NUT charge, anti–de Sitter space, and entropy

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It has been proposed that spacetimes with a $U(1)$ isometry group have contributions to the entropy from Misner strings as well as from the area of $d-2$ dimensional fixed point sets. In this paper we test this proposal by constructing Taub-NUT-AdS and Taub-bolt-AdS solutions which are examples of a new class of asymptotically locally anti–de Sitter space. We find that with the additional contribution from the Misner strings, we exactly reproduce the entropy calculated from the action by the usual thermodynamic relations. This entropy has the right parameter dependence to agree with the entropy of a conformal field theory on the boundary, which is a squashed three-sphere, at least in the limit of large squashing. However, the conformal field theory and the normalization of the entropy remain to be determined. [S0556-2821(99)06402-4]

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I. INTRODUCTION

It has been known for quite some time that black holes have entropy. The entropy is

$$S=\frac{\mathcal{A}}{4G}, \tag{1.1}$$

where $\mathcal{A}$ is the area of the horizon and G is Newton's constant. In any dimension d, this formula holds for black holes or black branes that have a horizon, which is a $d-2$ dimensional fixed point set of a $U(1)$ isometry group. However, it has recently been shown [1] that entropy can be associated with a more general class of spacetimes. In these metrics, the $U(1)$ isometry group can have fixed points on surfaces of any even co-dimension, and the spacetime need not be asymptotically flat or asymptotically anti–de Sitter space. In this more general class, the entropy is not just a quarter the area of the $d-2$ dimensional fixed point set.

Among the more general class of spacetimes for which entropy can be defined, an interesting case is those with nut charge. Nut charge can be defined in four dimensions [2] and can be regarded as a magnetic type of mass. Solutions with nut charge are not asymptotically flat (AF) in the usual sense. Instead, they are said to be asymptotically locally flat (ALF). In the Euclidean regime, in which we shall be working, the difference can be described as follows. An AF metric, such as a Euclidean Schwarzschild metric, has a boundary at infinity that is an S^2 of radius r times an S^1, whose radius is asymptotically constant. To get finite values for the action and Hamiltonian, one subtracts the values for periodically identified flat space. In ALF metrics, on the other hand, the boundary at infinity is an S^1 bundle over S^2. These bundles are labeled by their first Chern number, which is proportional to the nut charge. If the first Chern number is zero, the boundary is the product $S^2\times S^1$, and the metric is AF. However, if the first Chern number is k, then the boundary is a squashed S^3 with $|k|$ points identified around the S^1 fibers. Such ALF metrics cannot be matched to flat space at infinity to give a finite action and Hamiltonian, despite a number of papers that claim it can be done. The best that one can do is match to the self-dual multi-Taub-NUT (Newman-Unti-Tamburins) solutions [3]. These can be regarded as defining the vacuums for ALF metrics.

In the self-dual Taub-NUT solution, the $U(1)$ isometry group has a zero-dimensional fixed point set at the center, called a nut. However, the same ALF boundary conditions admit another Euclidean solution, called the Taub-bolt metric [4], in which the nut is replaced by a two-dimensional bolt. The interesting feature is that, according to the new definition of entropy, the entropy of the Taub-bolt metric is not equal to a quarter the area of the bolt, in Planck units. The reason is that there is a contribution to the entropy from the Misner string, the gravitational counterpart to a Dirac string for a gauge field.

The fact that black hole entropy is proportional to the area of the horizon has led physicists to try to identify the microstates with states on the horizon. After years of failure, success seemed to come in 1996, with the paper of Strominger and Vafa [5], which connected the entropy of certain black holes with a system of D-branes. With hindsight, this can now be seen as an example of a duality between a gravitational theory in asymptotically anti–de Sitter space and a conformal field theory on its boundary. It would be interesting if similar dualities could be found for solutions with nut charge, so that one could verify that the contribution of the Misner string was present in the entropy of a conformal field theory. This would be particularly significant for solutions like Taub-bolt, which do not have a spin structure.

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It would show that the duality between anti–de Sitter space and conformal field theories on its boundary did not depend on supersymmetry or string theory.

In this paper, we will describe the progress we have made towards establishing such a duality. We have found a family of Taub-bolt anti–de Sitter (AdS) solutions. These Euclidean metrics are characterized by an integer k and a positive real parameter s. The boundary at large distances is an S^1 bundle over S^2, with first Chern number k. If $k=0$, the boundary is a product, $S^1 \times S^2$, and the space is asymptotically anti–de Sitter, in the usual sense. But if k is not zero, the metrics are what may be called asymptotically locally anti–de Sitter (ALAdS). The boundary is a squashed S^3, with k points identified around the $U(1)$ direction. This is just like ALF metrics. But unlike the ALF case, the squashing of the S^3 tends to a finite limit as one approaches infinity. This means that the boundary has a well-defined conformal structure. One can then ask whether the partition function and entropy of a conformal field theory on the boundary is related to the action and entropy of these ALAdS solutions.

To make this question well posed we have to specify the reference backgrounds with respect to which the actions and Hamiltonians are defined. Like in the ALF case, a squashed S^3 cannot be embedded in Euclidean anti–de Sitter space. Therefore one cannot use it as a reference background to regularize the action and Hamiltonian. Instead, one has to use Taub-NUT anti–de Sitter space, which is a limiting case of our family. If $|k|$ is greater than 1, there is an orbifold singularity in the reference backgrounds, but not in the Taub-bolt anti–de Sitter solutions. These orbifold singularities in the backgrounds could be resolved by replacing a small neighborhood of the nut by an asymptotically local Euclidean (ALE) metric. We shall therefore take it that the orbifold singularities are harmless.

Another issue that has to be resolved is what conformal field theory to use on the squashed S^3. Here we are on shakier ground. For five-dimensional anti–de Sitter space, there are good reasons to believe that the boundary theory is a large-N Yang-Mills theory. But on the three-dimensional boundaries of four-dimensional anti–de Sitter space, Yang-Mills theory is not conformally invariant. The best that we can do is calculate the determinants of free fields on the squashed S^3, and see if they have the same dependence on the squashing as the action. Note that as the boundary is odd dimensional, there is no conformal anomaly. The determinant of a conformally invariant operator will just be a function of the squashing. We can then interpret the squashing as the inverse temperature, and get the number of degrees of freedom from a comparison with the entropy of ordinary black holes in four-dimensional anti–de Sitter space.

II. ENTROPY

We now turn to the question of how one can define the entropy of a spacetime. A thermodynamic ensemble is a collection of systems whose charges are constrained by Lagrange multipliers. One such charge is the energy or mass M, with the Lagrange multiplier being the inverse temperature, β. But one can also constrain the angular momentum J and gauge charges q_i. The partition function for the ensemble is the sum over all states,

$$\mathcal{Z}=\sum e^{-\mu_i K_i}, \tag{2.1}$$

where μ_i is the Lagrange multiplier associated with the charge K_i. Thus, it can also be written as

$$\mathcal{Z}=\mathrm{Tr}\, e^{-Q}. \tag{2.2}$$

Here Q is the operator that generates a Euclidean time translation $\Delta\tau=\beta$, a rotation $\Delta\phi=\beta\Omega$ and a gauge transformation $\alpha_i=\beta\Phi_i$, where Ω is the angular velocity and Φ_i is the gauge potential for q_i. In other words, Q is the Hamiltonian operator for a lapse that is β at infinity, a shift that is a rotation through $\Delta\phi$, and gauge rotations α_i. This means that the partition function can be represented by a Euclidean path integral over all metrics which are periodic at infinity under the combination of a Euclidean time translation by β, a rotation through $\Delta\phi$, and a gauge rotation α_i. The lowest order contributions to the path integral for the partition function will come from Euclidean solutions with a $U(1)$ isometry that agree with the periodic boundary conditions at infinity.

The Hamiltonian in general relativity or supergravity can be written as a volume integral over a surface of constant τ, plus surface integrals over its boundaries. The notation used will be that of [1]. The volume integral is

$$H_c=\int_{\Sigma_\tau} d^{d-1}x\left[N\mathcal{H}+N^i\mathcal{H}_i+A_0(D_iE^i-\rho)+\sum_{A=1}^{M}\lambda^A C^A\right], \tag{2.3}$$

and vanishes by the constraint equations. Thus the numerical value of the Hamiltonian comes entirely from the surface terms,

$$H_b=-\frac{1}{8\pi G}\int_{B_\tau}\sqrt{\sigma}[Nk+u_i(K^{ij}-Kh^{ij})N_j+2A_0F^{0i}u_i + f(N,N^i,h_{ij},\phi^A)]. \tag{2.4}$$

The action can be related to the Hamiltonian in the usual way,

$$I=\int d\tau\left[\int_{\Sigma_\tau} d^{d-1}x\left(P^{ij}\dot{h}_{ij}+E^i\dot{A}_i+\sum_{A=1}^{N}\pi^A\dot{\phi}^A\right)+H\right]. \tag{2.5}$$

Because the metric has a $U(1)$ isometry, all quantities with an overdot vanish. Thus

$$I=\beta H. \tag{2.6}$$

If the solution can be foliated by a family of surfaces that agree with Euclidean time at infinity, the only surface terms will be at infinity. In this case, a solution can be identified under any time translation, rotation, or gauge transformation at infinity. This means that the action will be linear in β, $\Delta\phi$, and α_i,

$$I=\beta H_\infty=\beta M+(\Delta\phi)J+\alpha_i q_i. \tag{2.7}$$

If one takes such a linear action to be $(-\log\mathcal{Z})$, and applies the standard thermodynamic relations, one finds the entropy is zero.

The situation is very different, however, if the solution cannot be foliated by surfaces of constant τ, where τ is the parameter of the $U(1)$ isometry group that agrees with the periodic identification at infinity. The breakdown of foliation can occur in two ways. The first is at fixed points of the $U(1)$ isometry group. These occur on surfaces of even co-dimension. Fixed point sets of co-dimension 2 play a special role. We shall refer to them as bolts. Examples include the horizons of non-extreme black holes and p-branes, but there can be more complicated cases, as in the Taub-bolt metric.

The other way the foliation by surfaces of constant τ can break down is if there are what are called Misner strings. To explain what they are, we write the metric in the Kaluza-Klein form with respect to the $U(1)$ isometry group,

$$ds^2=\exp\left[-\frac{4\sigma}{\sqrt{d-2}}\right](d\tau+\omega_i dx^i)^2 +\exp\left[\frac{4\sigma}{(d-3)\sqrt{d-2}}\right]\gamma_{ij}dx^i dx^j. \tag{2.8}$$

The one-form, ω_i, the dilaton, σ, and the metric, γ_{ij}, can be regarded as fields on Ξ, the space of orbits of the isometry group. If Ξ has homology in dimension 2, the Kaluza-Klein field strength F can have non-zero integrals over two-cycles. This means that the one-form, ω_i, will have Dirac strings in Ξ. In turn, this will mean that the foliation of the spacetime $\mathcal{M}$ by surfaces of constant τ will break down on surfaces of co-dimension 2, called Misner strings.

In order to do a Hamiltonian treatment using surfaces of constant τ, one has to cut out small neighborhoods of the fixed point sets and the Misner strings. This modifies the treatment in two ways. First, the surfaces of constant τ now have boundaries at the fixed point sets and Misner strings, as well as the usual boundary at infinity. This means there can be additional surface terms in the Hamiltonian. In fact, the surface terms at the fixed point sets are zero, because the shift and lapse vanish there. On the other hand, at a Misner string the lapse vanishes, but the shift is non-zero. The Hamiltonian can therefore have a surface term on the Misner string, which is the shift times a component of the second fundamental form of the constant τ surfaces. The total Hamiltonian will be

$$H=H_\infty+H_{\rm MS}, \tag{2.9}$$

i.e., the sum of this Misner string Hamiltonian and the Hamiltonian surface term at infinity. As before, the action will be βH. However, this will be the action of the spacetime with the neighborhoods of the fixed point sets and Misner strings removed. To get the action of the full spacetime, one has to put back the neighborhoods. When one does so, the surface term associated with the Einstein-Hilbert action will give a contribution to the action of minus area over $4G$, for both the bolts and Misner strings, that is,

$$I=\beta H_\infty+\beta H_{\rm MS}-\frac{1}{4G}(\mathcal{A}_{\rm bolt}+\mathcal{A}_{\rm MS}). \tag{2.10}$$

Here G is Newton's constant in the dimension one is considering. The surface terms around lower dimensional fixed point sets make no contribution to the action.

The action of the spacetime, I, will be the lowest order contribution to $(-\log\mathcal{Z})$. But

$$\log\mathcal{Z}=S-\beta H_\infty. \tag{2.11}$$

So the entropy is

$$S=\frac{1}{4}(\mathcal{A}_{\rm bolt}+\mathcal{A}_{\rm MS})-(\Delta\psi)H_{\rm MS}. \tag{2.12}$$

In other words, the entropy is the amount by which the action is less than the value, βH_∞, that it would have if the surfaces of constant τ foliated the spacetime.

Formula (2.12) for the entropy applies in any dimension and for any class of boundary conditions at infinity. In particular, we can apply it to ALF metrics in four dimensions that have nut charge. In this case, the reference background is the self-dual Taub-NUT solution. The Taub-bolt solution has the same asymptotic behavior, but with the zero-dimensional fixed point replaced by a two-dimensional bolt. The area of the bolt is $12\pi N^2$, where N is the nut charge. The area of the Misner string is $-12\pi N^2$. That is to say, the area of the Misner string in Taub-bolt is infinite, but it is less than the area of the Misner string in Taub-NUT, in a well-defined sense. The Hamiltonian on the Misner string is $-N/8$. Again the Misner string Hamiltonian is infinite, but the difference from Taub-NUT is finite. And the period, β, is $8\pi N$. Thus the entropy is

$$S=\pi N^2. \tag{2.13}$$

Note that this is less than a quarter the area of the bolt, which would give $3\pi N^2$. It is the effect of the Misner string that reduces the entropy.

III. ENTROPY OF THE TAUB-BOLT-AdS METRIC

The Taub-NUT-AdS metric can be obtained as a special case of the complex metrics given in [6] (see also [7]). The line element is

$$ds^2=b^2E\left[\frac{F(r)}{E(r^2-1)}(d\tau+E^{1/2}\cos\theta d\phi)^2+\frac{4(r^2-1)}{F(r)}dr^2 +(r^2-1)(d\theta^2+\sin^2\theta d\phi^2)\right], \tag{3.1}$$

where

$$F_N(r,E)=Er^4+(4-6E)r^2+(8E-8)r+4-3E, \tag{3.2}$$

E is an arbitrary constant which parametrizes the squashing, $b^2=-3/4\Lambda$, and $\Lambda<0$ is the cosmological constant. The Euclidean time coordinate, τ, has period $\beta=4\pi E^{1/2}$ and has a nut at $r=1$, which is the origin of the ψ-r plane. Asymptotically, the metric is ALAdS since the boundary is a squashed S^3, rather than $S^1\times S^2$.

We can obtain another family of metrics from [6] that have the same asymptotic behavior. They are the Taub-bolt-AdS metrics, which have the same form as Eq. (3.1) but the function $F(r)$ is

$$F_B(r,s)=Er^4+(4-6E)r^2+\left[-Es^3+(6E-4)s+\frac{3E-4}{s}\right]r+4-3E, \tag{3.3}$$

where

$$E=\frac{2ks-4}{3(s^2-1)}, \tag{3.4}$$

k is the Chern number of the S^1 bundle and s is an arbitrary parameter. In order to avoid curvature singularities, we must take $s>1$, $s>2/k$ and $r>s$. The periodicity of the imaginary time is $4\pi E^{1/2}/k$, and it has a bolt at $r=s$, with area

$$\mathcal{A}_{\mathrm{bolt}}=\frac{8}{3}b^2\pi(ks-2). \tag{3.5}$$

The boundary at infinity is a squashed S^3 with $|k|$ points identified on the S^1 fiber.

The action calculation is a fairly trivial combination of the original Schwarzschild-AdS action calculation [8] and the more recent understanding of the actions of metrics with nut charge [9]. As mentioned in Sec. I, in order to regularize the action and Hamiltonian calculations, we need to choose a reference background. Since the Taub-bolt-AdS metric cannot be embedded in AdS space, we cannot use this as a background. However, we can use a suitably identified and scaled Taub-NUT-AdS metric as a reference background. We need the periodicity of the imaginary time coordinates to agree. This means that for a Taub-Bolt-AdS metric with parameters (k,s) we must take the orbifold obtained by identifying k points on the S^1 as the reference background, rather than just the Taub-NUT-AdS metric. This will have a conical singularity at the origin; however, as mentioned before, we can smooth it out in a simple way, and hence we can just ignore it, and treat the space as non-singular. We then need to scale the background imaginary time by $E^{1/2}/\tilde{E}^{1/2}$ so that both imaginary time coordinates have the same periodicity, namely $\beta=4\pi E^{1/2}/k$. Finally, we require that the induced metrics agree sufficiently well on a hypersurface of constant radius R, as we take R to infinity. This yields equations for both the S^1 and the S^2 metric components,

$$\frac{EF_B(r,s)}{r^2-1}=\frac{\tilde{E}F_N(\tilde{r},\tilde{E})}{\tilde{r}^2-1} \tag{3.6}$$

and

$$E(r^2-1)=\tilde{E}(\tilde{r}^2-1). \tag{3.7}$$

To sufficient order, this has the solution $\tilde{E}=\eta E$ and $\tilde{r}=\lambda r$, where

$$\eta=1-\frac{2\rho}{R^3},\quad \lambda=1+\frac{\rho}{R^3},$$

$$\rho=\frac{(s-1)^2[E(s-1)(s+3)+4]}{2sE}. \tag{3.8}$$

Hence the matched background metric is

$$ds^2=b^2\eta E\left[\frac{F_N(\lambda r,\eta E)}{E(\lambda^2r^2-1)}(d\psi+E^{1/2}\cos\theta\, d\phi)^2+\frac{4(\lambda^2r^2-1)}{F_N(\lambda r,\eta E)}\lambda^2dr^2+(\lambda^2r^2-1)(d\theta^2+\sin^2\theta\, d\phi^2)\right], \tag{3.9}$$

with the function

$$F_N(\lambda r,\eta E)=E\eta\lambda^4r^4+(4-6E\eta)\lambda^2r^2+(8E\eta-8)\lambda r+4-3E\eta. \tag{3.10}$$

Calculating the action, we find that the surface terms cancel, just like in the Schwarzschild-AdS case, so that the action is given entirely by the difference in volumes of the metrics,

$$I=-\frac{2\pi b^2}{9k}\frac{(ks-2)[k(s^2+2s+3)-4(2s+1)]}{(s+1)^2}. \tag{3.11}$$

We see that the action will have zeros at up to 3 points,

$$s_{\pm}=\frac{4-k\pm\sqrt{16-4k-2k^2}}{k}\quad\text{and}\quad s_0=\frac{2}{k}. \tag{3.12}$$

For the case $k=1$, there will only be one valid zero, $s_+=3+\sqrt{10}$. The action will be positive for $s<s_+$, and negative for $s>s_+$. When $k=2$, all the zeros will coincide at the lowest value of $s=1$, and the action is negative for any other value of s. For larger values of k, $s_\pm$ will be imaginary, $s_0<1$, and hence the action will always be negative. The action for $k=1$ is plotted in Fig. 1.

The Hamiltonian calculation is more complicated than the simple action calculation completed above. There will be two non-zero contributions to the Hamiltonian—from the boundary at infinity and from the boundary along the Misner string. There is a third boundary, around the bolt, but the

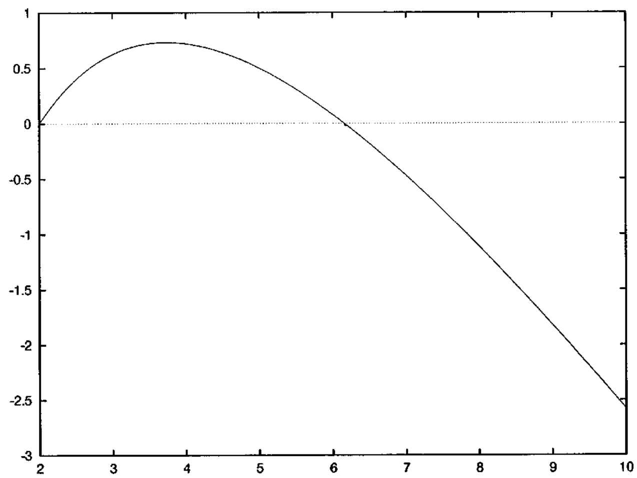

FIG. 1. The action I as a function of s for $k=1$ and $b^2=9/2\pi$, as given by Eq. (3.11). The zero is at $s=3+\sqrt{10}$.

Hamiltonian will vanish there. Using the matched Taub-NUT-AdS metric from above, we find that

$$H_\infty=\frac{b^2}{9}\frac{(s-1)(ks-2)[k(s+3)+4]}{E^{1/2}(s+1)^2} \tag{3.13}$$

and

$$H_{\rm MS}=\frac{b^2}{3}\frac{(k-2s)(ks-2)}{E^{1/2}(s+1)^2}. \tag{3.14}$$

The area of the Misner string is larger in the background, and hence the net area is negative,

$$\mathcal{A}_{\rm MS}=-\frac{32\pi b^2}{3}\frac{ks-2}{s+1}, \tag{3.15}$$

while the area of the bolt is

$$\mathcal{A}_{\rm bolt}=\frac{8\pi b^2}{3}(ks-2). \tag{3.16}$$

Substituting these values into the formula for the action (2.10) we regain the expression (3.11).

We are now in a position to use Eq. (2.12) for the entropy. We find that

$$S=\frac{2\pi b^2}{3k}\frac{(ks-2)[k(s^2+2s-1)-4]}{(s+1)^2}. \tag{3.17}$$

Similar to the action, the entropy will have three possible zeros,

$$s_\pm=\frac{-k\pm\sqrt{2k^2+4k}}{k} \quad \text{and} \quad s_0=\frac{2}{k}. \tag{3.18}$$

For $k=1$, all the zeros satisfy $s\leqslant 2$, while for $k=2$, the zeros are at $s\leqslant 1$. Hence in these cases the entropy is never negative, and is only zero at $(s=2, k=1)$ and $(s=1, k=2)$, which are exactly the two points where the action vanishes. For larger values of k, the zeros are all strictly less than 1, and hence the entropy is always positive.

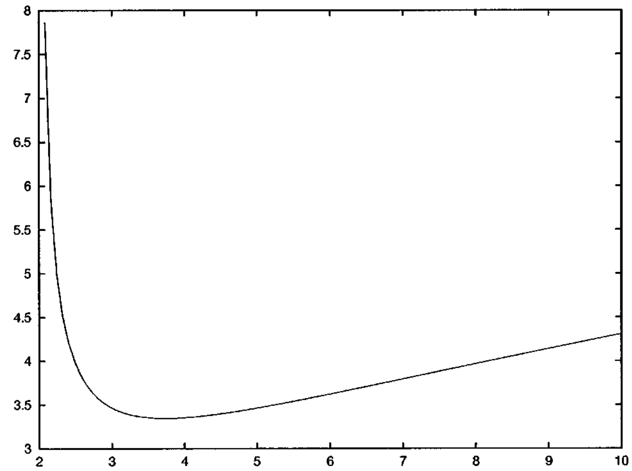

FIG. 2. The temperature $T=1/\sqrt{E}$ as a function of s for $k=1$ and $b^2=9/2\pi$. The minimum value is at $s=2+\sqrt{3}$.

One can regard $\mathcal{Z}$ as the partition function at a temperature

$$T=\beta^{-1}=\frac{k}{4\pi E^{1/2}}. \tag{3.19}$$

If one then assumes that mass is the only charge that is constrained by a Lagrange multiplier (nut charge is fixed by the boundary conditions and hence does not need a Lagrange multiplier), then one can calculate the entropy from the standard thermodynamic relation

$$S=\beta\frac{\partial I}{\partial\beta}-I=2E\frac{\partial I}{\partial E}-I, \tag{3.20}$$

where we have made the approximation $I=-\log\mathcal{Z}$. This yields the same value as in Eq. (3.17) and so acts as a consistency check on our formula for entropy.

One can also calculate the energy or mass of the system,

$$M=\frac{\partial I}{\partial\beta}=\frac{b^2}{9}\frac{(s-1)(ks-2)[k(s+3)+4]}{E^{1/2}(s+1)^2}=H_\infty. \tag{3.21}$$

Again, this agrees with the Hamiltonian calculation.

Identical to the AdS case, there is a phase transition in the ALAdS system (for $k=1$). This can be seen by considering the behavior of the Taub-NUT-AdS and Taub-bolt-AdS solutions as a function of temperature. There are no restrictions on the temperature of the Taub-NUT-AdS metric, but as can be seen from Fig. 2, the temperature of the Taub-bolt-AdS metric has a minimum value $T_0=\sqrt{6+3\sqrt{3}}/(4\pi)\approx 0.836516303738/\pi$.

Hence, if we have $T<T_0$, the system will be in the Taub-NUT-AdS ground state. As we increase T above T_0, there are two possible Taub-bolt metrics with different mass values but the same temperature. The one with lower s will be

thermodynamically unstable, since it has negative specific heat, $\partial M/\partial T$, while the one with larger s has positive specific heat, and hence will be stable. The lower s branch has positive action, and hence will be less likely than the Taub-NUT-AdS background. The behavior of the larger s branch will depend on T. At temperatures below $T_1 = \sqrt{7+2\sqrt{10}}/(4\pi) \approx 0.912570384968/\pi$, the action will be positive and the Taub-NUT-AdS background will be favored. But for T greater than T_1, the negative action implies that the Taub-bolt-AdS solution is preferred, and hence the Taub-NUT-AdS background will inevitably decay into it.

We can compare the local temperatures at the phase transition for the Schwarzschild-AdS ($k=0$) and the Taub-bolt-AdS ($k=1$ and the degenerate case $k=2$) metrics. In order to compare the temperatures in the different metrics, we want to rescale them so that the radii of the S^2 parts of their boundaries at infinity are 1. Hence, rescaling the $S^2 \times S^1$ boundary of the Schwarzschild-AdS case corresponds to multiplying the temperatures given in [8] by the quantity $b = \sqrt{-3/\Lambda}$ used in that paper, which is twice the b used in our present paper. In that case one gets $T_0^{k=0} = \sqrt{3}/(2\pi)$ and $T_1^{k=0} = 1/\pi$. In the Taub-bolt-AdS case, the temperature at the boundary with this rescaling is simply $(4\pi\sqrt{E})^{-1}$, as we have defined it above. The corresponding temperatures for the $k=1$ metric are $T_0^{k=1} = \sqrt{2+\sqrt{3}}\,T_0^{k=0}/2 \approx 0.96593 T_0^{k=0}$ and $T^{k=1} = \sqrt{7+2\sqrt{10}}/(4\pi) T_1^{k=0} \approx 0.91257 T_1^{k=0}$ respectively. For $k=2$, the minimum and critical temperatures coincide, and they are $T^{k=2} = T_0^{k=0}/\sqrt{2} = \sqrt{3/8}\,T_1^{k=1}$. The results are summarized in the table below:

k	πT_0	πT_1
0	0.86660	1.0
1	0.83652	0.91257
2	0.61237	0.61237

It is interesting that the first two results are much closer together than they are to the $k=2$ value.

IV. CONFORMAL FIELD THEORY

Formally at least, one can regard Euclidean conformal field theory on the squashed S^3 as a twisted $2+1$ theory on an S^2 of unit radius at a temperature $T = \beta^{-1}$. Thus, one would expect the entropy to be proportional to β^{-2} for small β. This dependence agrees with the expression that we have for the gravitational entropy of the Taub-bolt-AdS metric. To go further and obtain the normalization and sub-leading dependence on β would require a knowledge of the conformal field theory that we do not have. The best that we can do is calculate the determinants of conformally invariant free fields on the squashed S^3 and compare with the results for $S^2 \times S^1$ and Schwarzschild-AdS space. On $S^2 \times S^1$ the determinants of conformally invariant free fields will be the same function of β, but this cannot be the case on the squashed S^3 because fermions have zero modes at an infinite number of values of the squashing, whereas a scalar field has a zero mode only at one value. Furthermore, Taub-bolt-AdS solutions with k odd do not have spin structures. Thus, if they are dual to a conformal field theory, it should be one without fermions.

Similar work on Taub-NUT-AdS and Taub-bolt-AdS metrics for $k=1$ has been performed independently [10].

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