Hawking radiation by effective two-dimensional theories

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The recently proposed two-dimensional anomaly induced effective actions for the matter-gravity system are critically reviewed. Their failure to reproduce correctly Hawking's black hole radiation or the stability of Minkowski space-time leads us to a modification of the relevant ''quantum'' matter stress energy tensor that allows physically meaningful results to be extracted.
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I. INTRODUCTION

Hawking's remarkable discovery [1] that black holes emit quantum thermal radiation at a temperature inversely proportional to their mass (i.e., $T_H = 1/8\pi M$ in units where $\hbar = c = G = k_B = 1$) triggered, in the mid 1970s, a large scale investigation of quantum effects in strong gravitational fields (see for example [2]). The framework used was quantum field theory in curved space, a semiclassical approach in which only the matter fields are quantized, whereas gravity is still described classically according to general relativity. Its dynamical evolution is driven by the expectation value of the renormalized energy-momentum tensor operator of the quantized matter fields, i.e. $\langle T_{\mu\nu}\rangle$, according to the semiclassical Einstein equations

$$G_{\mu\nu}(g_{\mu\nu}) = 8\pi\langle T_{\mu\nu}(g_{\mu\nu})\rangle. \tag{1.1}$$

The left-hand side (LHS) is the Einstein tensor for the space-time metric $g_{\mu\nu}$, while the right-hand side RHS represents the expectation value of the stress tensor of the matter fields propagating on that space-time.

According to Wald's axioms [3] $\langle T_{\mu\nu}\rangle$ must be conserved, $\nabla_\mu \langle T^\mu_\nu\rangle = 0$, and vanishing for Minkowski space-time, so that Eq. (1.1) can make sense. One further important thing to note is the presence in $\langle T_{\mu\nu}\rangle$ of a trace anomaly (see for instance [4]). For conformally invariant fields, the expectation value of the trace $\langle T^\alpha_\alpha\rangle$ is nonzero, unlike its classical counterpart, and independent of the state in which the expectation value is taken. It is completely expressed in terms of geometrical objects as

$$\langle T^\alpha_\alpha\rangle = (2880\pi^2)^{-1}\{aC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + b(R_{\alpha\beta}R^{\alpha\beta} - \tfrac{1}{3}R^2) + c\Box R + dR^2\}, \tag{1.2}$$

where the coefficients in front of each of the above geometrical tensors are known and depend on the spin of the quantum field under consideration [2]. These features will be of fundamental importance throughout the paper.

Technically, one has to construct $\langle T_{\mu\nu}(g_{\mu\nu})\rangle$ for a sufficiently large class of metrics (according, for example, to the symmetry of the problem) and then solve Eq. (1.1) self-consistently for the metric. Unfortunately, general expressions for $\langle T_{\mu\nu}(g_{\mu\nu})\rangle$ are not available, except when the degree of symmetry of the problem is sufficiently high, for instance conformally invariant fields in homogeneous and isotropic space-times [5] where that the trace anomaly determines completely $\langle T_{\mu\nu}\rangle$. This is not the case, however, for black holes. $\langle T_{\mu\nu}(g_{\mu\nu})\rangle$ for a sufficiently arbitrary (even spherically symmetric) black hole spacetime is not known even approximately. So the evolution of the black hole as it Hawking emits (the so called backreaction) is an open problem. Much effort has been devoted to understand all the features of $\langle T_{\mu\nu}\rangle$ for the Schwarzschild black hole geometry in order to get some insight in the backreaction. Note, however, that the Schwarzschild spacetime is not a solution to Eq. (1.1) since the LHS vanishes, unlike the RHS.

Using analytical methods (which can be improved numerically) one can find reasonable approximations of $\langle T_{\mu\nu}\rangle$ for various kinds of quantum fields propagating on the Schwarzschild space-time (see for example [6]). Within this context, three quantum states might be proposed as a suitable candidate for the vacuum:

(i) the Boulware state $|B\rangle$ [7], defined by requiring normal modes to be positive frequency with respect to the Killing vector $\partial/\partial t$, according to which the region exterior to the horizon is static. The stress tensor in this zero temperature state describes the vacuum polarization outside a static star whose radius is bigger than the Schwarzschild one (i.e., $r > 2M$). As $r \to \infty \langle B|T_{\mu\nu}|B\rangle \to 0$. The Boulware state corresponds to our familiar concept of an empty state for large radii. Symbolically, $|B\rangle \to |M\rangle$, where $|M\rangle$ is Minkowski vacuum. However, $|B\rangle$ is pathological at the horizon as it diverges when evaluated in a free falling frame.

(ii) the Hartle-Hawking state [8], defined by taking incoming modes to be positive frequency with respect to the canonical affine parameter on the future horizon (Kruskal coordinate V) and outgoing modes to be positive frequency with respect to the canonical affine parameter on the past horizon (Kruskal's U). $\langle H|T_{\mu\nu}|H\rangle$ is well behaved on both future and past horizons. This state is not empty at infinity,

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corresponding to a thermal distribution of quanta at the Hawking temperature $T_H = 1/8\pi M$, i.e.,

$$\langle H|T^{\mu}_{\nu}|H\rangle \sim \frac{1}{2\pi^2}\int_0^{\infty}\frac{w^2 dw}{e^{8\pi M w}-1}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}. \tag{1.3}$$

That is, the state $|H\rangle$ corresponds to a black hole in equilibrium with an infinite reservoir of black body radiation.

(iii) the Unruh state $|U\rangle$ [9], defined by taking modes that are incoming from past null infinity to be positive frequency with respect to $\partial/\partial t$, while those that emanate from the past horizon to be positive frequency with respect to U. At infinity this state corresponds to an outgoing flux of blackbody radiation at the black hole temperature T_H

$$\langle U|T^{\mu}_{\nu}|U\rangle \sim \frac{L}{4\pi r^2}\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.4}$$

where L is the luminosity factor of the hole. $\langle U|T_{\mu\nu}|U\rangle$ is regular, in a free falling frame, on the future horizon, but not on the past horizon. As $r \to 2M$, the (r,t) part reads

$$\langle U|T^{\mu}_{\nu}|U\rangle \sim \frac{L}{4\pi}\begin{pmatrix} (1-2M/r)^{-1} & -r^2 \\ r^2(1-2M/r)^{-2} & -(1-2M/r)^{-1} \end{pmatrix}. \tag{1.5}$$

The state $|U\rangle$ is supposed to best approximate the state of the quantum fields outside a collapsing star as its surface approaches the horizon. This implies that the divergence on the past horizon is spurious, since this portion of the Schwarzschild spacetime is not physical being covered by the collapsing body.

Starting from these results attempts have been made to solve at least perturbatively the backreaction for a black hole enclosed in a box [10]. The approach followed (Hartree-Fock like) is to write the backreaction equations (1.1) as follows

$$G_{\mu\nu}(g^{s}_{\mu\nu} + \delta g_{\mu\nu}) = 8\pi\langle H|T_{\mu\nu}(g^{s}_{\mu\nu})|H\rangle, \tag{1.6}$$

where $g^{s}_{\mu\nu}$ represents the Schwarzschild metric, and one solves Eqs. (1.6) linearizing in the static spherically symmetric perturbation $\delta g_{\mu\nu}$. A similar approach is much more difficult to implement for an evaporating black hole. Attempts have been made by modelling the time dependent geometry near the horizon (and also asymptotically) by a Vaidya space-time and some insights in the evaporation process can be extracted [11].

The main difficulty to attack the backreaction equations (1.1) is, as we already pointed out, the absence of an explicit expression of $\langle T_{\mu\nu}\rangle$ for a sufficiently general (for example spherically symmetric) evaporating black hole geometry. This fact can be circumvented in two space-time dimensions. For conformally invariant and minimally coupled scalar fields, one is able to obtain an expression for the two-dimensional (2D) stress tensor $\langle T_{ab}(g_{ab})\rangle$ for a generic 2D metric g_{ab} [2]. This expression can be formally obtained starting from the well known Polyakov action [12]. Explicit evaluation of $\langle T_{ab}\rangle$ for a 2D Schwarzschild geometry in the state $|B\rangle$, $|H\rangle$ and $|U\rangle$ gives results which are in good qualitative agreement with the four-dimensional (4D) $\langle T_{\mu\nu}\rangle$ described earlier.

This nice agreement and the possibility of having a $\langle T_{ab}\rangle$ for an arbitrary 2D metric has triggered extensive investigation of 2D versions of the backreaction equations (1.1) in the hope of learning something about physical (i.e., 4D) black hole evaporation. In all such 2D models gravity (also called dilaton gravity) is coupled to the $\langle T_{ab}\rangle$ of quantized 2D massless and minimal scalar fields by some sort of backreaction equations [13]. The physically more promising ones are those in which the 2D dilaton gravity is the spherically symmetric 2D reduction of 4D Einstein's general relativity [14], where the dilaton ϕ is simply related to the radius of the two spheres r by means of the simple relation $r = e^{-\phi}$. The effective theory one is then considering is described by a 2D action of the form

$$S = S_{cl} + S_P, \tag{1.7}$$

where

$$S_{cl} = \frac{1}{2\pi}\int d^2x\sqrt{-g^{(2)}}e^{-2\phi}[R^{(2)} + 2(\nabla\phi)^2 + 2e^{2\phi}] \tag{1.8}$$

is the spherically symmetric reduction of 4D Einstein gravity and S_P is the so called Polyakov action [12] and will be given in Sec. II. This approach may be criticized, since while the first term, S_{cl}, has a real 4D origin, the same cannot be said for S_P. Coupling 4D spherically symmetric general relativity to 2D quantum fields appears to be rather naive. In a more solid approach to the spherically symmetric case also, the quantum fields should come from dimensional reduction of 4D. The idea is then to start from 4D minimally coupled scalar fields, perform the dimensional reduction under spherical symmetry and evaluate an effective 2D action for this kind matter to replace S_P in Eq. (1.7) [15,16]. This 2D effective action, which we will call S_{aind}, is constructed by functionally integrating the trace anomaly (see also [17,18,19]). The hope is to obtain in this way a more realistic picture of black hole evaporation. Unfortunately, for the Schwarzschild space-time, the $\langle T_{ab}\rangle$ so deduced is not even in qualitative agreement with the 4D $\langle T_{\mu\nu}\rangle$: it predicts a *negative* Hawking flux for an evaporating black hole [15,18]. This fact shades serious doubts on the validity of this more ''sophisticated'' [as compared to Eq. (1.7)] 2D approach. Puzzled with this problem, the authors of Ref. [15] proposed to add Weyl invariant nonlocal terms to the above effective action S_{aind}. The resulting $\langle T_{ab}\rangle$ has the desired feature for a Schwarzschild black hole and correctly reproduces the Hawking flux at infinity. However, for Minkowski space-time, this $\langle T_{ab}\rangle$ badly diverges and Minkowski is not a solution of the backreaction equations.

The situation appears rather frustrating. Trying to improve the simple Polyakov action to obtain a more accurate description of physical black holes, one gets unacceptable results. However, the physical motivations for these improvements seem very reasonable. The possibility of implementing S_{aind} also by means of Weyl invariant terms should, nevertheless, lead to a stress tensor that in our opinion has to satisfy the following requirements:

(i) conservation equations (in 4D);

(ii) vanishing in the vacuum Minkowski spacetime;

(iii) a 4D trace anomaly that, like in Eq. (1.2), does not depend on the state in which the expectation values are taken.

In this paper we shall propose a $\langle T_{\mu\nu}\rangle$ derived in part by S_{aind} that indeed satisfies the above requirements and is also in good qualitative agreement with the 4D results.

In order to make our analysis more clear, in Secs. II and III, we will rederive with some detail all the known results about the Polyakov theory applied to the Schwarzschild black hole. We think that this part is necessary in order to understand better the rest of paper. In Sec. IV we will then apply the two-dimensional techniques just introduced to S_{aind} and see how they lead to physically inconsistent results (such a systematic derivation of these results is not present in the literature). In Sec. V we show why we think that the effection action proposed in [15] does not improve much the situation. In the last two Secs., VI and VII, we will propose a possible solution to the problem based on the four dimensional interpretation.

II. MINIMALLY COUPLED 2D FIELDS

The action describing a conformally and minimally coupled scalar field f in 2D is

$$S_m^{(2)}=-\frac{1}{4\pi}\int d^2x\sqrt{-g^{(2)}}(\nabla f)^2, \tag{2.1}$$

leading to the field equation

$$\Box f=0. \tag{2.2}$$

Quantization is achieved by expanding the field operator $\hat{f}$ in normal modes. Being every 2D metric locally conformally flat, one can introduce a coordinate system (not unique) in which the metric takes the form

$$ds^2=-e^{2\rho}dx^+dx^-. \tag{2.3}$$

We shall call this system the $\{x^\pm\}$ conformal frame. In this frame normalized positive frequency mode functions are of the form

$$(4\pi w)^{-1/2}e^{-iwx^+},\quad (4\pi w)^{-1/2}e^{-iwx^-}. \tag{2.4}$$

Expansion of $\hat{f}$ in the basis (2.4) selects a conformal state, call it $|x^\pm\rangle$, in which the expectation value of the renormalized stress energy tensor operator for the scalar fields is [20]

$$\langle x^\pm|T_{\pm\pm}|x^\pm\rangle=-\frac{1}{12\pi}(\partial_\pm\rho\partial_\pm\rho-\partial_\pm^2\rho),$$

$$\langle x^\pm|T_{+-}|x^\pm\rangle=-\frac{1}{12\pi}\partial_+\partial_-\rho. \tag{2.5}$$

This stress tensor, as can be easily checked, is conserved, namely

$$\partial_\mp\langle x^\pm|T_{\pm\pm}|x^\pm\rangle+\partial_\pm\langle x^\pm|T_{+-}|x^\pm\rangle -\Gamma^\pm_{\pm\pm}\langle x^\pm|T_{+-}|x^\pm\rangle=0 \tag{2.6}$$

but it has, unlike its classical counterpart, developed a trace

$$\langle x^\pm|T|x^\pm\rangle=\frac{R^{(2)}}{24\pi} \tag{2.7}$$

where $T\equiv T_a^a$ and $R^{(2)}$ (hereafter R) is the Ricci scalar associated to the metric $g_{ab}^{(2)}$. This is the so called trace anomaly [4], which signals the breaking of conformal invariance at the quantum level.

The choice of the normal modes (2.4) is by no means unique. One should equally well had chosen another set of normal modes, obtained by solving the field equation (2.2) in another conformal frame, say $\{\tilde{x}^\pm\}$ where $\tilde{x}^\pm=\tilde{x}^\pm(x^\pm)$, i.e.,

$$(4\pi\tilde{w})^{-1/2}e^{-i\tilde{w}\tilde{x}^+},\quad (4\pi\tilde{w})^{-1/2}e^{-i\tilde{w}\tilde{x}^-}. \tag{2.8}$$

The expansion of the field operator $\hat{f}$ in terms of these new modes selects a conformal state $|\tilde{x}^\pm\rangle$. The expectation value of the energy momentum tensor (EMT) in this state is [21]

$$\langle\tilde{x}^\pm|T_{\pm\pm}|\tilde{x}^\pm\rangle=\langle x^\pm|T_{\pm\pm}|x^\pm\rangle+\Delta_\pm(x^\pm),$$

$$\langle\tilde{x}^\pm|T_{+-}|\tilde{x}^\pm\rangle=\langle x^\pm|T_{+-}|x^\pm\rangle. \tag{2.9}$$

Here

$$\Delta_+(x^+)=\frac{1}{24\pi}\left(\frac{G''}{G}-\frac{1}{2}\frac{G'^2}{G^2}\right), \tag{2.10}$$

where

$$G(x^+)=\frac{dx^+}{d\tilde{x}^+} \tag{2.11}$$

and a prime indicates derivation with respect to x^+. Similarly,

$$\Delta_-(x^-)=\frac{1}{24\pi}\left(\frac{F''}{F}-\frac{1}{2}\frac{F'^2}{F^2}\right) \tag{2.12}$$

and

$$F(x^-)=\frac{dx^-}{d\tilde{x}^-}. \tag{2.13}$$

Note that in Eqs. (2.9) the components of the stress tensor are still expressed, as in Eq. (2.5), in the $\{x^\pm\}$ frame, but the expectation value is taken in the $|\tilde{x}^\pm\rangle$ state. From Eqs.

(2.10), (2.12) we can see that the $\Delta_\pm$ is proportional to the Schwarzian derivative between $x^\pm$ and $\tilde{x}^\pm$. From the physical point of view, $\Delta_\pm$ give the expectation values of $T_{\pm\pm}$ in the state $|\tilde{x}_\pm\rangle$ normal ordered with respect to $|x_\pm\rangle$.

We see from Eqs. (2.9) that $\Delta_\pm$ represents conserved massless (i.e., trace free) radiation and is the only difference in the expectation values of $\langle T_{ab}\rangle$ in two distinct conformal states, being the trace anomaly state independent [see Eq. (2.7)]. This difference is nonlocal in the sense that it does not depend on the local geometry, but rather on the global definition (through the normal modes) of the states. Therefore $\Delta_\pm$ represent a nonlocal contribution to $\langle T_{ab}\rangle$ that depends on the state in which the expectation values are taken.

The expectation values of the EMT [Eqs. (2.5)] can also be easily obtained by integrating the conservation equations $\nabla_a\langle T^a_b\rangle=0$ once the trace anomaly Eq. (2.7) is given [22]. In the generic conformal frame of Eq. (2.3) the only nonvanishing Christoffel symbols are

$$\Gamma^\pm_{\pm\pm}=2\partial_\pm\rho. \tag{2.14}$$

Inserting this and the second of Eqs. (2.5) in the conservation equations (2.6), straightforward integration leads to

$$\langle T_{\pm\pm}\rangle=-\frac{1}{12\pi}[\partial_\pm\rho\partial_\pm\rho-\partial^2_\pm\rho-t_\pm(x^\pm)], \tag{2.15}$$

where $t_\pm(x^\pm)$ are two arbitrary integration functions of their respective arguments. They signal the nonlocal character of $\langle T_{\pm\pm}\rangle$ because of its state dependence. In view of the preceding discussion, $t_\pm(x^\pm)$ are related to the Schwarzian derivatives $\Delta_\pm(x^\pm)$ of Eqs. (2.9). Note that, as the trace anomaly does not depend on the quantum state, the $t_\pm(x^\pm)$ are necessary, in Eq. (2.15), to specify in which quantum state the expectation values are taken. Another way of seeing the appearance of these terms is to consider that under the transformation $x^\pm\to\tilde{x}^\pm$, which is at the same time a conformal and a coordinate transformation, $\langle T_{ab}\rangle$ does not transform as a tensor. Because of the breaking of the conformal invariance at the quantum level, the transformation of $\langle T_{ab}\rangle$ involves an anomalous contribution, namely the Schwarzian derivative.

An elegant way of recovering the previous results is to functionally integrate the trace anomaly Eq. (2.7) obtaining Polyakov's nonlocal effective action [12]

$$S_P=-\frac{1}{96\pi}\int d^2x\sqrt{-g}R\frac{1}{\Box}R, \tag{2.16}$$

where $\Box$ is the covariant Dalambertian. Varying S_P with respect to g^{ab} gives

$$\langle T_{ab}\rangle=-\frac{1}{96\pi}\left\{-2\nabla_a\nabla_b\left(\frac{1}{\Box}R\right)+\nabla_a\left(\frac{1}{\Box}R\right)\nabla_b\left(\frac{1}{\Box R}\right)+g_{ab}\left[2R-\frac{1}{2}\nabla^c\left(\frac{1}{\Box}R\right)\nabla_c\left(\frac{1}{\Box}R\right)\right]\right\}. \tag{2.17}$$

Choosing now a conformal frame Eq. (2.3), where $1/\Box R=-2\rho$, we recover the previous expressions for $\langle T_{ab}\rangle$, namely Eqs. (2.5). The functions $t_\pm(x^\pm)$ can always be added to $\langle T_{ab}\rangle$ because they are compatible with the conservation equations. Furthermore, as before being the trace anomaly state independent, S_P is the same for every quantum state. So the inclusion of $t_\pm$ is necessary to specify the state. Within this respect, one should also note that the trace anomaly determines only the Weyl noninvariant part of the effective action, namely S_P. The complete effective action could in principle contain also Weyl invariant nonlocal terms. These, however, do not contribute to the trace $\langle T\rangle$ and by requiring the conservation equations one concludes that their contribution to $\langle T_{\pm\pm}\rangle$ should be of the form $t_\pm(x^\pm)$.

As a final remark, one should remind that the renormalization procedure which leads to S_P in principle also allows for the presence of a two dimensional cosmological constant term [12]. The importance of such a term will be considered in Sec. VI.

III. THE 2D SCHWARZSCHILD BLACK HOLE

We now apply the results of the previous section to the 2D Schwarzschild black hole. In the Eddington-Finkelstein null frame $\{u,v\}$, the metric reads

$$ds^2=-\left(1-\frac{2M}{r}\right)du\,dv, \tag{3.1}$$

where

$$v=t+r_*,\quad u=t-r_* \tag{3.2}$$

and

$$r_*=\int\frac{dr}{1-2M/r}=r+2M\ln\left|\frac{r}{2M}-1\right|. \tag{3.3}$$

M represents the mass of the black hole. Expansion of the field operator $\hat{f}$ in the modes

$$(4\pi w)^{-1/2}e^{-iwv},\quad (4\pi w)^{-1/2}e^{-iwu} \tag{3.4}$$

defines a conformal state known as the Boulware vacuum $|B\rangle$. Application of Eqs. (2.5) gives [23]

$$\langle B|T_{uu}|B\rangle=\langle B|T_{vv}|B\rangle=\frac{1}{24\pi}\left(-\frac{M}{r^3}+\frac{3}{2}\frac{M^2}{r^4}\right),$$
$$\langle B|T_{uv}|B\rangle=-\frac{1}{24\pi}\left(1-\frac{2M}{r}\right)\frac{M}{r^3}. \tag{3.5}$$

As one immediately sees, the modes in Eq. (3.4) reduce at infinity to the usual Minkowski ingoing and outgoing plane waves and there $\langle B|T_{ab}|B\rangle=0$. So the state $|B\rangle$ reproduces at infinity the familiar notion of an empty vacuum state as inferred from Minkowski field theory. One can think of this feature as the reason for selecting $|B\rangle$ among the various candidates for a reasonable vacuum state of the theory.

If the behavior of $|B\rangle$ at infinity seems quite reasonable, the same cannot be said for the horizon $r=2M$. One expects in fact that, if these regions belong to the physical space-time manifold, $\langle T_{ab}\rangle$ should be finite there with respect to a local orthonormal frame. It can be shown that $\langle T_{ab}\rangle$ is regular on the future horizon if as $r\to 2M$ [24]

$$|\langle T_{vv}\rangle|<\infty,$$
$$(r-2M)^{-1}|\langle T_{uv}\rangle|<\infty,$$
$$(r-2M)^{-2}|\langle T_{uu}\rangle|<\infty. \tag{3.6}$$

The regularity on the past horizon is expressed by similar inequalities with u and v interchanged. It is now clear that $\langle B|T_{ab}|B\rangle$ is not regular both on the future and the past horizons. This behavior is connected to the fact that the state $|B\rangle$ is defined in terms of the (u,v) modes in Eq. (3.4) which oscillate infinitely on the horizon. Physically, the state $|B\rangle$ is supposed to describe the vacuum polarization of the space-time exterior to a static massive body whose radius is bigger than $2M$.

A coordinate system regular on the horizons is the Kruskal $\{U,V\}$ one defined in terms of the $\{u,v\}$ frame as (for $r>2M$)

$$U=-4Me^{-u/4M}, \quad V=4Me^{v/4M}. \tag{3.7}$$

Expansion of the field operator $\hat{f}$ in the Kruskal modes

$$(4\pi\tilde{w})^{-1/2}e^{-i\tilde{w}V}, \quad (4\pi\tilde{w})^{-1/2}e^{-i\tilde{w}U} \tag{3.8}$$

defines the Hartle-Hawking state $|H\rangle$. Evaluating the Schwarzian derivatives between $U(V)$ and $u(v)$, from Eqs. (2.9)–(2.13), we obtain

$$\langle H|T_{uu}|H\rangle=\langle H|T_{vv}|H\rangle=\frac{1}{768\pi M^2}\left(1-\frac{2M}{r}\right)^2 \times\left(1+\frac{4M}{r}+\frac{12M^2}{r^2}\right),$$
$$\langle H|T_{uv}|H\rangle=-\frac{1}{24\pi}\left(1-\frac{2M}{r}\right)\frac{M}{r^3}. \tag{3.9}$$

This state leads therefore to expectation values regular on both future and past horizons. The Kruskal modes, however, do not reduce asymptotically to standard Minkowski plane-waves. As a consequence, $\langle H|T_{ab}|H\rangle$ does not vanish at infinity. $|H\rangle$ is a thermal state at the Hawking temperature

$$T_H=\frac{1}{8\pi M} \tag{3.10}$$

and describes the thermal equilibrium of a black hole enclosed in a box with its radiation.

The last example we shall present deals, unlike the previous ones, with a dynamical situation, namely the formation of a black hole by gravitational collapse. It will be of fundamental relevance for the subsequent discussion. Let us consider, to limit the mathematical complexity, the simple case where the black hole is formed by the collapse of a shock-wave at $v=v_0$ [25] (for the timelike case see for instance [26]). In the ''in'' region $v<v_0$ the space-time is flat

$$ds^2_{in}=-du_{in}dv_{in}, \tag{3.11}$$

where u_{in} and v_{in} are the usual retarded and advanced Minkowski null coordinates

$$u_{in}=t_{in}-r_{in}, \quad v_{in}=t+r_{in}. \tag{3.12}$$

For $v>v_0$ the ''out'' geometry describes a black hole of mass M

$$ds^2_{out}=-(1-2M/r)dudv. \tag{3.13}$$

Matching the two geometries at $v=v_0$, we have

$$v=v_{in},$$
$$u=u_{in}-4M\ln\left(\frac{v_0-u_{in}-4M}{4M}\right). \tag{3.14}$$

We choose the quantum state to correspond to Minkowski vacuum on past null infinity. Call this state $|in\rangle$. Therefore $\langle in|T_{ab}|in\rangle=0$ for $v<v_0$.

The evaluation of the expectation values for $v>v_0$ requires the Schwarzian derivative between u and u_{in}. The net result is, for $v>v_0$,

$$\langle in|T_{uu}|in\rangle=\frac{1}{24\pi}\left(-\frac{M}{r^3}+\frac{3}{2}\frac{M^2}{r^4}-\frac{8M}{(u_{in}-v_0)^3}-\frac{24M^2}{(u_{in}-v_0)^4}\right),$$
$$\langle in|T_{vv}|in\rangle=\frac{1}{24\pi}\left(-\frac{M}{r^3}+\frac{3}{2}\frac{M^2}{r^4}\right)=\langle B|T_{vv}|B\rangle,$$
$$\langle in|T_{uv}|in\rangle=-\frac{1}{24\pi}\left(1-\frac{2M}{r}\right)\frac{M}{r^3}=\langle B|T_{uv}|B\rangle. \tag{3.15}$$

In the limit $u_{in}\to v_0-4M$ (i.e., the shell is close to crossing the horizon) at infinity we find a net flux

$$\langle T_{uu}\rangle\to\frac{1}{768\pi M^2} \tag{3.16}$$

representing the Hawking flux of evaporation at the correct Hawking temperature T_H. In the above limit ($u_{in}\to v_0-4M$), all time dependence disappears

$$\langle in|T_{uu}|in\rangle=\frac{1}{768\pi M^2}\left(1-\frac{2M}{r}\right)^2\times\left(1+\frac{4M}{r}+\frac{12M^2}{r^2}\right) \tag{3.17}$$

and the state $|in\rangle$ becomes what is called the Unruh state $|U\rangle$, which is obtained by expanding the field operator $\hat{f}$ in modes obtained by using the coordinate U for the outgoing modes and the coordinate v for the ingoing ones. Note that $\langle U|T_{ab}|U\rangle$ is regular on the future horizon [see Eqs. (3.6)]. The singularity on the past horizon is completely spurious, since for the black hole formed by gravitational collapse there is no past horizon.

These three examples were given not only as an application of the formalism, but rather to show how for the Schwarzschild space-time, the two-dimensional $\langle T_{ab}\rangle$ given by the Polyakov action reproduces the basic qualitative features of the 4D $\langle T_{\mu\nu}\rangle$. This qualitative agreement of the simple two dimensional calculations with the more complicated four dimensional ones is quite amazing. It has stimulated investigations of the backreaction problem, namely the evolution of the black hole as it emits Hawking radiation, by means of two dimensional models where the classical dilaton-gravity action is improved by adding the Polyakov term S_P. This gives an effective action which describes the effect of the quantized matter on the geometry, i.e., the backreaction. The same problem cannot even be attacked in the physical 4D context.

The coupling of the minimal 2D massless scalar fields to the spherically symmetric reduced Einstein-Hilbert action (or similar) to evaluate the backreaction might sound too naive. One can cogently argue that in a ''realistic'' 2D matter-gravity theory also the matter sector should derive, through dimensional reduction, from a consistent 4D theory. However, before embarking in backreaction calculations one should assure that these more ''sophisticated'' 2D models produce a $\langle T_{ab}\rangle$ which for the Schwarzschild space-time is at least in qualitative agreement with the 4D $\langle T_{\mu\nu}\rangle$ as it was for the naive Polyakov theory. Otherwise these ''sophisticated'' models suffer from physical inconsistency.

IV. MINIMALLY COUPLED 4D FIELDS

As we have seen, while the gravitational part of the action has a four dimensional origin, the matter sector is two dimensional. It seems therefore natural, in the search for a more physical model, to require that also the matter fields should be defined *ab initio* in 4D [15,16,17,18,19]. Restricting the attention to minimally coupled 4D scalar fields, one has that the corresponding 4D action reads

$$S_M^{(4)}=-\frac{1}{(4\pi)^2}\int d^4x\sqrt{-g^{(4)}}(\nabla f)^2. \tag{4.1}$$

Under the assumption of spherical symmetry, the 4D metric can be written as

$$ds^2=g_{ab}dx^a dx^b+e^{-2\phi}d\Omega^2, \tag{4.2}$$

where $g_{ab}(x^a)$, $a,b=1,2$, is the two-dimensional metric and $d\Omega^2$ the line element of the unit two-sphere. Performing the dimensional reduction in Eq. (4.1) and using Eq. (4.2), we arrive at a 2D action for our scalar fields

$$S_M^{(2)}=-\frac{1}{4\pi}\int d^2x\sqrt{-g^{(2)}}e^{-2\phi}(\nabla f)^2, \tag{4.3}$$

leading to the field equation

$$\nabla^\alpha(e^{-2\phi}\nabla_\alpha f)=0. \tag{4.4}$$

Comparing the action (4.3) to Eq. (2.1), we see that the scalar fields, from a 2D point of view, are still conformal, but there is now a coupling between them and the dilaton ϕ. This makes the trace anomaly to differ from Eq. (2.7) by extra ϕ terms [15,16,17,18,19]

$$\langle T\rangle=\frac{1}{24\pi}[R-6(\nabla\phi)^2+6\Box\phi], \tag{4.5}$$

which is still state independent. The coefficient of the last term in Eq. (4.5) is not unambiguously given in the literature, depending on the functional measure used for the scalar fields, i.e., genuine 2D versus spherically symmetric reduced $4D$. The last choice is the one which leads to our Eq. (4.5). In a generic conformal frame $\{x^\pm\}$, we have

$$\langle T_{+-}\rangle=-\frac{1}{12\pi}(\partial_+\partial_-\rho+3\partial_+\phi\partial_-\phi-3\partial_+\partial_-\phi). \tag{4.6}$$

The problem we have to face now is to construct the other components of $\langle T_{ab}\rangle$ for this improved theory. Following the analysis of the previous section, one could integrate the 2D conservation equations $\nabla_a\langle T^a_b\rangle=0$, obtaining (this is the approach of [18])

$$\begin{aligned}\langle T_{\pm\pm}\rangle=&-\frac{1}{12\pi}(\partial_\pm\rho\partial_\pm\rho-\partial^2_\pm\rho-t_\pm)\\&-\frac{1}{4\pi}\left(\frac{1}{\partial_\pm}(2\partial_\pm\rho\partial_+\phi\partial_-\phi)-\frac{\partial_\pm}{\partial_\mp}(\partial_-\phi\partial_+\phi)\right)\\&+\frac{1}{4\pi}\left(\frac{1}{\partial_\mp}(2\partial_\pm\rho\partial_+\partial_-\phi)-\partial^2_\pm\phi\right),\end{aligned} \tag{4.7}$$

where we have used the shorthand notation

$$\frac{1}{\partial_\pm}=\int dx^\pm. \tag{4.8}$$

The functions $t_\pm(x^\pm)$ in Eqs. (4.7) are arbitrary integration functions. Comparing Eqs. (4.7) with Eqs. (2.5), we see the appearance of dilaton dependent terms.
The other approach that we can follow, again as in the previous section, is to functionally integrate the trace anomaly Eq. (4.5) to obtain the 2D effective action (anomaly induced effective action) [15,16,19]

$$S_{aind}=-\frac{1}{2\pi}\int d^2x\sqrt{-g}\left(\frac{1}{48}R\frac{1}{\Box}R-\frac{1}{4}(\nabla\phi)^2\frac{1}{\Box}R+\frac{1}{4}\phi R\right). \tag{4.9}$$

The first term in this action is S_P of Eq. (2.16), the Polyakov action. However, a new nonlocal term has now appeared in the effective action, the second one. The last term, on the contrary, is local. By varying S_{aind} with respect to the 2D metric g_{ab}, we find

$$\langle T_{ab}\rangle=\langle T^P_{ab}\rangle+\frac{1}{8\pi}\Bigg\{-\frac{g_{ab}}{2}\left[(\nabla\phi)^2\left(\frac{1}{\Box}R\right)+\nabla^c\left(\frac{1}{\Box}(\nabla\phi)^2\right)\nabla_c\left(\frac{1}{\Box}R\right)-2(\nabla\phi)^2\right]+\partial_a\phi\partial_b\phi\left(\frac{1}{\Box}R\right)+\frac{1}{2}\left[\nabla_a\left(\frac{1}{\Box}(\nabla\phi)^2\right)\nabla_b\left(\frac{1}{\Box}R\right)+\nabla_b\left(\frac{1}{\Box}(\nabla\phi)^2\right)\nabla_a\left(\frac{1}{\Box}R\right)\right]-\nabla_a\nabla_b\left(\frac{1}{\Box}(\nabla\phi)^2\right)\Bigg\}-\frac{1}{8\pi}(g_{ab}\Box\phi-\nabla_a\nabla_b\phi), \tag{4.10}$$

where $\langle T^P_{ab}\rangle$ was given Eqs. (2.17) and comes from the Polyakov term in S_{aind}. In the conformal frame $\{x^{\pm}\}$, Eqs. (4.10) read (see also [19])

$$\langle T_{\pm\pm}\rangle=\langle T^P_{\pm\pm}\rangle+\frac{1}{2\pi}\left(\rho\partial_\pm\phi\partial_\pm\phi+\frac{1}{2}\frac{\partial_\pm}{\partial_\mp}(\partial_+\phi\partial_-\phi)\right)-\frac{1}{4\pi}(-2\partial_\pm\rho\partial_\pm\phi+\partial_\pm^2\phi) \tag{4.11}$$

and the trace is obviously Eq. (4.5). Surprisingly, the two expressions of $\langle T_{ab}\rangle$ given, namely Eqs. (4.7) obtained by integrating the 2D conservation equations $\nabla_a\langle T^a_b\rangle=0$ and Eqs. (4.11) obtained by functional differentiation of S_{aind}, do not coincide (whatever the functions $t_\pm$ might be). We shall see in Sec. VI that the procedure followed to get Eqs. (4.7) is not justified. Therefore we shall discuss, here, only the $\langle T_{ab}\rangle$ given by Eqs. (4.6) and (4.11).

Before starting the calculation of $\langle T_{ab}\rangle$ for the Schwarzschild black hole, one has to implement Eq. (4.11) by a state dependent term which selects the state in which the expectation values are taken. Naively, one could just add a term $t_\pm(x^\pm)$ as in the previous section, since it is compatible with the 2D conservation equations satisfied by the Polyakov term $\nabla_a\langle T^{Pa}_b\rangle=0$. However more care is required. Extra terms in $\langle T_{\pm\pm}\rangle$ arise as a consequence of the ''anomalous'' transformation of $\langle T_{ab}\rangle$ under the transformation $x^\pm\to\tilde{x}^\pm$.

Let us identify the previous expression Eqs. (4.11) as expectation values of T_{ab} in the state $|x^\pm\rangle$, i.e., $\langle x^\pm|T_{ab}|x^\pm\rangle$. Consider now, as we did before, the $|\tilde{x}^\pm\rangle$ state. We already know how the first term in Eq. (4.11), the Polyakov one, transforms. It is also easy to verify that the terms obtained by variation of ϕR in S_{aind} (like the trace) is state independent. We come now to the remaining term in Eq. (4.11), the second one; call it $T^{(2)}_{\pm\pm}$. We find

$$\langle\tilde{x}^\pm|T^{(2)}_{--}|\tilde{x}^\pm\rangle=\langle x^\pm|T^{(2)}_{--}|x^\pm\rangle+\frac{1}{4\pi}\left(\partial_-\phi\partial_-\phi\ln(FG)+\frac{F'}{F}\int dx^+\partial_+\phi\partial_-\phi\right) \tag{4.12}$$

and similarly for $T^{(2)}_{++}$ by interchanging $-$ with $+$ and F with G. F and G are defined as in Eqs. (2.11) and (2.13). Summing up, we have that for the theory described by S_{aind}

$$\langle\tilde{x}^\pm|T_{++}|\tilde{x}^\pm\rangle=\langle x^\pm|T_{++}|x^\pm\rangle+\frac{1}{24\pi}\left(\frac{G''}{G}-\frac{1}{2}\frac{G'^2}{G^2}\right)+\frac{1}{4\pi}\left(\partial_+\phi\partial_+\phi\ln(FG)+\frac{G'}{G}\int dx^-\partial_+\phi\partial_-\phi\right), \tag{4.13}$$

$$\langle\tilde{x}^\pm|T_{--}|\tilde{x}^\pm\rangle=\langle x^\pm|T_{--}|x^\pm\rangle+\frac{1}{24\pi}\left(\frac{F''}{F}-\frac{1}{2}\frac{F'^2}{F^2}\right)+\frac{1}{4\pi}\left(\partial_-\phi\partial_-\phi\ln(FG)+\frac{F'}{F}\int dx^+\partial_+\phi\partial_-\phi\right), \tag{4.14}$$

$$\langle\tilde{x}^\pm|T_{-+}|\tilde{x}^\pm\rangle=\langle x^\pm|T_{-+}|x^\pm\rangle. \tag{4.15}$$

So going from one conformal state to another $\langle T_{ab}\rangle$ does not only acquire a term proportional to the Schwarzian derivative, but also the last two terms in Eqs. (4.13) and (4.14). These do not represent, unlike $t_\pm$, massless 2D radiation and are much more complicated in this more ''sophisticated'' 2D model. Being these new state dependent terms nonlocal, there is a serious danger that they destroy the nice qualitative agreement in a Schwarzschild background between the prediction of the Polyakov EMT $\langle T^P_{ab}\rangle$ and the 4D $\langle T_{\mu\nu}\rangle$. In order to see if this is the case, we now calculate, using Eqs. (4.13), (4.14) and (4.15), the $\langle T_{ab}\rangle$ for the three states ($|B\rangle$, $|H\rangle$ and $|U\rangle$) defined on the Schwarzschild space-time in Sec. III and compare the result we obtain to the 4D $\langle T_{\mu\nu}\rangle$ described in the introduction.

For the Boulware state $|B\rangle$, we have $\tilde{x}^{\pm}=x^{\pm}=(u,v)$ where (u,v) are Eddington-Finkelstein coordinates [see Eqs. (3.2)]. Equations (4.13), (4.14) and (4.15) give

$$\langle B|T_{uu}|B\rangle=\langle B|T_{vv}|B\rangle=\frac{1}{24\pi}\left(-\frac{M}{r^3}+\frac{3}{2}\frac{M^2}{r^4}\right)+\frac{1}{16\pi}\left(1-\frac{2M}{r}\right)^2\frac{1}{r^2}\ln\left(1-\frac{2M}{r}\right),$$

$$\langle B|T_{uv}|B\rangle=-\frac{1}{24\pi}\left(1-\frac{2M}{r}\right)\frac{M}{r^3}+\frac{1}{8\pi}\left(1-\frac{2M}{r}\right)\frac{M}{r^3}. \tag{4.16}$$

Note first that for $M=0$ $|B\rangle$ becomes the usual Minkowski vacuum $|M\rangle$ and Eqs. (4.16) tell us that

$$\langle M|T_{ab}|M\rangle=0. \tag{4.17}$$

This result, as we shall see in the next section, is not so trivial as it appears. It implies that Minkowski space-time is a consistent solution of the semiclassical field equations. Needless to say that any other choice in the coefficient of the $\Box\phi$ term in the trace anomaly would lead to a nonvanishing $\langle T_{ab}\rangle$ in Minkowski space. Coming back to the Schwarzschild case ($M\neq 0$), we see that Eqs. (4.16) contain, in addition to the terms obtained by solely S_P, a term proportional to $(f^2/r^2)\ln f$, where $f=1-2M/r$. This gives, in Kruskal coordinates, a ''weak'' logarithmic divergence on the horizon [see eqs. (3.6)]. This divergence is however subleading when compared to the ''strong'' divergence $\sim 1/V^2$ or $\sim 1/U^2$ coming from $\langle B|T^P_{ab}|B\rangle$. Therefore, the physical features of the state $|B\rangle$ remain unaltered; the ''sophisticated'' S_{aind} introduces just extra vacuum polarization terms in the stress tensor in addition to those obtained by S_P. $|B\rangle$ can reasonably describe even in this theory the vacuum polarization of the space-time outside a static star. The qualitative agreement between $\langle B|T_{ab}|B\rangle$ and the 4D $\langle B|T_{\mu\nu}|B\rangle$ is still satisfactory.

Let us now consider the state $|H\rangle$ obtained by choosing $\tilde{x}^{\pm}=(U,V)$ [Kruskal coordinates, see Eqs. (3.7)] and $x^{\pm}=(u,v)$. Here, as we shall see, things are not so ''nice'' as before. In this state we get, from Eqs. (4.13)–(4.15),

$$\langle H|T_{uu}|H\rangle=\langle H|T_{vv}|H\rangle=\frac{1}{768\pi M^2}\left(1-\frac{2M}{r}\right)^2\times\left(1+\frac{4M}{r}+\frac{12M^2}{r^2}\right)+\frac{1}{16\pi}\left[\left(1-\frac{2M}{r}\right)^2\frac{1}{r^2}\times\left(-\ln r-\frac{r}{2M}\right)-\frac{1}{2M}\left(-\frac{1}{r}+\frac{M}{r^2}+\frac{1}{4M}\right)\right],$$

$$\langle H|T_{uv}|H\rangle=\langle B|T_{uv}|B\rangle. \tag{4.18}$$

Inspection of Eqs. (4.18) reveals that the uu and vv components of $\langle H|T_{ab}|H\rangle$ vanish like $(r-2M)^2$ on the horizon. Therefore $\langle H|T_{ab}|H\rangle$ is regular on both the future and past horizons $r=2M$, as expected. However its behavior as $r\to\infty$ is quite surprising

$$\langle H|T_{uu}|H\rangle=\langle H|T_{vv}|H\rangle\to\frac{1}{768\pi M^2}(1-6), \tag{4.19}$$

where the first term on the RHS comes from $\langle T^P_{ab}\rangle$, whereas the unexpected negative contribution (the -6) comes from the second nonlocal term in S_{aind} [see Eq. (4.9)] [15]. This result looks rather unphysical, since it would suggest that the black hole is in thermal equilibrium with a thermal bath of negative energy. Some clarifications are necessary to understand the validity of Eqs. (4.18) and their asymptotic limit. The lower bound r_0 of integration in the r integral present in Eqs. (4.13), (4.14) was taken to be $r_0=2M$. Any other choice [some of them might eliminate the negativity of the net flux in the asymptotic limit Eq. (4.19)] leads to a $\langle H|T_{ab}|H\rangle$ singular on the horizon. For the state $|B\rangle$ the stress tensor does not depend on the choice of r_0. Furthermore, as said before, the coefficient in front of the $\Box\phi$ term in the trace anomaly in Eq. (4.5) has been source of debate in the literature. For the problem at hand, we stress that the $\Box\phi$ term affects only the local part of $\langle T_{ab}\rangle$ giving no extra contribution to the Hawking radiation and, therefore, has nothing to do with the puzzling result we have in Eq. (4.19). So we have firm evidence that the 2D stress tensor $\langle T_{ab}\rangle$ constructed from S_{aind} in the $|H\rangle$ state is in strong qualitative disagreement with the well established result of the 4D $\langle H|T_{\mu\nu}|H\rangle$ which, we remind, describes a black hole in thermal equilibrium with a positive energy bath of radiation at the temperature T_H.

One can indeed find an equilibrium state $|\tilde{x}^{\pm}\rangle$ which is regular on the horizons and unlike $|H\rangle$ has a positive flux at infinity [27]. Mathematically, this is done by fine tuning two constants. One is the lower bound r_0 in the r integration. The second, say α, is related to the definition of the $\{\tilde{x}^{\pm}\}$ frame

$$\tilde{x}^{+}=\alpha e^{v/\alpha}, \quad \tilde{x}^{-}=-\alpha e^{-u/\alpha}. \tag{4.20}$$

The choice of exponential relation is imposed by the need of having a constant Schwarzian derivative as required for equilibrium. The outgoing flux can then be parametrized by a third constant β (which depends on r_0 and α) as (in the limit $r\to\infty$)

$$\langle\tilde{x}^{\pm}|T_{uu}|\tilde{x}^{\pm}\rangle=\langle\tilde{x}^{\pm}|T_{vv}|\tilde{x}^{\pm}\rangle\sim\frac{\beta}{768\pi M^2}. \tag{4.21}$$

According to the previous calculations, no regular solution for $\beta=1$ exists. For $\beta\neq 1$ and positive, one can find α ($\neq 4M$) and r_0 ($\neq 2M$) which allows regularity of $\langle\tilde{x}^{\pm}|T_{ab}|\tilde{x}^{\pm}\rangle$ on the horizons. Needless to say that the state so constructed has nothing to do with $|H\rangle$ and its physical significance, if it exists, is completely obscure.

Complete disagreement between the prediction of S_{aind} and the real 4D theory emerges also when considering our last example: the collapsing shell. Performing the calculation along the lines of the previous section, we have

$$\langle in|T_{ab}|in\rangle=0 \tag{4.22}$$

for $v<v_0$. When, instead, $v>v_0$

$$\langle in|T_{uu}|in\rangle=\frac{1}{12\pi}\left(\frac{ff''}{8}-\frac{f'^2}{16}-\frac{3}{4}\frac{M^2}{r^4(u,v_0)}+\frac{M}{2r^3(u,v_0)}\right)+\frac{1}{16\pi}\frac{f^2}{r^2}\left[\ln\frac{f}{f(u,v_0)}-\frac{f^2(u,v_0)}{r^2(u,v_0)}+\frac{2M}{r^2(u,v_0)}\times\left(-\frac{1}{r}+\frac{M}{r^2}+\frac{1}{r(u,v_0)}-\frac{M}{r^2(u,v_0)}\right)\right],$$

$$\langle in|T_{vv}|in\rangle=\langle B|T_{vv}|B\rangle,$$

$$\langle in|T_{uv}|in\rangle=\langle B|T_{uv}|B\rangle, \tag{4.23}$$

where $f=1-2M/r$ and $r(u,v_0)=(v_0-u_{in})/2$. In the first of Eqs. (4.23) the lower bound in the v integration has been taken as v_0, the position of the shell, since this appears as the more natural choice. The stress tensor is regular on the future horizon. As the shell approaches the horizon, the outgoing flux at infinity looks like Eq. (4.19) (as in [15])

$$\langle in|T_{uu}|in\rangle\to\frac{1}{768\pi M^2}(1-6) \tag{4.24}$$

indicating that the black hole ''antievaporates'' absorbing energy from the vacuum. On the other hand, as $r\to 2M$

$$\langle in|T_{vv}|in\rangle\to-\frac{1}{768\pi M^2}, \tag{4.25}$$

i.e., one has the usual negative energy inflow, which makes the interpretation even more puzzling.

Finally, it is worth noting that if we used the $\langle T_{ab}\rangle$ of eqs. (4.7), the one constructed by integrating the conservation equations $\nabla_a\langle T^a_b\rangle=0$, we would obtain (see [18]), instead of Eq. (4.24) and in the same limit,

$$\langle T_{uu}\rangle\to\frac{1}{768\pi M^2}(1-3) \tag{4.26}$$

which unfortunately does not improve the situation.

Concluding this section, we arrive at the unsatisfactory situation in which the ''sophisticated'' 2D theory described by S_{aind} produces a $\langle T_{ab}\rangle$ for the Schwarzschild black hole which, apart from the $|B\rangle$ state, not only is in qualitative disagreement with all that is known about the 4D $\langle T_{\mu\nu}\rangle$, but, even more seriously, it is physically unacceptable. Its use in backreaction models is therefore highly questionable.

V. AN IMPROVED THEORY

As already said, the conformal anomaly determines only the Weyl noninvariant part of the effective action, namely S_{aind}. The complete effective action should also contain a part invariant under Weyl transformations. The authors of Ref. [15] tried to calculate this part perturbatively, since unlike S_{aind} it cannot be computed exactly. Using a simple classical approximation to the heat kernel, they proposed to add to S_{aind} the following nonlocal Weyl invariant term of the Coleman-Weinberg type

$$S_{wi}=\frac{1}{8\pi}\int d^2x\sqrt{-g}\left[-(\nabla\phi)^2\frac{1}{\Box}R+\phi R+[-\Box\phi+(\nabla\phi)^2]\left(1-\ln\frac{[-\Box\phi+(\nabla\phi)^2]}{\mu^2}\right)\right], \tag{5.1}$$

where μ is an arbitrary renormalization scale. One sees that the nonlocal term in Eq. (5.1) cancels exactly the second nonlocal term in S_{aind} [see eq. (4.9)], leaving as unique nonlocal term the Polyakov one

$$S_{imp}=S_{aind}+S_{wi}=\frac{1}{8\pi}\int d^2x\sqrt{-g}\left[-\frac{1}{12}R\frac{1}{\Box}R+[-\Box\phi+(\nabla\phi)^2]\left(1-\ln\frac{[-\Box\phi+(\nabla\phi)^2]}{\mu^2}\right)\right]. \tag{5.2}$$

At first sight, the advantage of this new formulation of the 2D theory is clear: the second nonlocal term in S_{aind}, responsible for the appearance of the unphysical -6 in the Hawking flux [see eqs. (4.19) and (4.24)] has disappeared. This is the main argument used in [15] to show the accordance of this model with the 4D picture of Hawking black hole evaporation. The flux at infinity is now given by the Polyakov term as in the naive theory of Secs. II and III, leading to the expected value $1/768\pi M^2$. However, let us analyze in some detail the components of $\langle T^{imp}_{ab}\rangle$ in this theory. We have

$$\langle T^{imp}_{ab}\rangle=\langle T^P_{ab}\rangle+\frac{1}{4\pi}\left[-\frac{g_{ab}}{2}\left(-\Box\phi+(\nabla\phi)^2-(\nabla\phi)^2\ln\frac{[-\Box\phi+(\nabla\phi)^2]}{\mu^2}-\nabla^c\phi\nabla_c\ln\frac{[-\Box\phi+(\nabla\phi)^2]}{\mu^2}\right)-\partial_a\phi\partial_b\phi\ln\frac{[-\Box\phi+(\nabla\phi)^2]}{\mu^2}-\frac{1}{2}\partial_a\phi\partial_b\ln\frac{[-\Box\phi+(\nabla\phi)^2]}{\mu^2}-\frac{1}{2}\partial_b\phi\partial_a\ln\frac{[-\Box\phi+(\nabla\phi)^2]}{\mu^2}\right]. \tag{5.3}$$

As usual, choosing a conformal frame $\{x^{\pm}\}$, we find

$$\langle T^{imp}_{\pm\pm}\rangle=\langle T^{P}_{\pm\pm}\rangle-\frac{1}{4\pi}\Bigg\{\partial_{\pm}\phi\partial_{\pm}\phi\ln\frac{[-\Box\phi+(\nabla\phi)^2]}{\mu^2}+\partial_{\pm}\phi\partial_{\pm}\ln\frac{[-\Box\phi+(\nabla\phi)^2]}{\mu^2}\Bigg\},$$

$$\langle T^{imp}_{+-}\rangle=\langle T_{+-}\rangle, \tag{5.4}$$

where the RHS of the second of Eqs. (5.4) is still given by Eq. (4.6), since S_{wi} does not alter, by construction, the trace anomaly. In the above

$$[-\Box\phi+(\nabla\phi)^2]=4e^{-2\rho}(\partial_+\partial_-\phi-\partial_+\phi\partial_-\phi). \tag{5.5}$$

From Eqs. (5.4) we see that being the term in the curl brackets local, the difference of $\langle T^{imp}_{ab}\rangle$ between two states is simply the Schwarzian derivative as in Eqs. (2.9)–(2.13). Inserting the Schwarzschild solution in Eq. (5.4), we find ($f=1-2M/r$)

$$\langle T^{imp}_{uu}\rangle=\langle T^{P}_{uu}\rangle+\frac{1}{16\pi}\left[-\frac{f^2}{r^2}\ln\frac{f'}{\mu^2 r}+\frac{f^2}{r}\left(\frac{f''}{f'}-\frac{1}{r}\right)\right],$$

$$\langle T^{imp}_{vv}\rangle=\langle T^{P}_{vv}\rangle+\frac{1}{16\pi}\left[-\frac{f^2}{r^2}\ln\frac{f'}{\mu^2 r}+\frac{f^2}{r}\left(\frac{f''}{f'}-\frac{1}{r}\right)\right], \tag{5.6}$$

where a prime indicates derivative with respect to r and $\langle T^{P}_{uu}\rangle$, $\langle T^{P}_{vv}\rangle$ are given in Sec. III for the different states. At first sight the above expression seems reasonable, just local vacuum polarization added to the Polyakov term. Let us consider, however, the case $f=1$, i.e., Minkowski space-time. One immediately sees in Eqs. (5.6) that the argument of the ln vanishes and $\langle T^{imp}_{ab}\rangle$ diverges. Therefore Minkowski vacuum is no longer a solution of the theory. The calculation in the shell collapse case of $\langle in|T^{imp}_{ab}|in\rangle$ for $v<v_0$ (i.e., in the flat portion of the spacetime inside the shell) becomes meaningless in this context. This divergence is analogous to the infrared divergence of the Coleman-Weinberg potential in the massless case.

However, in addition to the Minkowski problem, we will have dangerous divergences of $\langle T^{imp}_{ab}\rangle$ for static spacetimes in regions where the surface gravity f' vanishes. Nonextreme Reissner-Nordström spacetime is one such example. The surface gravity vanishes for $r=Q^2/M$ which lies between the inner and the outer horizon and there $\langle T_{ab}\rangle$ diverges. A similar situation happens for the Schwarzschild–de Sitter spacetime. All such features are not expected on physical ground and up to now there is no 4D evidence of such phenomena.

VI. THE FOUR DIMENSIONAL INTERPRETATION

For the reasons previously explained (Minkowski as ground state), we prefer to come back to the action S_{aind} and try to understand whether it is possible or not to extract physically sensible results. Let us consider the $\langle T_{ab}\rangle$ in Eqs. (4.11) and (4.6) and calculate its covariant divergence $\nabla_a\langle T^a_b\rangle$. As expected on the basis of the difference between Eqs. (4.7) and (4.11), the result is nonzero and reads, in the conformal frame $\{x^{\pm}\}$,

$$\nabla_a\langle T^a_{\pm}\rangle=\frac{N}{2\pi}(2\rho\partial_{\pm}\phi\partial_+\partial_-\phi+\partial_{\pm}\phi\partial_{\pm}\phi\partial_{\mp}\rho+\partial_{\pm}\phi\partial_{\mp}\phi\partial_{\pm}\rho+\partial_{\pm}\phi\partial_+\partial_-\rho). \tag{6.1}$$

The RHS of these equations are proportional to the quantum part (not to all) of the equation of motion of ϕ. We can in fact rewrite them in a more elegant form

$$\partial_{\mp}\langle T_{\pm\pm}\rangle+\partial_{\pm}\langle T_{+-}\rangle-\Gamma^{\pm}_{\pm\pm}\langle T_{+-}\rangle+\partial_{\pm}\phi\frac{\delta S_{aind}}{\delta\phi}=0. \tag{6.2}$$

Equation (6.2) can be written in covariant way as

$$\nabla_a\langle T^a_b\rangle+\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\phi}\nabla_b\phi=0. \tag{6.3}$$

This relation has a general validity and applies for all theories described by an action $S=S[g_{\mu\nu},\phi]$.[1] It shows, for instance, that the 2D conservation equations are automatically satisfied by $\langle T^{P}_{ab}\rangle$ for the simple reason that the Polyakov action S_P does not depend on ϕ. For the other theories we are concerned with in this paper a similar result is no longer valid.

The situation could seem therefore rather unsatisfactory: we started from a 4D classical theory, reduced it to 2D by assuming spherical symmetry and we are now left with a 2D effective theory where the basic ingredient, the matter energy momentum tensor, is not conserved. Equation (6.3) has however an elegant interpretation as seen from the 4D point of view. Consider the 4D action $S^{(4)}_{aind}$ which, by dimensional reduction under spherical symmetry, gives S_{aind}. We can then define the 4D energy momentum tensor $\langle T^{(4)}_{\mu\nu}\rangle$ (see also [15])

$$\langle T^{(4)}_{\mu\nu}\rangle=\frac{1}{\sqrt{-g^{(4)}}}\frac{\delta S^{(4)}_{aind}}{\delta g^{\mu\nu}_{(4)}}. \tag{6.4}$$

Under spherical symmetry, these equations translate into the following definitions ($a,b=1,2$)

$$\langle T^{(4)}_{ab}\rangle=\frac{\langle T^{(2)}_{ab}\rangle}{4\pi e^{-2\phi}},$$

$$\langle T_{\theta\theta}\rangle=\frac{\langle T_{\phi\phi}\rangle}{\sin^2\theta}=\frac{1}{8\pi\sqrt{-g^{(2)}}}\frac{\delta S_{aind}}{\delta\phi}, \tag{6.5}$$

[1]This, as W. Kummer pointed out to us, is a consequence of diffeomorphism invariance and holds in any dimension for an arbitrary dilaton gravity theory.

where we have explicitly inserted the superscripts $^{(2)}$ and $^{(4)}$ for clarification. These allow us to reinterpret Eqs. (6.3) as the conservation equations of the 4D stress tensor $\langle T^{(4)}_{\mu\nu}\rangle$, i.e., Eqs. (6.3) can be rewritten simply as

$$\nabla_\mu \langle T^{(4)\mu}_\nu \rangle = 0. \tag{6.6}$$

From the explicit form of S_{aind} Eq. (4.9) we obtain, in a conformal frame $\{x^\pm\}$,

$$8\pi\langle T_{\theta\theta}\rangle = -\frac{1}{2\pi\sqrt{-g^{(2)}}}(2\rho\partial_+\partial_-\phi + \partial_-\rho\partial_+\phi + \partial_+\rho\partial_-\phi + \partial_+\partial_-\rho). \tag{6.7}$$

The above discussion and the 4D interpretation of the failure of the 2D conservation equations can be repeated step by step for the improved theory of Eq. (5.2). In that case the angular component of $\langle T^{(4)}_{\mu\nu}\rangle$ is

$$\begin{aligned} 8\pi\langle T_{\theta\theta}\rangle = -\frac{1}{4\pi\sqrt{-g^{(2)}}}\Bigg[& -2\partial_+\partial_-\phi\left(-2\rho + \ln\frac{[4(\partial_+\partial_-\phi - \partial_+\phi\partial_-\phi)]}{\mu^2}\right) \\ & -\partial_+\phi\{-2\partial_-\rho + \partial_- \ln[4(\partial_+\partial_-\phi - \partial_+\phi\partial_-\phi)]\} \\ & -\partial_-\phi\{-2\partial_+\rho + \partial_+ \ln[4(\partial_+\partial_-\phi - \partial_+\phi\partial_-\phi)]\} \\ & + 2\partial_+\partial_-\rho - \partial_+\partial_- \ln[4(\partial_+\partial_-\phi - \partial_+\phi\partial_-\phi)]\Bigg]. \end{aligned} \tag{6.8}$$

Let us finally write in full the action of the theories we have examined. The first model (anomaly induced) is described by

$$S = S_g + S_{aind}, \tag{6.9}$$

where

$$S_g = \frac{1}{2\pi}\int d^2x\sqrt{-g}e^{-2\phi}[R + 2(\nabla\phi)^2 + 2e^{2\phi} - 2\Lambda] \tag{6.10}$$

(Λ is the 4D cosmological constant) and

$$S_{aind} = -\frac{1}{2\pi}\int d^2x\sqrt{-g}\left(\frac{1}{48}R\frac{1}{\Box}R - \frac{1}{4}(\nabla\phi)^2\frac{1}{\Box}R + \frac{1}{4}\phi R\right). \tag{6.11}$$

The resulting field equations are

$$\begin{aligned} 2r\nabla_a\nabla_b r + g_{ab}\left(1 - (\nabla r)^2 - 2r\Box r + \frac{1}{2}\Lambda r^2\right) &= 2\pi\langle T_{ab}\rangle, \\ r\Box r - \frac{1}{2}r^2R - \frac{\Lambda r^2}{2} &= -4\pi^2\langle T_{\theta\theta}\rangle, \end{aligned} \tag{6.12}$$

where we have used $r \equiv e^{-\phi}$, $\langle T_{ab}\rangle$ is given in Eqs. (4.10) and $\langle T_{\theta\theta}\rangle$ in Eq. (6.7). The improved theory of Sec. V is described by

$$S = S_g + S_{imp}, \tag{6.13}$$

where

$$S_{imp} = \frac{1}{8\pi}\int d^2x\sqrt{-g}\left[-\frac{1}{12}R\frac{1}{\Box}R + [-\Box\phi + (\nabla\phi)^2] \times\left(1 - \ln\frac{[-\Box\phi + (\nabla\phi)^2]}{\mu^2}\right)\right] \tag{6.14}$$

and the field equations are the same [with obvious substitution of the source terms with Eqs. (5.3) and (6.8)]. For the improved theory Minkowski, as we have said, is not a self-consistent solution of the equations of motion. The LHS of Eqs. (6.12) vanishes identically (for $\Lambda = 0$) whereas $\langle T^{(4)imp}_{\theta\theta}\rangle$ and $\langle T^{imp}_{ab}\rangle$ diverge, as can be seen explicitly in Eqs. (6.8) and (5.3). However, the improved theory has other interesting solutions.

The presence of a scale in the theory depletes, in this case, Minkowski space-time of its central role in favor of other geometries. Let us consider de Sitter spacetime, which is a classical solution of S_g with $\Lambda \neq 0$. One can then show, by fine tuning the arbitrary renormalization scale μ in Eq. (5.2) (i.e., $\mu^2 = \frac{2}{3}\Lambda$) and the 2D cosmological constant (that can always be added to the Polyakov term S_P), that for the de Sitter spacetime

$$\begin{aligned} \langle dS|T^{imp}_{ab}|dS\rangle &= 0, \\ \langle dS|T^{imp}_{\theta\theta}|dS\rangle &= 0, \end{aligned} \tag{6.15}$$

where $|dS\rangle$ means de Sitter invariant state, obtained by choosing $\{\tilde{x}^\pm\}$ as Gibbons-Hawking null coordinates [28]. The de Sitter spacetime does not acquire, in the improved theory of Eq. (6.13), quantum corrections and is therefore a self-consistent solution of the semiclassical equations. Despite this fact, we feel rather uneasy with the unphysical results that this improved theory predicts for Minkowski space. The same can be said for Hawking black hole evaporation as described by S_{aind}. In the next section we shall outline how, in our opinion, an effective 2D theory which can positively deal with black hole evaporation should look like.

VII. THE PHYSICAL STRESS TENSOR: A PROPOSAL

The satisfactory interpretation of $\langle T^{(2)}_{ab}\rangle$ and $\delta S/\delta\phi$ as part of $\langle T^{(4)}_{\mu\nu}\rangle$ along with the conservation equations (6.6) encourage us to adopt a 4D point of view. An ''acceptable'' 2D effective action deduced from the trace anomaly (S_{aind}) and additional Weyl invariant terms should reproduce at least the qualitative features of $\langle T^{(4)}_{\mu\nu}\rangle$ for the Schwarzschild space time. We stress that the comparison can only be qualitative, since the exact analytic expression can of course not be met by a simple 2D theory. In particular, the 4D anomalous trace $\langle T^{(4)\alpha}_\alpha\rangle$ is a local expression involving $R^{(4)}$, $R^{(4)}_{\mu\nu}$,

$C_{\alpha\beta\gamma\delta}$ [see Eq. (1.2)]: a much more complicated expression than our 2D analogous Eq. (4.5). Nevertheless, we require that some characteristic features of $\langle T^{(4)}_{\mu\nu}\rangle$ should be reproduced by an ''acceptable'' 2D theory, namely, in the spirit of the Wald's axioms,

(i) conservation equations $\nabla_\mu\langle T^{(4)\mu}_{\nu}\rangle=0$;

(ii) vanishing of $\langle T^{(4)}_{\mu\nu}\rangle$ for Minkowski vacuum;

(iii) locality (by this we mean state independence) of the 4D trace $\langle T^{(4)\alpha}_{\alpha}\rangle$.

Using the definitions given in Eqs. (6.5), we have

$$\langle T^{(4)}\rangle=e^{2\phi}\left(\frac{\langle T^{(2)}\rangle}{4\pi}+2\langle T_{\theta\theta}\rangle\right). \tag{7.1}$$

Since the 2D trace anomaly $\langle T^{(2)}\rangle$ given by Eq. (4.5) is already local, one has to require, in order to satisfy (iii), that $\langle T_{\theta\theta}\rangle$ has to enjoy the same property. Let us consider the $\langle T_{\theta\theta}\rangle$ given by S_{aind}, namely Eq. (6.7). Under the scaling $\{x^\pm\}\to\{\tilde{x}^\pm\}$, it transforms in an ''anomalous'' way, i.e.,

$$\langle\tilde{x}^\pm|T_{\theta\theta}|\tilde{x}^\pm\rangle=\langle x^\pm|T_{\theta\theta}|x^\pm\rangle-\frac{e^{-2\rho}}{8\pi^2}\left((\partial_+\partial_-\phi)\ln FG+\frac{1}{2}\partial_-\phi\frac{G'}{G}+\frac{1}{2}\partial_+\phi\frac{F'}{F}\right). \tag{7.2}$$

Because of the curl brackets term, $\langle T_{\theta\theta}\rangle$ and, hence, $\langle T^{(4)\alpha}_{\alpha}\rangle$ is state dependent, contrary to our assumption (iii). Note that $\langle T^{imp}_{\theta\theta}\rangle$ of Eq. (6.8) is state independent, however we do not consider it as a good starting point since it diverges for Minkowski spacetime, making the improved theory of Sec. V incompatible with the requirement (ii).

Our task will be to find a modified version of the stress tensor, call it $\langle T^{(4)new}_{\mu\nu}\rangle$, which does indeed fulfill all our three requirements. This tensor should in principle derive by an effective action S^{new} which is obtained implementing, as in Sec. V, S_{aind} with Weyl invariant terms. We are not able to construct S^{new} explicitly, but we shall give a sketch of how the new tensor should look like.

Let us select a conformal frame $\{x^\pm=u,v\}$ of reference which we will specify later. In an arbitrary conformal frame $\{\tilde{x}^\pm=U,V\}$, related to the previous one by the functions F and G as in Eqs. (2.11), (2.13), we have, in the state $|\tilde{x}^\pm\rangle$,

$$\langle\tilde{x}^\pm|T_{\theta\theta}|\tilde{x}^\pm\rangle=-\frac{e^{-2\tilde{\rho}}}{4\pi^2}\left(\tilde{\rho}\partial_U\partial_V\phi+\frac{1}{2}\partial_U\phi\partial_V\tilde{\rho}+\frac{1}{2}\partial_V\phi\partial_U\tilde{\rho}+\frac{1}{2}\partial_V\partial_U\tilde{\rho}\right). \tag{7.3}$$

Now define

$$\langle\tilde{x}^\pm|T^{new}_{\theta\theta}|\tilde{x}^\pm\rangle=\langle\tilde{x}^\pm|T_{\theta\theta}|\tilde{x}^\pm\rangle+\frac{e^{-2\tilde{\rho}}}{4\pi^2}\left((\partial_U\partial_V\phi)\frac{1}{2}\ln FG+\frac{1}{4}(\partial_V\phi)\frac{\dot{F}}{F}+\frac{1}{4}(\partial_U\phi)\frac{\dot{G}}{G}\right), \tag{7.4}$$

where $\dot{F}\equiv dF/dU$ and $\dot{G}\equiv dG/dV$. Using Eq. (7.2) and $\tilde{\rho}=\rho+\frac{1}{2}\ln FG$, one can show that

$$\langle\tilde{x}^\pm|T^{new}_{\theta\theta}|\tilde{x}^\pm\rangle=-\frac{e^{-2\rho}}{4\pi^2}\left((\partial_u\partial_v\phi)\rho+\frac{1}{2}\partial_u\phi\partial_v\rho+\frac{1}{2}\partial_v\phi\partial_u\rho+\frac{1}{2}\partial_u\partial_v\rho\right)=\langle x^\pm|T_{\theta\theta}|x^\pm\rangle. \tag{7.5}$$

This means that in every state the expectation value of $\langle T^{new}_{\theta\theta}\rangle$ is given by $\langle x^\pm|T_{\theta\theta}|x^\pm\rangle$. So we have achieved the state independence: under the conformal transformation $\{x^\pm\}\to\{\tilde{x}^\pm\}\langle T_{\theta\theta}\rangle$ remains unchanged as required by (iii). The reference state $|x^\pm\rangle$ is chosen, in view of the requirement (ii), such that $\{x^\pm\}$ are Minkowskian coordinates at infinity. For the Schwarzschild spacetime this implies that $|x^\pm\rangle=|B\rangle$.

Having defined $\langle T^{new}_{\theta\theta}\rangle$ which allows $\langle T^{(4)\alpha}_{\alpha}\rangle$ to be state independent, in order to enforce the conservation equations $\nabla_\mu\langle T^{(4)\mu}_{\nu}\rangle=0$, as required by (i), we have to redefine $\langle T_{\pm\pm}\rangle$ as well. In the $\{\tilde{x}^\pm=U,V\}$ frame, we then have

$$\begin{aligned}\langle\tilde{x}^\pm|T^{new}_{UU}|\tilde{x}^\pm\rangle=&\frac{1}{2\pi}\Bigg[\left((\partial_U\phi)^2\tilde{\rho}+\frac{1}{2}\partial_U\int dV\partial_U\phi\partial_V\phi\right)\\&-\frac{1}{4\pi}\left((\partial_U\phi)^2\ln FG+\dot{F}\int dV\partial_V\phi\partial_U\phi\right)\\&-\frac{1}{4\pi}(-2\partial_U\tilde{\rho}\partial_U\phi+\partial_U^2\phi)\Bigg]\\&+\langle\tilde{x}^\pm|T^P_{UU}|\tilde{x}^\pm\rangle\end{aligned} \tag{7.6}$$

where the last term is given in detail in Sec. II. Similarly,

$$\begin{aligned}\langle\tilde{x}^\pm|T^{new}_{VV}|\tilde{x}^\pm\rangle=&\frac{1}{2\pi}\Bigg[\left((\partial_V\phi)^2\tilde{\rho}+\frac{1}{2}\partial_V\int dU\partial_U\phi\partial_V\phi\right)\\&-\frac{1}{4\pi}\left((\partial_V\phi)^2\ln FG+\dot{G}\int dU\partial_V\phi\partial_U\phi\right)\\&-\frac{1}{4\pi}(-2\partial_V\tilde{\rho}\partial_V\phi+\partial_V^2\phi)\Bigg]\\&+\langle\tilde{x}^\pm|T^P_{VV}|\tilde{x}^\pm\rangle.\end{aligned} \tag{7.7}$$

Note that under the conformal transformation $\{U,V\}\to\{x^\pm=u,v\}$ the terms under curl brackets transform like a tensor, whereas the Polyakov term picks up the usual Schwarzian derivative. Summarizing, from Eq. (7.6) and (7.7) we have

$$\langle\tilde{x}^{\pm}|T_{uu}^{new}|\tilde{x}^{\pm}\rangle=\frac{1}{2\pi}\left((\partial_u\phi)^2\rho+\frac{1}{2}\partial_u\int dv\,\partial_u\phi\partial_v\phi\right)-\frac{1}{4\pi}(-2\partial_u\rho\partial_u\phi+\partial_u^2\phi)+\frac{1}{12\pi}(\partial_u^2\rho-\partial_u\rho\partial_u\rho)+\frac{1}{24\pi}\left(\frac{F''}{F}-\frac{1}{2}\frac{F'^2}{F^2}\right)=\langle x^{\pm}|T_{uu}|x^{\pm}\rangle+\frac{1}{24\pi}\left(\frac{F''}{F}-\frac{1}{2}\frac{F'^2}{F^2}\right)\quad(7.8)$$

and similarly for u interchanged with v and F with G. The 2D trace part remains unchanged

$$\langle\tilde{x}^{\pm}|T_{uv}^{new}|\tilde{x}^{\pm}\rangle=\langle\tilde{x}^{\pm}|T_{uv}|\tilde{x}^{\pm}\rangle.\quad(7.9)$$

One can check that $\langle T_{\mu\nu}^{new}\rangle$ is conserved, has a trace which does not depend on the state and vanishes for Minkowski space-time, defined by $\rho=0$ and $e^{-\phi}=(v-u)/2$: $\langle M|T_{\mu\nu}^{new}|M\rangle=0$.

We should remind that our $\langle T_{\mu\nu}^{new}\rangle$ is defined modulo additional local terms which come from local Weyl invariant contributions that can be added to the 2D effective action. These extra terms have to vanish in Minkowski space and being local do not contribute to the Hawking radiation.

Let us now see how our procedure works for the Schwarzschild black hole. Being the reference vacuum the Boulware one, $\langle B|T_{\mu\nu}^{new}|B\rangle$ has the same form as given in Sec. IV, in particular (omitting the superscript ''new'')

$$\langle B|T_{uu}|B\rangle=\frac{1}{12\pi}\left(\frac{ff''}{8}-\frac{f'^2}{16}\right)+\frac{1}{16\pi}\frac{f^2}{r^2}\ln f,$$

$$\langle B|T_{vv}|B\rangle=\frac{1}{12\pi}\left(\frac{ff''}{8}-\frac{f'^2}{16}\right)+\frac{1}{16\pi}\frac{f^2}{r^2}\ln f,$$

$$\langle B|T_{uv}|B\rangle=\frac{1}{96\pi}ff''+\frac{1}{16\pi}\frac{ff'}{r},\quad(7.10)$$

where $f=1-2M/r$. As expected, these expressions vanish for $M=0$, confirming that Minkowski space-time is a solution of the backreaction equations. In the Hartle-Hawking state, we have

$$\langle H|T_{uu}|H\rangle(=\langle H|T_{vv}|H\rangle)=\langle B|T_{uu}|B\rangle+\frac{1}{768\pi M^2}.\quad(7.11)$$

As $r\to\infty$ the first term on the RHS of the above equation vanishes, confirming that $|H\rangle$ asymptotically describes radiation in thermal equilibrium at the correct Hawking temperature T_H. Note that we have a logarithmic divergence on the horizon (in Kruskal coordinates). This is however integrable and does not affect the regularity of the semiclassical geometry.

Finally, for the dynamical situation of a black hole formed by the gravitational collapse of a shock-wave at $v=v_0$ we have, from Eq. (7.8), (for $v>v_0$)

$$\langle in|T_{uu}|in\rangle=\frac{1}{12\pi}\left(\frac{ff''}{8}-\frac{f'^2}{16}-\frac{3}{4}\frac{M^2}{r^4(u,v_0)}+\frac{M}{2r^3(u,v_0)}\right)+\frac{1}{16\pi}\left(\frac{f^2}{r^2}\ln f-\frac{f^2(u,v_0)}{r^2(u,v_0)}\right),$$

$$\langle in|T_{vv}|in\rangle=\langle B|T_{vv}|B\rangle.\quad(7.12)$$

As the shell radius approaches the horizon $|in\rangle\to|U\rangle$ and we have

$$\langle U|T_{uu}|U\rangle=\frac{1}{768\pi M^2}\left(1-\frac{2M}{r}\right)^2\left(1+\frac{4M}{r}+\frac{12M^2}{r^2}\right)\quad(7.13)$$

leading to the expected flux at infinity (in the limit $u\to+\infty$)

$$\langle T_{uu}\rangle\to\frac{1}{768\pi M^2}.\quad(7.14)$$

VIII. CONCLUSIONS

The purpose of this paper was to extend the analysis of quantum black holes from the framework of Polyakov theory to more appealing 2D theories (S_{aind},S_{imp}) whose link to the physical four dimensions appears more direct. Despite the appeal of these ''sophisticated theories,'' their predictions turned out to be unacceptable: negative Hawking flux (S_{aind})—nonzero (diverging) renormalized stress tensor for Minkowski space-time (S_{imp}).

Given these astonishing results, we have attempted to modify the matter stress energy tensor by imposing three requirements on it. The first two, conservation equations and vanishing in Minkowski space, are quite obvious; the third (state independence of the 4D trace) escapes from the strict two-dimensional point of view of all these models. However, as the discussion of the conservation equations has clearly shown, a correct handling and understanding of these theories can only be four dimensional. Within our approach sensible results emerge that can be positively compared to the 4D ones. We are well aware that our method may appear rather rough being not based on an elegant effective action like S_{aind} and S_{imp}. Unfortunately, as they stand S_{aind} and S_{imp} cannot be the final answer.

One should however not exclude the possibility that there is no way of extracting sensible results from these hybrid lower dimensional theories and the only true improvement of the Polyakov theory for the description of quantum black holes has to be genuinely 4D.

Note added. Following our analysis Lombardo *et al.* [29] have recently argued that an expansion of the Weyl invariant part of the effective action in powers of the operator $P=\Box\phi-(\nabla\phi)^2$ might solve the problems that the ''improved theory'' of our Sec. V has when dealing with Minkowski space-time.

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