{n;1}^{[1/2]}(g^+,\ldots,g^+). \quad (42)$$

The index $[j]$ labels the spin content of the internal states within the loop (complex scalar, Weyl fermion, and gluon). By the indices n; 1 we mean the single-trace structure in a non-Abelian gauge theory using the color-ordered Feynman rules [2] derived from the action (11). The result (39) is normally found through a supersymmetric identity which states that the contribution of a virtual supersymmetric multiplet to the scattering gives zero, i.e., $A_{n;1}^{[N>0]}(g^+,\ldots,g^+)=0$; as discussed in [3,4], taking linear combinations of states at one loop in different supersymmetric multiplets then gives Eq. (42).

We examine the derivation of Eq. (42) in the following. Clearly, the only violation of this identity between the $j=1/2$ and $j=0$ contributions ($j=1$ will be discussed in the sequel paper) can come from the coupling of trees from the anti-self-dual field strength $F_{\alpha\beta}$ to the loop. However, as in case of the first identity discussed, we may take $F_{\alpha\beta}=0$ in the amplitudes $A_{n;1}^{[j]}(g^+,\ldots,g^+)$. The reduced action formulation we present in this work is thus naturally suited to describing self-dual Yang-Mills theory, which has an *S*-matrix coinciding with the Wick-rotated one-loop MHV gauge theory amplitudes.

VII. DISCUSSION

For completeness, we also give the reduced Lagrangian describing the theory of fermions obtaining their mass through a Higgs effect. The Yukawa coupling of the Majorana fermion is

$$\mathcal{L}_y=\tfrac{1}{2}\lambda\,\phi\psi^\alpha\psi_\alpha+\tfrac{1}{2}\lambda\,\overline{\phi\psi^{\dot{\alpha}}}\,\overline{\psi}_{\dot{\alpha}}. \quad (43)$$

In this theory, integrating out the dotted spinor components of the Lagrangian (9) with the mass terms replaced by Eq. (43) leads to nonlinearities in the final action. We find, after eliminating $\bar{\psi}$ and rescaling the ψ field as in Eq. (13), the fermion contributions

$$\mathcal{L} = -\tfrac{1}{2}\psi^{\alpha}(\Box - \lambda^{2}\bar{\phi}\phi)\psi_{\alpha} - (i/4)\psi^{\alpha}F_{\alpha}^{\beta}\psi_{\beta} + \tfrac{1}{4}\psi^{\alpha}\psi_{\alpha}(\nabla \ln \bar{\phi})^{2} - \tfrac{1}{2}\psi^{\alpha}\psi_{\alpha}(\Box \ln \bar{\phi}) - \tfrac{1}{2}\psi^{\alpha}(\nabla_{(\alpha}{}^{\dot{\gamma}} \ln \bar{\phi})\nabla_{\beta)\dot{\gamma}}{}^{\psi^{\beta}}. \tag{44}$$

Expression (44) is defined by a perturbation about the vacuum value $\langle\phi\rangle$ of the Higgs particle. Similar simplifications for amplitude calculations, for example in the electroweak sector of the standard model, are expected using the reduced action above.

We expect that the use of these rules will aid substantially in the future computation of higher-point loop amplitudes involving external fermions. The simplifications obtained in eliminating complicated intermediate algebra in gluon scattering amplitudes should persist in these cases as well. Furthermore, it would be interesting to generalize these examples of reduced Lagrangians to theories containing higher-spin fields. We have already looked at the case of spin 1 and found similar simplifications; details will be presented elsewhere. As a final note, it would be interesting to find the local symmetry responsible for the ambiguity in choosing the fermionic reference momenta; the analogous invariance responsible for the simplifications involving the vector reference momenta is gauge symmetry. The appearance of such fermionic gauge symmetry is suggested by a first-quantized approach to a spin-1/2 field [15].

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