Perturbative Gross-Neveu model coupled to a Chern-Simons field: A renormalization group study

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In 2+1 dimensions, for low momenta, using dimensional renormalization we study the effect of a Chern-Simons field on the perturbative expansion of fermions self-interacting through a Gross-Neveu coupling. For the case of just one fermion field, we verify that the dimension of operators of canonical dimension lower than three decreases as a function of the Chern-Simons coupling. [S0556-2821(98)07224-5]

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I. INTRODUCTION

Effective field theories are a subject of great interest in theoretical physics not only due to their potential applications but also because they provide new insight into the way we look at field theories [1]. From this perspective nonrenormalizable models have acquired a new status as they may become physically relevants at low energies [2]. The point is that, if the scale of energy one is interested is low enough, the ambiguities due to the virtual states of high energy do not show up or, equivalently, are not meaningful. On the energy interval where this happens the theory proceeds as a usual renormalizable one. Nonetheless, as observed in [3], the use of a mass-independent regularization is almost mandatory to guarantee that high order counterterms can be effectively neglected.

The ultraviolet behavior of the Green functions may be changed by a rearrangement of the perturbative series. In fact, the incorporation of vacuum polarization effects in general improves the convergence properties of the resumed series; this mechanism is well known to be operative in the context of the $1/N$ expansion. In particular, Gross-Neveu- [4] or Thirring- [5] like four-fermion interactions which in 2+1 dimensions are perturbatively nonrenormalizable become renormalizable within the framework of the $1/N$ expansion [4]. This result has motivated a series of investigations on the properties of these theories [6]. In particular, using renormalization group (RG) methods, it has been proved that the N-component Gross-Neveu model in 2+1 dimensions is infrared stable at low energies but has also a nontrivial ultraviolet stable fixed point. These facts indicate that the theory could be perturbatively investigated if the momentum is low enough. This actually would be the only remaining possibility for small N.

It has recently been conjectured that in 2 +1 dimensions, besides the $1/N$ expansion, there is another way to improve the ultraviolet behavior of Feynman amplitudes. By coupling fermion fields to a Chern-Simons (CS) field the scale dimension of field operators could be lowered, possibly turning nonrenormalizable interactions into renormalizable or, better, super-renormalizable ones. Using a sharp cutoff to regulate divergences, this idea was tested in [7] where the effect of the CS field over massless self-coupled fermions with a quartic, Gross-Neveu-like, interaction was studied.

In this paper we will pursue this study further by considering massive fermions and adopting dimensional renormalization [8] as a tool to render finite the Feynman amplitudes. In this way we evade the ambiguity problem associated with the routing of the momentum flowing through the associated Feynman graphs [9]. Nevertheless, it should be stressed that our calculations are valid insofar, as said above, the effect of the higher order counterterms can be neglected. Otherwise, new couplings should be introduced. Our investigations, restricted to the case of fermions of just one flavor, i.e., $N=1$, show that, differently to what happens for large N, the renormalization group beta function has only a trivial infrared stable fixed point. Moreover, the operator dimensions of the basic field and composites of canonical dimension lower than 3 are monotonic decreasing functions of the CS parameter. This indicates that the Feynman amplitudes have a better ultraviolet behavior if the underlying theory is renormalizable. However, no improvement in the ultraviolet behavior seems to occur if the composite operators have canonical dimension bigger than 3.

Our work is organized as follows. In Sec. II some basic properties of the model as Feynman rules, ultraviolet behavior of Feynman diagrams, and comments on the regularization procedure are presented. The derivation and calculation of the renormalization group parameters are indicated in Sec. III. Section IV contains a discussion of our results as well as our conclusions. Details of the calculations of the pole part of the relevant amplitudes are described in Appendixes A and B.

II. QUARTIC INTERACTION

We consider a self-interacting two-component spinor field minimally coupled to a CS field. The Lagrangian density is given by

$$\mathcal{L}=\frac{1}{2\pi\alpha}\epsilon^{\mu\nu\alpha}\partial_\mu A_\nu A_\alpha+\bar{\psi}(i\partial\!\!\!/-m)\psi+\bar{\psi}\gamma^\mu\psi A_\mu -G(\bar{\psi}\psi)(\bar{\psi}\psi)+\frac{1}{2\lambda}(\partial_\mu A^\mu)^2. \tag{2.1}$$

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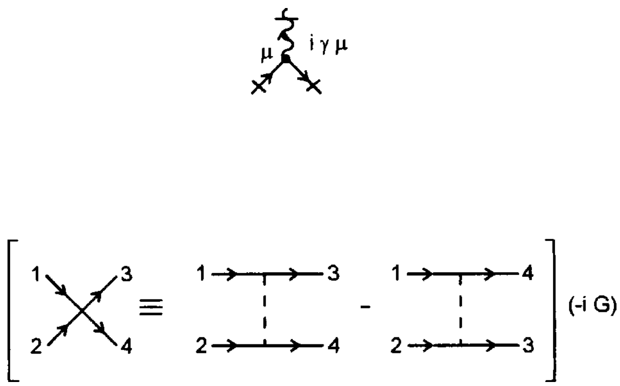

FIG. 1. Feynman rules for the interaction vertices. Solid and wavy lines represent the fermion and vector propagators, respectively.

The Dirac field ψ_α represents particles and antiparticles of spin up and the same mass m (the parameter m is to be taken positive) [10]. The Gross-Neveu (GN) term in Eq. (2.1) is the most general Lorentz-covariant quartic self-interaction, for the Thirring-like vector interaction is not independent but satisfies $:(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi):=-3:(\bar{\psi}\psi)(\bar{\psi}\psi):$. λ is a gauge-fixing parameter but for simplicity we will always work in the Landau gauge, formally obtained by letting $\lambda\to 0$. In this gauge, the Green functions may be computed using the Feynman rules depicted in Fig. 1. For convenience, we have introduced auxiliary dotted lines, hereafter called auxiliary GN lines, to clarify the structure of the four-fermion vertex.

Divergences show up, the degree of superficial divergence of a generic graph γ being

$$d(\gamma)=3-N_A-N_F+V, \tag{2.2}$$

where N_A and N_F are the number of external lines associated with the propagators for the Chern-Simons and the fermion fields, respectively; V denotes the number of quartic vertices in γ. The model is of course nonrenormalizable and the number of counterterms necessary to render the amplitudes finite increases with the order of perturbation but, to a given order, the number of counterterms is finite. To do calculations we will employ dimensional renormalization using a space-time dimension d. It is therefore convenient to introduce a dimensionless coupling g and a renormalization parameter μ through $G=(g/\Lambda)\mu^\epsilon$ and $\alpha\to\alpha\mu^\epsilon$, where $\epsilon=3-d$ must be set to zero at the end. The massive parameter Λ must be considered much bigger than any typical momenta and than the fermion mass m; it sets the scale which limits the region where our results are valid. Divergences will appear as poles in ϵ and a renormalized amplitude is given by the ϵ-independent term in the Laurent expansion of the corresponding regularized integral. However, at the one-loop level no infinities will remain after the remotion of the regulator. This is so because the poles for a graph γ may occur only at even values of the degree of superficial divergence of γ [11]. Moreover, it is easy to check that asymptotically, i.e., for zero external momenta, one-loop graphs with even degree of divergence are odd functions of the loop momentum and therefore vanish, after symmetric integration.

III. RENORMALIZATION GROUP

As known, Green functions of renormalizable models, which have been made finite by the subtraction of pole terms in the dimensionally regularized amplitudes, satisfy a 't Hooft–Weinberg-type renormalization group equation [12]. Nonrenormalizable models require special consideration since the form of the effective Lagrangian changes with the order of perturbation. However, at sufficiently small momenta, such that the effect of the new counterterms may be neglected, the Green functions will still approximately satisfy the RG equation. Thus, although being nonrenormalizable, for small enough momenta, the Green function of the theory (2.1) satisfies the following renormalization group equation:

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\mu\frac{\partial}{\partial\mu}+\delta m\frac{\partial}{\partial m}+\beta\frac{\partial}{\partial g}-N\gamma\right]\Gamma^{(N)}(p_1,\ldots,p_N)\approx 0, \tag{3.1}$$

where $\Gamma^{(N)}(p_1,\ldots,p_N)$ denotes the vertex function of N fermion fields (since A_μ is not a dynamical field, we shall not consider vertex functions having external vector fields). The symbol $\approx$ means equality in the region where all counterterms different from those terms already present in Eq. (2.1) can be neglected. For example, order-$G\alpha$ contributions to the two-point function are quadratically divergent and would require a counterterm proportional to p^2 besides the usual mass and wave function renormalizations. For momenta not small this would modify the right hand side of Eq. (3.1). However, if $p\ll\Lambda$, this additional term can be safely neglected.

As a consequence of the Coleman-Hill theorem, which states that all radiatives corrections to the CS term are finite [13], the *beta* function for the CS coupling α vanishes identically; that explains why the term with a derivative with respect to α is absent from (3.1).

The coefficients δ, β, and γ in Eq. (3.1) may be obtained by formally computing the action of the differential operator over the two- and four-point Green functions. For the two-point function, up to second order in the coupling constants, we have

$$\Gamma^{(2)}(p)=i(\not{p}-m)+\frac{g}{\Lambda}\mu^\epsilon I_1^{(2)}+\alpha\mu^\epsilon I_2^{(2)}+\alpha^2(1-\mathcal{T})\mu^{2\epsilon}I_3^{(2)} + \frac{g\alpha}{\Lambda}(1-\mathcal{T})\mu^{2\epsilon}I_4^{(2)}+\frac{g^2}{\Lambda^2}(1-\mathcal{T})\mu^{2\epsilon}I_5^{(2)}, \tag{3.2}$$

where the limit $\epsilon\to 0$ must be understood. In the above expression I_i, $i=1,\ldots,5$, denote the regularized Feynman amplitudes. In particular, the graphs ascribed to I_3, I_4, and I_5 have been depicted in Figs. 2, 3 and 4, respectively; $\mathcal{T}$ is an operator to remove the pole term in the amplitudes to

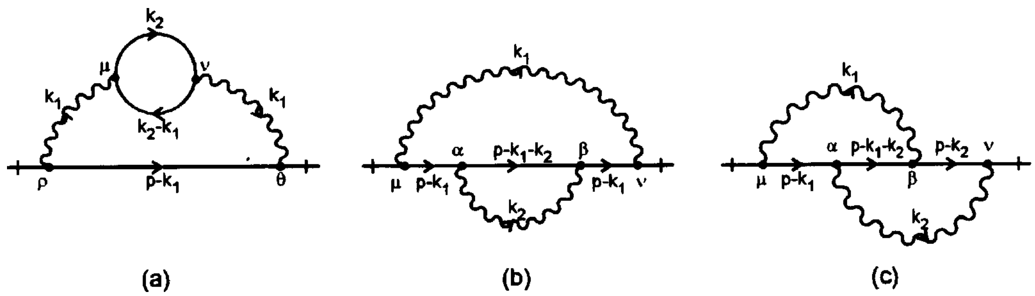

FIG. 2. Order-α^2 graphs contributing to the two-point function.

which it is applied. As mentioned earlier, the amplitudes $I_1^{(2)}$ and $I_2^{(2)}$ which are associated with one-loop diagrams are finite. Inserting Eq. (3.2) into Eq. (3.1) allows us to determine the coefficients δ and γ as follows. Initially notice that as Λ enters into the perturbative expansion only in the combination g/Λ, fixing β in lowest order as being equal to g eliminates all contributions of the term with the derivative with respect to Λ in Eq. (3.1). After that, up to the order we will study, in Eq. (3.1) there will be no mixing of higher order contribution to β with those to γ and δ.

Using the expansions

$$\delta = \sum_{i,j} \delta_{i,j} g^i \alpha^j, \tag{3.3}$$

$$\gamma = \sum_{i,j} \gamma_{i,j} g^i \alpha^j, \tag{3.4}$$

where the sum is restricted to $i+j \leqslant 2$, we get

$$\delta_{1,0} = \delta_{0,1} = \gamma_{1,0} = \gamma_{0,1} = 0, \tag{3.5}$$

$$\delta_{0,2} = -2i(A_3 + B_3), \quad \gamma_{0,2} = -iB_3, \tag{3.6}$$

$$\delta_{1,1} = -\frac{2m}{\Lambda}(A_4 + B_4), \quad \gamma_{1,1} = -\frac{m}{\Lambda}B_4, \tag{3.7}$$

$$\delta_{2,0} = \frac{2im^2}{\Lambda^2}(A_5 + B_5), \quad \gamma_{2,0} = \frac{im^2}{\Lambda^2}B_5, \tag{3.8}$$

where A_i and B_i, for $i = 3,4,5$, are defined by writing the pole term for the amplitude $I_i^{(2)}$ as

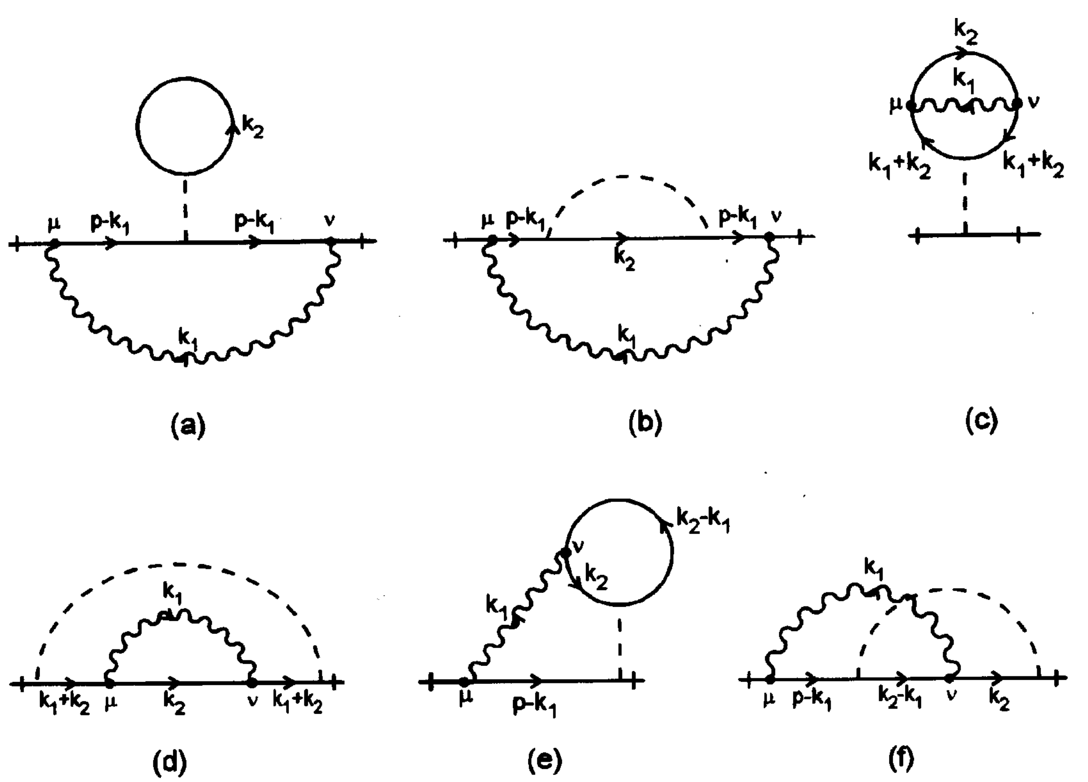

FIG. 3. Fermionic self-energy graphs of order $g\alpha$.

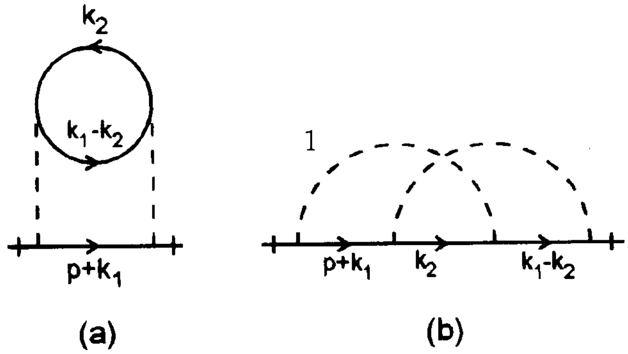

FIG. 4. Order-g^2 fermionic self-energy graphs.

$$\text{pole term of } I_3^{(2)} = (mA_3 + \not{p}B_3)\frac{1}{\epsilon}, \tag{3.9}$$

$$\text{pole term of } I_4^{(2)} = -i[m^2A_4 + m\not{p}B_4 + O(p^2)]\frac{1}{\epsilon}, \tag{3.10}$$

$$\text{pole term of } I_5^{(2)} = -[m^3A_5 + m^2\not{p}B_5 + O(p^2)]\frac{1}{\epsilon}. \tag{3.11}$$

Appendix A presents a detailed analysis of the various contributions to these parameters. From Eqs. (A19), (A22), and (A24), the final result is

$$\delta = -\frac{8}{3}\alpha^2 - \frac{11}{8\pi}\frac{m}{\Lambda}g\alpha + \frac{7}{12\pi^2}\frac{m^2}{\Lambda^2}g^2, \tag{3.12}$$

$$\gamma = -\frac{1}{12}\alpha^2 - \frac{1}{8\pi}\frac{m}{\Lambda}g\alpha + \frac{5}{48\pi^2}\frac{m^2}{\Lambda^2}g^2. \tag{3.13}$$

Letting $m\to 0$, we note that our determination of γ agrees with [7]. In the region $m \ll \Lambda$, however, our result presents corrections for nonvanishing fermion mass.

To fix β we look now at the four-point Green function, which up to third order is (we omit contributions which by power counting are finite)

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4)$$

$$= \mu^{\epsilon}\Bigg(-i\frac{g}{\Lambda} + \frac{g\alpha}{\Lambda}\mu^{\epsilon}I_1^{(4)} + \frac{g^2}{\Lambda^2}\mu^{\epsilon}I_2^{(4)} + \frac{g}{\Lambda}\alpha^2(1-\mathcal{T})\mu^{2\epsilon}I_3^{(4)} + \frac{g^2\alpha}{\Lambda^2}(1-\mathcal{T})\mu^{2\epsilon}I_4^{(4)} + \frac{g^3}{\Lambda^3}(1-\mathcal{T})\mu^{2\epsilon}I_5^{(4)}\Bigg). \tag{3.14}$$

Here, again, the one-loop amplitudes $I_1^{(4)}$ and $I_2^{(4)}$ are finite because of the use of dimensional regularization. Inserting Eq. (3.14), the expansion $\beta = \Sigma_{i,j}\beta_{i,j}g^i\alpha^j$, and δ and γ given in Eqs. (3.4) and (3.3) into Eq. (3.1), we obtain $\beta_{1,0} = 1$, $\beta_{0,1} = \beta_{1,1} = \beta_{2,0} = \beta_{0,2} = 0$, and

$$\beta_{1,2} = -4iB_3 - 2C_3, \tag{3.15}$$

$$\beta_{2,1} = \frac{2im}{\Lambda}C_4 - \frac{4m}{\Lambda}B_4, \tag{3.16}$$

$$\beta_{3,0} = \frac{2m^2}{\Lambda^2}C_5 + \frac{4im^2}{\Lambda^2}B_5. \tag{3.17}$$

In the above expressions, C_i, for $i=3,4,5$, are related to the pole part of the amplitudes $I_i^{(4)}$ through

$$\text{pole part of } I_3^{(4)} = -iC_3/\epsilon, \tag{3.18}$$

$$\text{pole part of } I_4^{(4)} = -[mC_4 + O(p)]/\epsilon, \tag{3.19}$$

$$\text{pole part of } I_5^{(4)} = i[m^2C_5 + O(p)]/\epsilon. \tag{3.20}$$

In Appendix B we have collected the results of the calculations of the pole part of the relevant graphs. Using Eq. (B6) we obtain, finally,

$$\beta = g + \frac{20}{3}g\alpha^2 - \frac{21m}{2\pi\Lambda}g^2\alpha + \frac{161m^2}{12\pi^2\Lambda^2}g^3. \tag{3.21}$$

IV. DISCUSSION AND CONCLUSIONS

An inspection of Eq. (3.21) shows that the renormalization group *beta* function has $g=0$ as a fixed point. As, for $m \ll \Lambda$, $\beta \approx g(1+20\alpha^2/3)$, the origin is an infrared stable fixed point. Actually, this is the only existing fixed point. Here we are in disagreement with Ref. [7] where a line of fixed points was found. The diverse conclusions are perhaps due to the use of different regularizations but a more direct comparison of the methods seems infeasible as the calculations in [7] were not spelled out.

We will examine now the dimensions of some operators. As seen before the basic field ψ has operator dimension $d_\psi = 1-\alpha^2/12$, and so at $g=0$ the Green functions of the fermion field have an improved ultraviolet behavior as α increases. Similar results are obtained if one considers composite operators of canonical dimension less than 3. The simplest of them, the mass operator $\bar{\psi}\psi$, has an anomalous dimension given by

$$\gamma_{\bar{\psi}\psi} = 2\gamma - 2R_{\text{res}}, \tag{4.1}$$

where R_{res} is the residue coming from graphs contributing to the vertex function with the insertion of the mass operator $\bar{\psi}\psi$ and having two external fermionic lines. For practical purposes this residue may be computed by taking the mass derivative of the contributions calculated in item (1) of Appendix A. The result is $iA_3 = 5/4$. Thus the dimension of $\bar{\psi}\psi$ turns out to be equal to

$$d_{\bar{\psi}\psi}=2-\frac{8}{3}\alpha^2. \tag{4.2}$$

From the computation of the anomalous dimension of the operator ψ one could easily obtain the dimension of $\bar{\psi}\,\partial\!\!\!/\psi$, in the case $m=0$. Indeed, recalling that the integrated operator counts the number of internal fermionic lines, we get that the dimension of $\bar{\psi}\,\partial\!\!\!/\psi$ is given by (4.1) but with the replacement of $R_{\rm res}$ by $3iB_3=1/4$. Thus

$$d_{\bar{\psi}\,\partial\!\!\!/\psi}=3-2/3\alpha^2. \tag{4.3}$$

The determination of the anomalous dimension of the operator $(\bar{\psi}\psi)^2$ at $g=0$ is more complicated due to the fact that renormalization in general produces a mixing with other operators of dimension lower or equal to 4. However, if we restrict the calculation to the $m=0$ case, as we will do, only operators of dimension 4 need to be considered. A further simplification is obtained by considering only (formally) integrated operators. We have

$$\int d^3xN[(\bar{\psi}\psi)^2]=a_1\int d^3x(\bar{\psi}\psi)^2+a_2\int d^3x\bar{\psi}\partial^2\psi, \tag{4.4}$$

$$\int d^3xN[\bar{\psi}\partial^2\psi]=b_1\int d^3x(\bar{\psi}\psi)^2+b_2\int d^3x\bar{\psi}\partial^2\psi, \tag{4.5}$$

where the symbol N indicates a normal product prescription corresponding to the subtraction of the pole terms. A direct calculation gives that $a_2=b_1=0$, $a_1=1+C_3\alpha^2/\epsilon$ and $b_2=1-\alpha^2/3\epsilon$. A straightforward analysis shows now that the dimensions of $\int d^3xN[(\bar{\psi}\psi)^2]$ and $\int d^3xN[\bar{\psi}\partial^2\psi]$ are given by

$$4+4\gamma_{02}\alpha^2-2C_3\alpha^2=4+\frac{20}{3}\alpha^2, \tag{4.6}$$

$$4+2\gamma_{02}\alpha^2+\frac{2}{3}\alpha^2=4+\frac{\alpha^2}{2}, \tag{4.7}$$

respectively. One sees that, at least for the operators that we explicitly considered, the operator dimension decreases with α if the canonical dimension is lower or equal to 3, in accordance with [7]. However, if the canonical dimension is bigger than 3, the operator dimension increases with α so that no improvement for nonrenormalizable interactions results.

Our results are valid if the basic fermion field is flavorless. The N-flavor case is presently under investigation.

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APPENDIX A

In this appendix we shall present a detailed analysis of the contributions to the pole part of the two-point vertex function. Although there is some latitude in the choice of the rules which define the regularization scheme, in this work we have adopted an specific way of implementing the dimensional regularization. Initially, using the identities

$$\gamma^\mu\gamma^\nu=g^{\mu\nu}-i\epsilon^{\mu\nu\rho}\gamma_\rho \tag{A1}$$

and

$$\epsilon^{\mu\nu\rho}\epsilon_{\rho\sigma\lambda}=\delta^\mu_\sigma\delta^\nu_\lambda-\delta^\mu_\lambda\delta^\nu_\sigma, \tag{A2}$$

the Feynman integrands were simplified directly in $2+1$ dimensions. After this step, the integrals were promoted to d dimensions and performed by the usual rules. Because of the use of dimensional regularization, the poles only appear at the two-loop level, beginning at second order in the coupling constants. We will examine separately each order of perturbation, i.e., the orders α^2, $g\alpha$, and g^2. We have the following.

(1) Order α^2. In this order there are three diagrams which are shown in Fig. 2. These diagrams give the contributions

Figure 2(a):

$$I_3(a)=4\pi^2i\epsilon_{\rho\mu\lambda}\epsilon_{\nu\theta\alpha}\mathcal{T}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}k_1^\lambda k_1^\alpha\mathrm{Tr}[\gamma^\mu(k\!\!\!/_2+m)\gamma^\nu(k\!\!\!/_2-k\!\!\!/_1+m)]$$
$$\times\frac{[\gamma^\rho(p\!\!\!/-k\!\!\!/_1+m)\gamma^\theta]}{(k_2^2-m^2)[(k_2-k_1)^2-m^2][(p-k_1)^2-m^2](k_1^2)^2}=[mA_3(a)+p\!\!\!/B_3(a)]\frac{1}{\epsilon}, \tag{A3}$$

where, on the right hand side of the first equality, we have introduced the operator $\mathcal{T}$ to extract the pole part of the expression to which it is applied. From Eq. (A3) we obtain

$$\frac{A_3(a)}{\epsilon}=\frac{1}{2m}\mathrm{Tr}I_3(a)\big|_{p=0}=\frac{2i\pi^2}{m}\mathcal{T}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{-8m^3k_1^2-8m(k_1^2)^2-8mk_1^2(k_1\cdot k_2)+8m(k_1\cdot k_2)^2}{(k_2^2-m^2)[(k_2-k_1)^2-m^2][k_1^2-m^2](k_1^2)^2}. \tag{A4}$$

The trace was taken in the integrand of Eq. (A3) and calculated directly at $d=3$. This, of course, does not affect the result for the pole part of the integrals. Analogously,

$$\frac{B_3(a)}{\epsilon}=\frac{1}{2p^2}\mathrm{Tr}[I_3(a)\not{p}]=\frac{2i\pi^2}{p^2}\mathcal{T}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{\text{numerator}}{(k_2^2-m^2)[(k_2-k_1)^2-m^2][(p-k_1)^2-m^2](k_1^2)^2}, \tag{A5}$$

where

$$\begin{aligned}\text{numerator}=&-16\epsilon_{\rho\mu\lambda}\epsilon_{\nu\theta\alpha}k_1^\rho k_2^\mu p^\lambda k_1^\nu k_2^\theta p^\alpha-16m^2k_1^2(k_1\cdot p)-8k_1^2(k_1\cdot k_2)(k_1\cdot p)+8k_1^2k_2^2p^2+8m^2(k_1\cdot p)^2\\&+8(k_1\cdot k_2)(k_1\cdot p)^2-8(k_1\cdot p)^2k_2^2-8(k_1\cdot k_2)^2p^2+8(k_1\cdot k_2)^2(k_1\cdot p).\end{aligned} \tag{A6}$$

Here and in what follows we shall adopt the following procedure for performing the integrals. We first consider the k_2 integral and use Feynman's trick,

$$\frac{1}{a_1^{\alpha_1}a_2^{\alpha_2}}=\frac{\Gamma[\alpha_1+\alpha_2]}{\Gamma[\alpha_1]\Gamma[\alpha_2]}\int_0^1dx\,\frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{[xa_1+(1-x)a_2]^{\alpha_1+\alpha_2}}, \tag{A7}$$

to reduce denominators containing the variable of integration to only one denominator. After integrating in k_2 we use again Feynman's formula (A7) to combine the denominators that depend on k_1. We then integrate over k_1 and, finally, perform the parametric integrations. In the present case, after integrating on k_2, we get

$$\frac{A_3(a)}{\epsilon}=\mathcal{T}\frac{\pi^{(2-d/2)}}{2^{d-3}}\int_0^1dx\int\frac{d^dk_1}{(2\pi)^d}\frac{1}{(k_1^2-m^2)k_1^2}\left[\Gamma\left(1-\frac{d}{2}\right)\Delta^{d/2-1}+\Gamma\left(2-\frac{d}{2}\right)k_1^2\Delta^{d/2-2}(1+x-x^2)\right] \tag{A8}$$

and

$$\begin{aligned}\frac{B_3(a)}{\epsilon}=&-\mathcal{T}\frac{1}{p^2}\frac{\pi^{(2-d/2)}}{2^{d-1}}\int_0^1dx\int\frac{d^dk_1}{(2\pi)^d}\frac{k_1\cdot p}{[(p-k_1)^2-m^2](k^2)^2}\\&\times\left[4\Gamma\left(1-\frac{d}{2}\right)[(d-2)(k_1\cdot p)-k_1^2]\Delta^{d/2-1}+8x(1-x)\Gamma\left(2-\frac{d}{2}\right)[(k_1\cdot p)k_1^2-(k_1^2)^2]\Delta^{d/2-2}\right],\end{aligned} \tag{A9}$$

where $\Delta=m^2-x(1-x)k_1^2$. Continuing our calculation, we would introduce two new parametric integrations as there are now three different denominators (we take $1/\Delta$ as a new denominator) depending on k_1 in each of the terms of the above expressions. However, as the result does not depend on m and we are looking only for the pole part of the amplitudes, we can speed up the calculation by modifying the dependence on m of some denominators. For example, without changing the final result, we can replace the first term on the right hand side of Eq. (A8) by

$$\mathcal{T}\frac{\pi^{(2-d/2)}}{2^{d-3}}\int_0^1dx\int\frac{d^dk_1}{(2\pi)^d}\frac{\Gamma\left(1-\frac{d}{2}\right)\Delta^{d/2-1}}{(k_1^2-m^2)^2}. \tag{A10}$$

Similarly, in the computation of $B_3(a)$ one can set $m=0$ in the expression for Δ so that one has to use only one parametric integral. Following this recipe, after integrating in k_1 we obtain $[a=1-y-yx(1-x)]$

$$\frac{A_3(a)}{\epsilon}=\mathcal{T}\frac{\pi^{2-d}}{2^{2d-3}}\Gamma[3-d]\int_0^1dx\int_0^1dy(1-y)\left[\frac{(1-2y)m^2}{a}\right]^{d-3}\left[\frac{y^{-d/2}}{a^{3-d/2}}+\frac{d}{2}(1+x-x^2)\frac{y^{1-d/2}}{a^{4-d/2}}\right]=-\frac{5i}{8\epsilon} \tag{A11}$$

and

$$\frac{B_3(a)}{\epsilon}=-\mathcal{T}\frac{\pi^{2-d}}{2^{2d-3}}\Gamma[3-d]\int_0^1 dx\int_0^1 dy\,\frac{(1-y)^{2-d/2}[ym^2-y(1-y)p^2]^{d-3}}{[x(x-1)]^{1-d/2}}$$
$$\times\left[\frac{(1-5y)\Gamma[1-d/2]}{\Gamma[3-d/2]}-2\frac{\Gamma[2-d/2]\Gamma[5-d/2]}{\Gamma[4-d/2]^2}(1-y)(1-7y)\right]=-\frac{i}{24\epsilon}. \tag{A12}$$

Now, for the remaining graphs of order α^2, we have
Figure 2(b):

$$I_3(b)=-4i\pi^2\epsilon_{\mu\nu\lambda}\epsilon_{\alpha\beta\rho}\mathcal{T}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}k_1^\lambda k_2^\rho\frac{[\gamma^\mu(\not p-\not k_1+m)\gamma^\alpha(\not p-\not k_1-\not k_2+m)\gamma^\beta(\not p-\not k_1+m)\gamma^\nu]}{[(p-k_1)^2-m^2]^2[(p-k_1-k_2)^2-m^2]k_1^2k_2^2}$$
$$=[mA_3(b)+\not pB_3(b)]\frac{1}{\epsilon}. \tag{A13}$$

Following the same steps as in previous case, we obtain

$$\frac{A_3(b)}{\epsilon}=\frac{1}{2m}\mathrm{Tr}I_3(b)\big|_{p=0}=-\frac{i}{4\epsilon}, \tag{A14}$$

$$\frac{B_3(b)}{\epsilon}=\frac{1}{2p^2}Tr(I_3(b)\not p)=-\frac{i}{12\epsilon}. \tag{A15}$$

Figure 2(c):

$$I_3(c)=-4i\pi^2\epsilon_{\mu\beta\lambda}\epsilon_{\alpha\nu\rho}\mathcal{T}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}k_1^\lambda k_2^\rho\frac{[\gamma^\mu(\not p-\not k_1+m)\gamma^\alpha(\not p-\not k_1-\not k_2+m)\gamma^\beta(\not p-\not k_2+m)\gamma^\nu]}{[(p-k_1)^2-m^2][(p-k_1-k_2)^2-m^2][(p-k_2)^2-m^2]k_1^2k_2^2}$$
$$=[mA_3(c)+\not pB_3(c)]\frac{1}{\epsilon}. \tag{A16}$$

The computation of this expression is a bit more complicated because one has to introduce three Feynman parametric integrals. The final result is, nevertheless, simple:

$$A_3(c)=-\frac{3i}{8}, \tag{A17}$$

$$B_3(c)=\frac{i}{24}. \tag{A18}$$

Collecting these results, we obtain

$$A_3=A_3(a)+A_3(b)+A_3(c)=-\frac{5i}{4},\quad B_3=B_3(a)+B_3(b)+B_3(c)=-\frac{i}{12}. \tag{A19}$$

(2) Order $g\alpha$ graphs. There are six diagrams which have been drawn in Fig. 3. They give the following.

Figures 3(a) and 3(b). Both diagrams have the structure of a product of two one-loop graphs. The corresponding dimensionally regularized amplitudes does not have a pole at $d=3$.

Figure 3(c). Actually, this diagram does not contribute because the corresponding analytic expression is finite. Indeed, we have

$$I_4(c)=\mathcal{T}\frac{\partial}{\partial m}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{\mathrm{Tr}[\gamma^\mu(\not{k}_2+m)\gamma^\nu(\not{k}_2+\not{k}_1+m)]\epsilon_{\mu\nu\rho}}{(k_2^2-m^2)[(k_1+k_2)^2-m^2]}\frac{k_1^\rho}{k_1^2}$$

$$=-4i\mathcal{T}\frac{\partial}{\partial m}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{m}{(k_2^2-m^2)[(k_1+k_2)^2-m^2]}=0, \tag{A20}$$

since the integral in the second equality has the structure of a product of two one-loop integrals.

Figure 3(d). The same reasoning can be applied to this situation since no external momentum flows through the diagram. We conclude that there is not a pole term.

Figure 3(e). By Furry's theorem this diagram cancels with its charge-conjugated partner.

Figure 3(f). We have the contribution

$$I_4(f)=4\pi\epsilon_{\mu\nu\lambda}\mathcal{T}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\gamma^\mu(\not{p}-\not{k}_1+m)k_1^\lambda\frac{(\not{k}_2-\not{k}_1+m)\gamma^\nu(\not{k}_2+m)}{[(p-k_1)^2-m^2](k_2^2-m^2)[(k_2-k_1)^2-m^2]k_1^2}$$

$$=-i\frac{[m^2A_4(f)+m\not{p}B_4(f)]}{\epsilon}, \tag{A21}$$

from which one obtains

$$A_4=A_4(f)=\frac{9}{16\pi},\quad B_4=B_4(f)=\frac{1}{8\pi}. \tag{A22}$$

(3) Order g^2 graphs. There are only the two diagrams shown in Fig. 4. We get

$$I_5=-4i\mathcal{T}\int\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_1}{(2\pi)^d}\frac{\not{p}+\not{k}_1+m}{[(p+k_1)^2-m^2](k_2^2-m^2)[(k_2-k_1)^2-m^2]}$$

$$\times\{(\not{k}_2+m)(\not{k}_2-\not{k}_1+m)-\mathrm{Tr}[(\not{k}_2+m)(\not{k}_2-\not{k}_1+m)]\}=-\frac{(m^3A_5+m^2\not{p}B_5)}{\epsilon}, \tag{A23}$$

where the two terms in the second equality refer to the graphs 4(a) and 4(b), respectively. After a lengthy calculation one determines

$$A_5=-\frac{3i}{16\pi^2},\quad B_5=-\frac{5i}{48\pi^2}. \tag{A24}$$

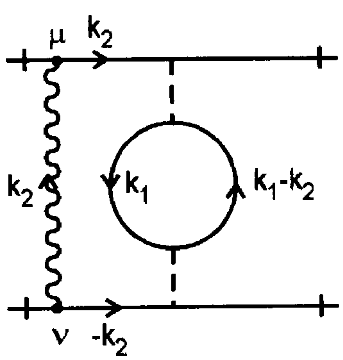

FIG. 5. Example of a two-loop diagram.

APPENDIX B

In this appendix we will discuss the calculation of the pole part of the four-point vertex function which is needed for fixing the renormalization group beta function. Actually, since all we need is the constant part of the residue, the calculation of the relevant graphs will be done at zero external momenta.

TABLE I. Pole part for four legs graphs with a closed fermionic loop.

Order of perturbation	Number of diagrams	Pole part
$g\alpha^2$	12	$-i/2\epsilon$
$g^2\alpha$	7	$-4i/\pi\epsilon$
g^3	8	$4i/\pi^2\epsilon$

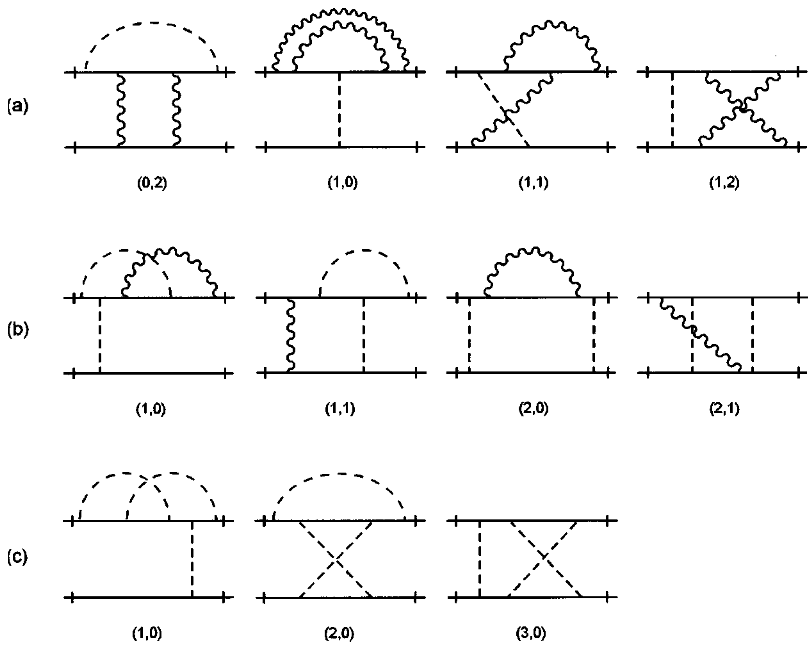

FIG. 6. Diagrams illustrating the various classes of graphs in the four-point vertex function.

The first observation is that, up to the order we are interested, i.e., third order, there are too many graphs. To be systematic, we will separate them according to whether they have or have not closed fermionic loops. Moreover, if they do not possess fermionic loops, we group them according to the number of CS or auxiliary GN lines linking the two fermion lines crossing the diagram. Many diagrams cancel because of Furry's theorem; this is the case if there is a fermionic loop with an odd number of attached CS lines. Other diagrams have the structure of a product of two one-loop graphs and therefore are finite. We shall not consider these two types of graphs any longer. The antisymmetrized amplitude for a graph γ has the generic structure of a product, $(A\otimes B)C$, where A and B refer to the propagators and vertices associated with the two fermion lines and C to the others factors ($\otimes$ indicates the antisymmetrized direct product). Using this notation, one can verify that

TABLE II. Pole part for four legs graphs without closed fermionic loops. The first column lists different types of diagrams (i,j), where i and j are the number of GN and CS lines joining the two fermion lines crossing the graph; the digits in parentheses after the pole parts are the number of contributing graphs.

Diagram type	Order $g\alpha^2$	Order $g^2\alpha$	Order g^3
(0,2)	$i/2\epsilon$(8)	—	—
(1,0)	$-5i/2\epsilon$ (12)	$3i/\pi\epsilon$ (18)	$-3i/4\pi^2\epsilon$ (6)
(1,1)	$8i/\epsilon$ (24)	$4i/\pi\epsilon$ (8)	—
(1,2)	$-2i/\epsilon$ (12)	—	—
(2,0)	—	$-i/\pi\epsilon$ (14)	$i/\pi^2\epsilon$ (12)
(2,1)	—	$-15i/2\pi\epsilon$ (9)	—
(3,0)	—	—	$5i/4\pi^2\epsilon$ (4)

$$\text{pole part of } \int d^dk_1d^dk_2(A\otimes B)C$$
$$=\frac{\mathcal{T}}{2}\int d^dk_1d^dk_2(\mathrm{Tr}[A]\mathrm{Tr}[B]-\mathrm{Tr}[AB])C. \quad \text{(B1)}$$

For example, from the analytic expression for the graph shown in Fig. 5,

$$\lambda\int d^dk_1d^dk_2[\gamma^\mu S_F(k_2)]$$
$$\otimes[\gamma^\nu S_F(-k_2)]\mathrm{Tr}[S_F(k_1)S_F(k_1-k_2)]\epsilon_{\mu\nu\lambda}\frac{k_2^\lambda}{k_2^2}, \quad \text{(B2)}$$

where λ is a combinatorial factor, we determine A, B, and C as

$$A=[\gamma^\mu S_F(k_2)], \quad \text{(B3)}$$

$$B=[\gamma^\nu S_F(-k_2)], \quad \text{(B4)}$$

$$C=\lambda\,\epsilon_{\mu\nu\lambda}\frac{k_2^\lambda}{k_2^2}. \quad \text{(B5)}$$

The results for the pole parts are summarized in Tables I and II, which correspond to the two cases mentioned above. In Table I are listed the results from graphs with one closed fermionic loop; these have been arranged according to the number of CS vertices in the loop. Table II exhibits the pole part of graphs without a fermionic loop. They have been collected into types (i,j), where i and j are the number auxiliary GN and CS lines, respectively, linking the two fermion lines crossing the diagram. Notice that there are no contributions from graphs of type (0,1) since they are not proper. Figure 6 furnishes examples of each one of these sets of diagrams. The final result for each order is obtained by summing the corresponding entries in each table. Thus we have

$$C_3=-\frac{7}{2},\quad C_4=\frac{11i}{2\pi},\quad C_5=\frac{13}{2\pi^2}. \tag{B6}$$

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[10] We use natural units ($c=\hbar=1$) and our metric is $g_{00}=-g_{11}=-g_{22}=1$. The fully antisymmetric tensor $\epsilon^{\mu\nu\lambda}$ is normalized such that $\epsilon^{012}=1$ and we define $\epsilon^{ij}\equiv\epsilon^{0ij}$. Repeated greek indices sum from 0 to 2, while repeated Latin indices from the middle of the alphabet sum from 1 to 2. For the γ matrices we adopt the representation $\gamma^0=\sigma^3,\gamma^1=i\sigma^1,\gamma^2=i\sigma^2$, where $\sigma^i,i=1,2,3$, are the Pauli spin matrices.

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