Field theory tests for correlators in the AdS-CFT correspondence

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The order g^2 radiative corrections to all two- and three-point correlators of the composite primary operators Tr X^k are computed in $\mathcal{N}=4$ supersymmetric Yang-Mills theory with the gauge group SU(N). Corrections are found to vanish for all N. For $k=2$ this is a consequence of known superconformal nonrenormalization theorems, and for general k the result confirms an $N\to\infty$, fixed large g^2N supergravity calculation and further conjectures by S. Lee *et al.*, hep-th/9806074. A three-point correlator involving $\mathcal{N}=4$ descendents of Tr X^2 is calculated, and its order g^2 contribution also vanishes, giving evidence for the absence of radiative corrections in correlators of descendent operators. [S0556-2821(99)00502-0]

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I. INTRODUCTION

The Maldacena conjecture [1–3] has brought fruitful insights into string theory, supergravity, and supersymmetric field theory while emphasizing a new and striking role of anti–de Sitter space (AdS) in theoretical physics. Many correlation functions [2–14] have now been computed and compared in the boundary $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) and bulk five-dimensional AdS (AdS$_5$) supergravity theories. Certain two- and three-point correlators of operators in the same multiplet as the stress tensor and SU(4) flavor currents satisfy superconformal nonrenormalization theorems [15,16] and thus have no radiative corrections.

In a very recent paper [13] the correlators of the chiral primary family[1] Tr X^k were studied. Three-point functions $\langle$Tr X^{k_1}Tr X^{k_2}Tr $X^{k_3}\rangle$ of normalized operators were evaluated using an $N\to\infty$ fixed large g^2N supergravity computation and shown to be equal to their free-field values. A conjecture was made that the correlators are independent of $\lambda = g^2N$ to leading order in N and also a final speculation that they are independent of g^2 even for finite N. Known nonrenormalization theorems apply only to the case $k_1=k_2=k_3=2$. Thus, these results and speculations hint at nonrenormalization properties which go beyond the known symmetries of $\mathcal{N}=4$ SYM theory.

In this paper, we check the nonrenormalization properties of the two- and three-point functions of general chiral primary operators Tr X^k to order g^2, by an explicit computation in the (boundary) $\mathcal{N}=4$ SYM theory. A further calculation of a three-point function involving $\mathcal{N}=4$ descendents of the operator Tr X^2 to the same order gives evidence that nonrenormalization also holds for descendents of the chiral primary operators.

The essential steps in the method are

(i) We use an $\mathcal{N}=1$ description of the 6 real fields X^i as 3 complex z^i and their conjugates $\bar{z}^i$ in **3** and $\bar{\textbf{3}}$ representations of the manifest SU(3) subgroup of the original SU(4) flavor group. Without loss of generality, a choice of highest weight operators in the rank k symmetric tensor representations of SU(3) is made, which simplifies the flavor combinatorics.

(ii) The essential part of the problem is then reduced to that of color combinatorics which we handle by summing over all permutations of color generators in traces. Within each such permutation of z^i lines we sum over choices of pairs of lines carrying interaction vertices and choices of lines carrying self-energy insertions.

(iii) In Sec. II, we apply these steps to $\langle$Tr X^kTr $X^k\rangle$ and show that the sum of all interactions gives an order g^2 amplitude proportional to that for the case $k=2$ which vanishes by a known nonrenormalization theorem.

(iv) In Sec. III, we study the combinatorics of the three-point function $\langle$Tr X^kTr X^kTr $X^2\rangle$. The net sum of

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[1] Tr X^k is shorthand for Tr $X^{\{i_1}X^{i_2}\cdots X^{i_k\}}$, an operator in the rank k traceless symmetric tensor product of the fundamental real scalar fields X^i of the $\mathcal{N}=4$ SYM theory. These fields are SO(6) vectors in the adjoint representation of the SU(N) gauge group. It is convenient to introduce $X^i=T^aX^i_a$, where T^a is a Hermitian generator of the SU(N) fundamental, with normalization Tr $T^aT^b=\delta^{ab}/2$. The structure constants f^{abc} of SU(N) are normalized by $[T^a,T^b]=if^{abc}T^c$.

59 045008-1

interactions produces amplitudes which again vanish by nonrenormalization theorems for $k=2$.

(v) The analysis of the general three-point correlator $\langle \mathrm{Tr}\, X^{k_1} \mathrm{Tr}\, X^{k_2} \mathrm{Tr}\, X^{k_3}\rangle$ is somewhat more involved and is presented in Sec. IV.

Specifically, our results are as follows. The free field contributions to the above correlators are polynomials in N of leading order N^k for two-point functions and order $N^{(k_1+k_2+k_3)/2-1}$ for three-point functions. The cancellations of radiative corrections that we find to order g^2 occur for all N, not just leading order. Therefore, we confirm the results and speculations of [13] in their strongest form. Concretely, the origin of the cancellations to order g^2 may be traced back to the fact that only interactions which involve at most two X-lines appear, so that nonrenormalization theorems for $k=2$ can be used in the proof for general values of k. To order g^4 however, there are also interactions involving three X-lines, and the situation is far more complicated. There may exist special symmetries or properties of the $\mathcal{N}=4$ theory which lead to nonrenormalization theorems in higher order, but the only present evidence is the large N calculations of [13].

Another question of interest within the $\mathcal{N}=4$ SYM theory is whether the absence of radiative corrections for two and three-point functions of $\mathrm{Tr}\, X^k$ established by nonrenormalization theorems for $k=2$ or by calculation for general k remains true for all supersymmetric partners (descendents) of these operators. The case $k=2$ is of special interest because the multiplet of descendents includes the SU(4) flavor currents and the stress tensor. In this case the formalism of [17] predicts that there is a unique $\mathcal{N}=4$ superconformal correlator for this multiplet, so that radiative corrections of descendents should vanish as well. To check this we note that the second descendent of $\mathrm{Tr}\, X^2$ has the schematic form $S^{ij} = \mathrm{Tr}\, \lambda^i\lambda^j + g(\mathrm{Tr}\, X^3)^{ij}$, where $\mathrm{Tr}\, X^2$ transforms in the **20'** of SU(4) while S^{ij} transforms in the **10**. We discuss the specific form of S^{ij} in Sec. V and show that both free-field and order g^2 contributions to the correlator $\langle S^{ij}S^{kl}\mathrm{Tr}\, X^2\rangle$ vanish.

Although our method requires knowledge of the structure of the $\mathcal{N}=4$ theory only at a schematic level, we record here for reference the (Euclidean signature) Lagrangian in an $\mathcal{N}=1$ component notation (where L and R are chirality projectors):

$$\mathcal{L} = \Big[\tfrac{1}{4}F_{\mu\nu}^2 + \tfrac{1}{2}\bar{\lambda}\slashed{D}\lambda + \overline{D_\mu z^i}D_\mu z^i + \tfrac{1}{2}\bar{\psi}^i\slashed{D}\psi^i$$

$$+ i\sqrt{2}gf^{abc}(\bar{\lambda}_a\bar{z}_b^i L\psi_c^i - \bar{\psi}_a^i R z_b^i\lambda_c)$$

$$- \frac{Y}{2}f^{abc}(\epsilon_{ijk}\bar{\psi}_a^i L z_b^j\psi_c^k + \epsilon_{ijk}\bar{\psi}_a^i R\bar{z}_b^j\psi_c^k)$$

$$- \tfrac{1}{2}g^2(f^{abc}\bar{z}_b^i z_c^i)^2 + \frac{Y^2}{4}f^{abc}f^{ade}\epsilon_{ijk}\epsilon_{ilm}z_b^j z_c^k\bar{z}_d^l\bar{z}_e^m\Big].$$
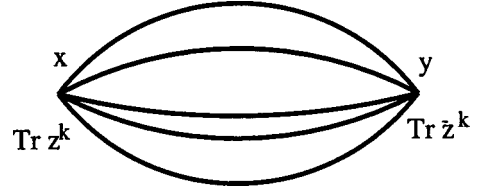
(1.1)

FIG. 1. Born graph for the $\langle \mathrm{Tr}(z^1)^k \,\mathrm{Tr}(\bar{z}^1)^k\rangle$ correlator.

We use separate coupling constants g and Y in order to distinguish interactions arising from the gauge and superpotential sectors. The relation $Y=\sqrt{2}g$ produces $\mathcal{N}=4$ supersymmetry.

II. THE CORRELATORS $\langle \mathrm{Tr}\, X^k\mathrm{Tr}\, X^k\rangle$

Without loss of generality, we may choose to evaluate the correlators of the SU(3) flavor highest weight field z^1 and its complex conjugate $\bar{z}^1$ only. This choice vastly simplifies the problem since flavor combinatorics becomes trivial and separates from color combinatorics. We therefore study the correlator $\langle \mathrm{Tr}(z^1)^k\mathrm{Tr}(\bar{z}^1)^k\rangle$ for which contributing Feynman diagrams are shown in Figs. 1 and 2. The free field contribution is a sum over permutations of ordering in the second operator trace relative to a fixed ordering in the first operator trace:

$$\langle \mathrm{Tr}(z^1)^k(x)\mathrm{Tr}(\bar{z}^1)^k(y)\rangle = G(x,y)^k P_{k,k,0}(N),$$

$$P_{k,k,0}(N) \equiv \sum_{\text{perms }\sigma} \mathrm{Tr}(T^{a_1}\cdots T^{a_k})\mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(k)}}),$$

(2.1)

where $G(x,y) = 1/[4\pi^2(x-y)^2]$ is the free scalar propagator, and the sum in the second line is over all permutations σ. The permutation sum of contracted traces defines the polynomial in N we have called $P_{k,k,0}(N)$. By cyclicity of the trace there are $(k-1)!$ classes of permutations each with k identical contributions. So $P_{k,k,0}(N)$ has an overall factor of k. The leading term comes from ''reverse order permutations'' $[\sigma(1)=k, \sigma(2)=k-1,$ etc.$]$ and is of order $P_{k,k,0}(N)\sim k(N/2)^k$, in agreement with [13].

From the Lagrangian one can easily see that there are two-particle interactions inside the ''rainbow'' coming from gauge boson exchange and D-term quartic vertices. (F-term vertices do not contribute.) The sum of these is a basic two particle interaction of the following structure:

FIG. 2. Order g^2 corrections to the two-point function.

$$(f^{pab}f^{pa'b'} + f^{pab'}f^{pa'b})B(x,y)G(x,y)^2 \qquad (2.2)$$

$$B(x,y) = b_0 + b_1 \ln(x-y)^2\mu^2$$

where a, a', b, b' indicate adjoint color indices. The indicated space-time dependence is the result of integrations which may be performed explicitly, but it turns out that we will not need the precise values of b_0, b_1. Self-energy corrections to order g^2 arise from two sources: a gauge boson exchange and a fermion loop. They are represented in the figures below by a dark dot, and may be expressed as follows:

$$= \delta^{aa'}NA(x,y)G(x,y) \qquad (2.3)$$

$$A(x,y) = a_0 + a_1 \ln(x-y)^2\mu^2.$$

The coefficients a_i, b_i are gauge dependent, but we know that the final color traced amplitude is gauge independent and that logarithms will also cancel, since the primary operators cannot have anomalous dimension.

We shall now show that the sum of the g^2 contributions to the two-point functions of Fig. 2 precisely cancel for all N and all k. The study of the color factors is facilitated by the following identity for any set of matrices M_i and N:

$$\sum_{i=1}^{n} \text{Tr}(M_1\cdots[M_i,N]\cdots M_n)=0. \qquad (2.4)$$

We will use this identity here and in following sections.

For each fixed permutation of final relative to initial z-lines we insert the two particle interaction structure Eq. (2.2) between all distinct pairs of lines. We then convert the two f symbol contractions to commutators within the final trace, and obtain the expression [omitting an overall factor of $G(x,y)^k$ throughout]

$$\frac{1}{4}(-2B)\text{Tr}(T^{a_1}\cdots T^{a_k})$$

$$\times \sum_{i\neq j=1}^{k} \text{Tr}(T^{a_{\sigma(1)}}\cdots[T^{a_{\sigma(i)}},T^p]\cdots[T^{a_{\sigma(j)}},T^p]\cdots T^{a_{\sigma(k)}}).$$

$$(2.5)$$

The factor of 1/4 is inserted because the expression would otherwise overcount pairs for two reasons: first by the expected factor of two for $\Sigma_{i\neq j}$ rather than $\Sigma_{i<j}$, and second since a given choice of the pair $\sigma(i), \sigma(j)$ occurs both in the fixed final trace permutation chosen and in the permutation with $\sigma(i)$ and $\sigma(j)$ interchanged. The $-$ sign appears because of the i's in $[T,T]=ifT$ and the factor of 2 because the two ff terms in Eq. (2.2) contribute equally.

We now use Eq. (2.4) on one of the commutators to rewrite Eq. (2.5) as

$$\frac{1}{2}B\ \text{Tr}(T^{a_1}\cdots T^{a_k})$$

$$\times \sum_{i=1}^{k} \text{Tr}(T^{a_{\sigma(1)}}\cdots[[T^{a_{\sigma(i)}},T^p],T^p]\cdots T^{a_{\sigma(k)}})$$

$$= \frac{1}{2}NB\ \text{Tr}(T^{a_1}\cdots T^{a_k})$$

$$\times \sum_{i=1}^{k} (\text{Tr}\ T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(i)}}\cdots T^{a_{\sigma(k)}}), \qquad (2.6)$$

where the last equality follows since $[[,T^p],T^p]$ is the Casimir operator in the adjoint representation, so that for any generator T^a we have $[[T^a,T^p],T^p]=NT^a$. The sum Σ_i is now over k identical terms and thus collapses to an overall factor k. The self-energy corrections also produce an obvious factor of k relative to the free field form. Adding interaction and self-energy contributions, we find the following result, including the sum over all permutations σ:

$$\frac{1}{2}kN(B+2A)\sum_{\text{perms }\sigma} \text{Tr}(T^{a_1}\cdots T^{a_k})\text{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(k)}}).$$

$$(2.7)$$

The nonrenormalization theorem for $k=2$ implies that $B+2A=0$, and this is enough to show that radiative corrections vanish for all values of k. Note that this argument does not require the explicit expressions for A and B.

III. THE CORRELATORS $\langle \text{Tr } X^k \text{Tr } X^k \text{Tr } X^2 \rangle$

As we noted in the previous section, flavor combinatorics is greatly simplified by considering the SU(3) flavor highest weight fields for the Tr X^k operators and a flavor nonsinglet for the operator Tr X^2. Without loss of generality, we therefore study the correlators $\langle \text{Tr}(z^1)^k\text{Tr}(\bar{z}^1)^k\text{Tr}(\bar{z}tz)\rangle$, where t is a diagonal SU(3) generator. This choice ensures that Tr$(\bar{z}tz)$ transforms as **8** of SU(3), which is a part of the **20'** of SU(4) and does not mix with the singlet. The free field contribution to this correlator is a sum of all possible pairings of fields into free propagators

$$\langle \text{Tr}(z^1)^k(x)\text{Tr}(\bar{z}^1)^k(y)\text{Tr}(\bar{z}tz)(w)\rangle$$

$$= G(x,y)^{k-1}G(x,w)G(y,w)P_{k,k,2}(N)t_{11},$$

$$P_{k,k,2}(N)\equiv k\sum_{\text{perms }\sigma} \text{Tr}(T^{a_1}\cdots T^{a_k})\text{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(k)}}).$$

$$(3.1)$$

The additional factor of k enters because of the distinguished line to the third vertex Tr$(\bar{z}tz)$. This line can appear at any position in the first trace, and may then be moved to first position using cyclicity. Since $P_{k,k,2}=kP_{k,k,0}(N)$, the leading term for large N is $P_{k,k,2}\sim k^2(N/2)^k$, in agreement with [13].

Explicit calculation of the order g^2 radiative corrections to Eq. (3.1) would not be necessary if, as might be expected [18,19], the mathematical form of the correlation function of $\mathcal{N}=1$ superfields $\langle S(z_1)\bar{S}(z_2)J(z_3)\rangle$ is uniquely determined by $\mathcal{N}=1$ superconformal symmetry as a function of the three superspace points $z_i=(x_i,\theta_i,\bar{\theta}_i)$, $i=1,2,3$. Here $S(z)$ is a chiral operator of dimension k, $\bar{S}(z)$ is its antichiral conjugate, and $J(z)$ is a flavor current superfield containing a conserved vector flavor current $J_\mu(x)$ as its $\theta\bar{\theta}$ component. A unique superconformal form means that all component correlators, among them $\langle\text{Tr}(z^1)^k\text{Tr}(\bar{z}^1)^k\text{Tr}(\bar{z}tz)\rangle$, contain a common coupling dependent factor. The same factor occurs in the two-point function $\langle\text{Tr}(z^1)^k\text{Tr}(\bar{z}^1)^k\rangle$ because of the Ward identity, and its order g^2 term would then be absent by the calculation of Sec. II. However, current understanding of superconformal correlators is still tentative, and we will
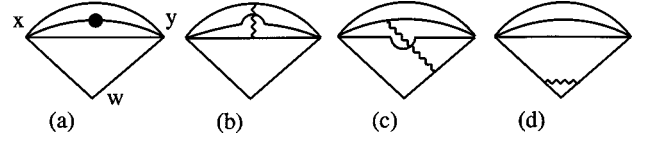

FIG. 3. Order g^2 corrections to the three-point function. In addition to gauge boson exchange diagrams (b), (c), and (d) there are analogous diagrams with quartic interactions.

therefore present an explicit calculation of the radiative corrections to $\langle\text{Tr }X^k\text{Tr }X^k\text{Tr }X^2\rangle$.

Previously described order g^2 interactions that contribute to the two-point function naturally appear in the computation of the corrections to the three-point function. In addition, there is an interaction that depends on all three space-time coordinates, which is of the form

$$(f^{pab}f^{pa'b'}+f^{pab'}f^{pa'b})C(x;y,w)G(x,y)G(x,w). \tag{3.2}$$

Again, the specific functional form of C will not be needed in the argument below.

We now analyze combinatorics and color factors present in different diagrams depicted in Fig. 3. The contribution of the self-energy diagrams is straightforward to evaluate. The sum of all such diagrams is

$$N[(k-1)A(x,y)+A(x,w)+A(y,w)], \tag{3.3}$$

where we indicated the space-time arguments of the function A defined in Eq. (2.3). As usual, we omit the part proportional to the Born contribution, which is identical for all diagrams.

Next we consider the insertion of the basic interaction (2.2) within the "rainbow" part of the correlator, as in the diagram of Fig. 3(b). We proceed as in Sec. II and insert the color factor into all pairs of lines i,j that belong to the "rainbow," and we replace the f symbols with commutators inside the second trace $\text{Tr}(T^{a\sigma(1)}\cdots T^{a\sigma(k)})$ in Eq. (3.1). We choose the index of the line to the lower vertex to be a_1 in the first trace and $a_{\sigma(k)}$ in the second, so we sum over $i\neq j=1,\ldots,k-1$. The sum over the "rainbow" lines is

$$-\frac{1}{2}\sum_{i\neq j=1}^{k-1}\text{Tr}(T^{a\sigma(1)}\cdots[T^{a\sigma(i)},T^p]\cdots[T^{a\sigma(j)},T^p]\cdots T^{a\sigma(k)})$$

$$=\frac{1}{2}\sum_{i=1}^{k-1}\text{Tr}(T^{a\sigma(1)}\cdots[[T^{a\sigma(i)},T^p],T^p]\cdots T^{a\sigma(k)})$$

$$+\frac{1}{2}\sum_{i=1}^{k-1}\text{Tr}(T^{a\sigma(1)}\cdots[T^{a\sigma(i)},T^p]\cdots[T^{a\sigma(k)},T^p]),$$

$$\tag{3.4}$$

where we have used Eq. (2.4). Using this in the second term as well as the identity $[[T^a,T^p],T^p]=NT^a$, we can express the above equation as

$$\frac{1}{2}(k-1)N\text{ Tr}(T^{a\sigma(1)}\cdots T^{a\sigma(k)})$$

$$-\frac{1}{2}\text{ Tr}(T^{a\sigma(1)}\cdots[[T^{a\sigma(k)},T^p],T^p])$$

$$=\frac{1}{2}(k-2)N\text{ Tr}(T^{a\sigma(1)}\cdots T^{a\sigma(k)}). \tag{3.5}$$

The last result has the same color factor as the Born term. After contracting with the trace for the vertex at x, and summing over permutations, one finds the previous color polynomial $P_{k,k,2}(N)$. Therefore this set of diagrams yields

$$\frac{1}{2}(k-2)NB(x,y)P_{k,k,2}(N). \tag{3.6}$$

The remaining order g^2 diagrams are obtained by inserting the interaction of Eq. (3.2) into the Born graph. The algebra is quite similar to the previous case. We first analyze diagrams depicted in Fig. 3(c). The color structure of the diagrams with gauge boson exchange between the single leg and the rainbow and their analogs with D-term quartic interactions is again obtained by replacing the f symbols with commutators inside the second trace. One of the commutators is attached to the distinguished line with the index $a_{\sigma(k)}$, the other corresponds to a line from the rainbow $a_{\sigma(i)}$ with $i=1,\ldots,k-1$:

$$-\sum_{i=1}^{k-1} \mathrm{Tr}(T^{a_{\sigma(1)}}\cdots[T^{a_{\sigma(i)}},T^p]\cdots[T^{a_{\sigma(k)}},T^p])$$

$$=\mathrm{Tr}(T^{a_{\sigma(1)}}\cdots[[T^{a_{\sigma(k)}},T^p],T^p])$$

$$=N\,\mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(k)}}). \tag{3.7}$$

Again, the final result is proportional to the Born graph thus the sum over all permutations is trivial. Both gluon exchange and quartic interactions from the D-term contribute to the diagrams of Fig. 3(c). Adding the diagrams obtained by symmetrizing in x,y, we obtain the result

$$2N[C(x;y,w)+C(y;x,w)]P_{k,k,2}(N). \tag{3.8}$$

The diagram depicted in Fig. 3(d) needs to be treated separately, because quartic interactions which contribute in this case come from both the D-term and the F-term. We denote the space-time function associated with the sum of gauge boson and quartic vertices by $C'(w;x,y)$. This is defined so that C' and C would coincide if F-term vertices are

excluded. All contributions to the diagram have the color factor N from a double contraction on ff and matrix element t^{11} as a flavor factor. The latter is included in the Born amplitude (3.1). It is now straightforward to see that the net contribution of the graph 3(d) is

$$-2NC'(w;x,y)P_{k,k,2}(N). \tag{3.9}$$

The nonrenormalization theorem for a three-point function with $k=2$ implies that

$$N[A(x,y)+A(x,w)+A(y,w)+2C(x;y,w)$$
$$+2C(y;x,w)-2C'(w;x,y)]=0. \tag{3.10}$$

We now combine this result with the relation $A+\frac{1}{2}B=0$, obtained from the nonrenormalization of the two-point function in the previous section to show that there are no order g^2 corrections to the $\langle\mathrm{Tr}\,X^k\,\mathrm{Tr}\,X^k\,\mathrm{Tr}\,X^2\rangle$ correlator. Specifically the sum of all interaction diagrams computed in Eqs. (3.3), (3.6), (3.8), and (3.9) is the Born term (3.1) multiplied by Eq. (3.11)

$$N[(k-1)A(x,y)+A(x,w)+A(y,w)+\tfrac{1}{2}(k-2)B(x,y)+2C(x;y,w)+2C(y;x,w)-2C'(w;x,y)]$$

$$=N\{[A(x,y)+A(x,w)+A(y,w)+2C(x;y,w)+2C(y;x,w)-2C'(w;x,y)]+(k-2)[A(x,y)+\tfrac{1}{2}B(x,y)]\}=0, \tag{3.11}$$

and this vanishes by the nonrenormalization theorems for all k and N.

IV. THE CORRELATORS $\langle\mathrm{Tr}\,X^{k_1}\,\mathrm{Tr}\,X^{k_2}\,\mathrm{Tr}\,X^{k_3}\rangle$

In this section we show that a general three-point function of $\mathrm{Tr}\,X^k$ does not receive corrections at order g^2. The basic ingredients of our arguments were outlined in two previous sections, but the algebra is more complicated for the case at hand. We first define the symmetric trace as follows:

$$\mathrm{Str}(T^{a_1}\cdots T^{a_k})\equiv\sum_{\mathrm{perms}\,\sigma}\frac{1}{k!}\,\mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(k)}}). \tag{4.1}$$

The Born diagram for the correlator $\langle\mathrm{Tr}\,X^{k_1}\,\mathrm{Tr}\,X^{k_2}\,\mathrm{Tr}\,X^{k_3}\rangle$ contains $\alpha_3=(k_1+k_2-k_3)/2$ lines connecting the operators $\mathrm{Tr}\,X^{k_1}$ and $\mathrm{Tr}\,X^{k_2}$, etc. It is convenient and sufficiently general to work with highest weight components of the first two operators and to choose the third operator

$$\mathrm{Tr}(\bar z t z)^{\alpha_1\alpha_2}=t_{i_1\cdots i_{\alpha_1};j_1\cdots j_{\alpha_2}}\mathrm{Str}(\bar z^{i_1}\cdots\bar z^{i_{\alpha_1}}z^{j_1}\cdots z^{j_{\alpha_2}}),$$

where t is any tensor which is symmetric in its i and j indices and traceless upon contraction of an i and j index. This ensures that the operator belongs to an irreducible SU(3) component of $\mathrm{Tr}\,X^{k_3}$, with $k_3=\alpha_1+\alpha_2$.

The free field contribution to the correlator we study is then

$$\langle\mathrm{Tr}(\bar z^1)^{k_1}(x)\mathrm{Tr}(z^1)^{k_2}(y)\mathrm{Tr}(\bar z t z)^{\alpha_1\alpha_2}(w)\rangle$$

$$=G(x,y)^{\alpha_3}G(y,w)^{\alpha_1}G(w,x)^{\alpha_2}P_{k_1,k_2,k_3}(N)t_{1\cdots1;1\cdots1},$$

$$P_{k_1,k_2,k_3}(N)=\frac{k_1!k_2!}{\alpha_3!}\,\mathrm{Str}(T^{a_1}\cdots T^{a_{\alpha_3}}T^{b_1}\cdots T^{b_{\alpha_2}})$$

$$\times\mathrm{Str}(T^{b_1}\cdots T^{b_{\alpha_2}}T^{c_1}\cdots T^{c_{\alpha_1}})$$

$$\times\mathrm{Str}(T^{c_1}\cdots T^{c_{\alpha_1}}T^{a_1}\cdots T^{a_{\alpha_3}}). \tag{4.2}$$

There are three classes of diagrams contributing at order g^2 to the correlator. These are self-energy insertions described in Eq. (2.3), gauge boson exchanges within each "rainbow" [Eq. (2.2)] and gauge boson exchanges from one rainbow to another [Eq. (3.2)]. The self-energy graphs are straightforward to evaluate. The sum of all such diagrams equals to

$$N[\alpha_3 A(x,y)+\alpha_2 A(x,w)+\alpha_1 A(y,w)]P_{k_1,k_2,k_3}(N), \tag{4.3}$$

where we omitted the propagator and flavor matrix factors of the Born amplitude.

As the next step we evaluate diagrams with the insertions of the gauge boson and D-term quartic interactions described in Eq. (3.2) between different rainbows. The color structure of these diagrams is

$$\sum_{i=1}^{\alpha_3}\sum_{j=1}^{\alpha_2}(f^{pc_ic_i'}f^{pb_jb_j'}+f^{pc_ib_j'}f^{pb_jc_i'})$$

$$\times\{\mathrm{Str}(T^{c_1}\cdots T^{c_i'}\cdots T^{c_{\alpha_3}}T^{b_1}\cdots T^{b_j'}\cdots T^{b_{\alpha_2}})\mathrm{Str}(T^{b_1}\cdots T^{b_{\alpha_2}}T^{a_1}\cdots T^{a_{\alpha_1}})\mathrm{Str}(T^{a_1}\cdots T^{a_{\alpha_1}}T^{c_1}\cdots T^{c_{\alpha_3}})\}$$

$$=-2\sum_{i=1}^{\alpha_3}\sum_{j=1}^{\alpha_2}\mathrm{Str}(T^{c_1}\cdots[T^{c_i},T^p]\cdots T^{c_{\alpha_3}}T^{b_1}\cdots[T^{b_j},T^p]\cdots T^{b_{\alpha_2}})\mathrm{Str}(T^{b_1}\cdots T^{b_{\alpha_2}}T^{a_1}\cdots T^{a_{\alpha_1}})\mathrm{Str}(T^{a_1}\cdots T^{a_{\alpha_1}}T^{c_1}\cdots T^{c_{\alpha_3}}).$$

$$(4.4)$$

We will now show that the color structure arising from insertions of the f symbols on any pairs of indices is identical.

$$\sum_{i=1}^{\alpha_3}\sum_{j=1}^{\alpha_2}\mathrm{Str}(T^{c_1}\cdots[T^{c_i},T^p]\cdots T^{c_{\alpha_3}}T^{b_1}\cdots[T^{b_j},T^p]\cdots T^{b_{\alpha_2}})\mathrm{Str}(T^{b_1}\cdots T^{b_{\alpha_2}}T^{a_1}\cdots T^{a_{\alpha_1}})\mathrm{Str}(T^{a_1}\cdots T^{a_{\alpha_1}}T^{c_1}\cdots T^{c_{\alpha_3}})$$

$$=\mathrm{Str}(T^{c_1}\cdots T^{c_{\alpha_3}}T^{b_1}\cdots T^{b_{\alpha_2}})\sum_{j=1}^{\alpha_2}\mathrm{Str}(T^{b_1}\cdots[T^{b_j},T^p]\cdots T^{b_{\alpha_2}}T^{a_1}\cdots T^{a_{\alpha_1}})\sum_{i=1}^{\alpha_3}\mathrm{Str}(T^{a_1}\cdots T^{a_{\alpha_1}}T^{c_1}\cdots[T^{c_i},T^p]\cdots T^{c_{\alpha_3}})$$

$$=-\mathrm{Str}(T^{c_1}\cdots T^{c_{\alpha_3}}T^{b_1}\cdots\cdots T^{b_{\alpha_2}})\sum_{k=1}^{\alpha_1}\mathrm{Str}(T^{b_1}\cdots T^{b_{\alpha_2}}T^{a_1}\cdots[T^{a_k},T^p]\cdots T^{a_{\alpha_1}})\sum_{i=1}^{\alpha_3}\mathrm{Str}(T^{a_1}\cdots T^{a_{\alpha_1}}T^{c_1}\cdots[T^{c_i},T^p]\cdots T^{c_{\alpha_3}})$$

$$=\mathrm{Str}(T^{c_1}\cdots T^{c_{\alpha_3}}T^{b_1}\cdots\cdots T^{b_{\alpha_2}})\mathrm{Str}(T^{b_1}\cdots T^{b_{\alpha_2}}T^{a_1}\cdots T^{a_{\alpha_1}})\sum_{i=1}^{\alpha_3}\sum_{k=1}^{\alpha_1}\mathrm{Str}(T^{a_1}\cdots[T^{a_k},T^p]\cdots T^{a_{\alpha_1}}T^{c_1}\cdots[T^{c_i},T^p]\cdots T^{c_{\alpha_3}}).$$

$$(4.5)$$

Using this last result we may now define the above quantity as a function $W_{k_1,k_2,k_3}(N)$, which is independent of the location of insertion of the two commutators. Adding all three sets of diagrams with interactions among two different ''rainbows,'' we obtain

$$-2W_{k_1,k_2,k_3}(N)[C(x;y,w)+C(y;x,w)-C(w;x,y)],\qquad(4.6)$$

since the color factors are identical for the contributions associated with each of the three vertices, except for a sign at the Tr X^{k_3} vertex because the operator contains both z and $\bar{z}$ fields.

The next step is the evaluation of diagrams with gauge boson exchanges within one ''rainbow.'' The color structure of such diagrams is of the form

$$-\frac{1}{2}\sum_{i\neq j=1}^{\alpha_2}\mathrm{Str}(T^{c_1}\cdots T^{c_{\alpha_3}}T^{b_1}\cdots[T^{b_i},T^p]\cdots[T^{b_j},T^p]\cdots T^{b_{\alpha_2}})\,\mathrm{Str}(\cdots)\mathrm{Str}(\cdots)$$

$$=\frac{1}{2}\left\{\sum_{i=1}^{\alpha_3}\sum_{j=1}^{\alpha_2}\mathrm{Str}(T^{a_1}\cdots[T^{a_i},T^p]\cdots T^{a_{\alpha_3}}T^{b_1}\cdots[T^{b_j},T^p]\cdots T^{b_{\alpha_2}})\right.$$

$$\left.+\sum_{j=1}^{\alpha_2}\mathrm{Str}(T^{a_1}\cdots T^{a_{\alpha_3}}T^{b_1}\cdots[[T^{b_j},T^p],T^p]\cdots T^{b_{\alpha_2}})\right\}\mathrm{Str}(\cdots)\mathrm{Str}(\cdots)$$

$$=\frac{1}{2}W_{k_1,k_2,k_3}(N)+\frac{1}{2}N\alpha_2 P_{k_1,k_2,k_3}(N).\qquad(4.7)$$

Finally, we evaluate the contribution of the quartic F-term vertex in the Lagrangian (1.1). There are nonvanishing Wick-contractions with one z and one $\bar{z}$ field of the operator Tr X^{k_3}, and one line is attached to each of the two other vertices. One can see by contracting the ϵ symbols in the F-term of (1.1), using the tracelessness of t, that this contribution also has the flavor factor $t_{1\cdots1;1\cdots1}$. The space-time function associated with the F-term will combine with gauge boson and D-term contributions at the Tr X^{k_3} vertex to give the same function $C'(w;x,y)$ used in our discussion of the $k-k-2$ correlator in Sec. III. It remains to determine the color structure of this contribution, which is again obtained from the quartic Lagrangian vertex. The result is

$$\frac{1}{2} \sum_{i=1}^{\alpha_1} \sum_{j=1}^{\alpha_2} \text{Str}(T^{a_1} \cdots T^{a_i} \cdots T^{a_{\alpha_1}} T^{b_1} \cdots T^{b_j} \cdots T^{b_{\alpha_2}}) f^{pa'_i b_j} f^{pb'_j a_i}$$

$$\text{Str}(T^{c_1} \cdots T^{c_{\alpha_3}} T^{b_1} \cdots T^{b'_j} \cdots T^{\alpha_2}) \text{Str}(T^{c_1} \cdots T^{\alpha_3} T^{a_1} \cdots T^{a'_i} \cdots T^{\alpha_1}). \tag{4.8}$$

Replacing $f^{pa'_i b_j} T^{b_j} = -i[T^p, T^{a'_i}]$ and $f^{pb'_j a_i} T^{a_i} = -i[T^p, T^{b'_j}]$, we recognize this double sum as proportional to the quantity $W_{k_1,k_2,k_3}(N)$, introduced earlier.

We can now combine all contributions together to obtain

$$N[\alpha_3(A + \tfrac{1}{2}B)(x,y) + \alpha_2(A + \tfrac{1}{2}B)(x,w) + \alpha_1(A + \tfrac{1}{2}B)(y,w)]P_{k_1,k_2,k_3}(N)$$

$$-[2(C(x;y,w) + C(y;x,w) - C'(w;x,y)) - \tfrac{1}{2}(B(x,y) - B(x,w) - B(y,w))]W_{k_1,k_2,k_3}(N). \tag{4.9}$$

This can be seen to vanish when the nonrenormalization theorems used in previous sections are applied, specifically when $A + \frac{1}{2}B = 0$ is combined with Eq. (3.11). This result shows that order g^2 corrections cancel for arbitrary N in all three-point functions of the operators $\text{Tr}\, X^k$.

V. A DESCENDENT CORRELATOR

It is commonly believed that the three-point correlators of chiral primary operators determine those of descendents. For $\mathcal{N}=1$ SUSY the form of superconformal correlators has been discussed by [20,18,19]. If there is a unique superconformal "tensor" form for a given superfield correlator, then the correlation functions of all components are determined. For $\mathcal{N}=4$ SUSY no off-shell superspace (or auxiliary field) formalism is known, but there is an on-shell formalism [17] which predicts a unique three-point supercorrelator for the multiplet containing the primary $\text{Tr}\, X^2$ and descendents. Other useful results have been deduced within this formalism [21], but the rules of applicability are not completely clear (at least to us), so it is desirable to check its predictions. Absence of radiative corrections is known for the three-point function of flavor currents [6] and for the stress tensor [22], and we shall study here an amplitude involving the second descendent of $\text{Tr}\, X^2$.

We shall discuss this descendent from an $\mathcal{N}=1$ viewpoint. For a chiral multiplet $(z, L\psi, F)$, where ψ is a Majorana spinor and L is the left chiral projector, the second descendent of z^2 is the operator $S = \bar{\psi}L\psi - 2zF$. For the Lagrangian (1.1) the $\mathcal{N}=1$ auxiliary fields are $F_i^a = \frac{1}{2}Yf^{abc}\epsilon_{ijk}z_b^j z_c^k$. This defines the descendent operator

$$S^{ij} = \bar{\psi}_a^i L\psi_a^j - \tfrac{1}{2}Yf^{abc}\epsilon^{mn\{i}z_a^{j\}}\bar{z}_b^m \bar{z}_c^n, \tag{5.1}$$

which transforms in the **6** representation of SU(3) flavor, which is contained in the decomposition of the **10** of SU(4), whose components can be denoted by S^{ij}, $i,j=1,\ldots,4$. We will study the correlator $\langle S^{ij}S^{kl}\text{Tr}\, X^2 \rangle$ which vanishes in free field approximation but could have an order g^2 contribution. Indeed, there is one SU(4) invariant coupling of $\mathbf{10} \otimes \mathbf{10} \otimes \mathbf{20}'$ representations, so to check for the absence of radiative corrections, it is sufficient to choose the convenient set of components $\langle S^{44}(x)S^{11}(y)\text{Tr}(\bar{z}^1(w))^2 \rangle$. Here, S^{44} is the SU(4) partner of S^{ij} whose bifermion term is $\text{Tr}\, \bar{\lambda}L\lambda$, where λ is the gaugino of the $\mathcal{N}=1$ description. To find the correctly normalized triboson term we use the fact that the **10** representation is both the symmetric second rank tensor in the SU(4) description and the self-dual third rank tensor of the SO(6) description of flavor. We use an SO(6) Clifford algebra construction[2] to connect these descriptions. Omitting tedious details, the result for the descendent is

$$S^{44} = \bar{\lambda}_a L\lambda_a - \tfrac{1}{3}Yf^{abc}\epsilon_{ijk}z_a^i z_b^j z_c^k. \tag{5.2}$$

(Of course, to enforce $\mathcal{N}=4$ SUSY, we should set $Y = \sqrt{2}g$, but we postpone this until the end of the calculation.)

There are two diagrams for the two-loop correlator $\langle S^{44}(x)S^{11}(y)\text{Tr}(\bar{z}^1(w))^2 \rangle$, a nonplanar diagram with a fermion loop and a purely bosonic Born-like diagram. Diagrams are drawn in Fig. 4. The eight-dimensional integral in the nonplanar amplitude can be evaluated using the technique of conformal inversion [23,24], and the other amplitude is elementary. Attention to combinatoric factors is essential. The results are
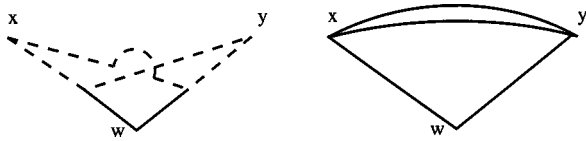

FIG. 4. Two graphs contributing to the $\langle S^{44}S^{11}\text{Tr}(\bar{z}^1)^2 \rangle$ correlator. Solid lines represent the boson propagators, broken ones fermion propagators.

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[2] In six Euclidean dimensions there are six Hermitean 8×8 gamma matrices Γ^i and a chirality matrix $\bar{\Gamma} = i\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\Gamma^6$. There is a symmetric charge conjugation matrix C, which anticommutes with $\bar{\Gamma}$. Let Γ^{ijk} denote the antisymmetric third rank tensor of the Clifford algebra. The $\Gamma^{ijk}C$ are symmetric matrices, while $(1 \pm \bar{\Gamma})\Gamma^{ijk}C$ are symmetric, self-dual, and effectively 4×4. The contraction of these matrices with $\text{Tr}\, X^iX^jX^k$ produces a symmetric second rank SU(4) tensor. The relative normalization of the bosonic admixtures in Eqs. (5.1) and (5.2) may be obtained from a specific construction of these matrices.

$$\text{nonplanar} = -g^2 N(N^2-1)G(x,y)^2 G(x,w)G(y,w), \tag{5.3}$$

$$\text{bosonic} = \tfrac{1}{2} Y^2 N(N^2-1)G(x,y)^2 G(x,w)G(y,w). \tag{5.4}$$

We see that these two graphs cancel one another only when the $\mathcal{N}=4$ relation $Y=\sqrt{2}g$ is imposed, so this is clearly not a consequence of just $\mathcal{N}=1$ SUSY.

One should note that each graph is itself the lowest order contribution to a three-point correlator of gauge invariant operators in the $\mathcal{N}=4$ SYM theory. In each amplitude only two of the three operators have scale dimension protected by R symmetry. These correlators are thus not accessible from AdS$_5$ calculations with supergravity or Kaluza-Klein fields. Perhaps they can be computed using appropriate string modes and interactions.

VI. CONCLUSIONS

The main result of this paper is that order g^2 radiative corrections to the correlators $\langle \text{Tr } X^k \text{Tr } X^k \rangle$ and $\langle \text{Tr } X^{k_1} \text{Tr } X^{k_2} \text{Tr } X^{k_3} \rangle$ cancel to all orders in N. This confirms the $N \to \infty$ fixed large $g^2 N$, AdS$_5$ supergravity calculation of Ref. [13] and the additional conjecture and speculation made in that paper concerning finite N behavior. The result goes beyond known nonrenormalization theorem for two- and three-point functions of Tr X^2 and thus suggests an unsuspected additional simplicity or symmetry of the $\mathcal{N}=4$ SYM theory.

Our method used careful combinatorics to reduce the interaction terms of general Tr X^k correlators to multiples of those of Tr X^2, which vanish by nonrenormalization theorems. It seems unlikely that the method can be applied in higher order. For example in order g^4 there are interactions which involve the exchange of two gluons among three adjacent scalar lines of the ''rainbow'' of Fig. 1. It does seem clear, though, that the method and the order g^2 result are valid for the $\mathcal{N}=4$ theory with any gauge group.

The Montonen-Olive duality of $\mathcal{N}=4$ SYM theory relates weak and strong coupling behavior of the theory at fixed N. The gauge invariant correlators we study are invariant under the S-duality group so the absence of order g^2 radiative corrections for small g implies that there are no order $1/g^2$ terms in the strong coupling expansion. This is valid for all N and therefore consistent with but stronger than the conclusions of Ref. [9]. In this paper it was argued that the lowest order nonperturbative string corrections to the type IIB effective supergravity action are of order $\alpha'^3 R^4$ where α' is the string tension and R^4 denotes a quartic contraction of the ten-dimensional curvature tensor. This correction term therefore makes no contribution to two- and three-point boundary correlators of the graviton and its supersymmetric partners, including Kaluza-Klein modes, in the AdS$_5 \times S_5$ theory [25]. Using the Maldacena correspondence, a cubic correction of the form $\alpha'^2 R^3$ would produce order $1/g^2 N$ corrections as $N \to \infty$ in the correlators $\langle \text{Tr } X^{k_1} \text{ Tr } X^{k_2} \text{ Tr } X^{k_3} \rangle$.

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