Gravitational entropy and global structure

S. W. Hawking* and C. J. Hunter†
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom

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The underlying reason for the existence of gravitational entropy is traced to the impossibility of foliating topologically non-trivial Euclidean spacetimes with a time function to give a unitary Hamiltonian evolution. In d dimensions the entropy can be expressed in terms of the $d-2$ obstructions to foliation, bolts and Misner strings, by a universal formula. We illustrate with a number of examples including spaces with nut charge. In these cases, the entropy is not just a quarter the area of the bolt, as it is for black holes.
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I. INTRODUCTION

The first indication that gravitational fields could have entropy came when investigations [1] of the Penrose process for extracting energy from a Kerr black hole showed that there was a quantity called the irreducible mass which could go up or stay constant, but which could never go down. Further work [2] showed that this irreducible mass was proportional to the area of the horizon of the black hole and that the area could never decrease in the classical theory, even in situations where black holes collided and merged together. There was an obvious analogy with the second law of thermodynamics, and indeed black holes were found to obey analogues of the other laws of thermodynamics as well [3]. But it was Bekenstein who took the bold step [4] of suggesting that the area actually was the physical entropy, and that it counted the internal states of the black hole. The inconsistencies in this proposal were removed when it was discovered that quantum effects would cause a black hole to radiate like a hot body [5,6].

For years people tried to identify the internal states of black holes in terms of fluctuations of the horizon. Success seemed to come with the paper of Strominger and Vafa [7] which was followed by a host of others. However, in light of recent work on anti–de Sitter space [8], one could reinterpret these papers as establishing a relation between the entropy of the black hole and the entropy of a conformal field theory on the boundary of a related anti–de Sitter space. This work, however, left obscure the deep reason for the existence of gravitational entropy. In this paper we trace it to the fact that general relativity and its supergravity extensions allow spacetime to have more than one topology for given boundary conditions at infinity. By topology, we mean topology in the Euclidean regime. The topology of a Lorentzian spacetime can change with time only if there is some pathology, such as a singularity, or closed time-like curves. In either of these cases, one would expect the theory to break down.

The basic premise of quantum theory is that time translations are unitary transformations generated by the Hamiltonian. In gravitational theories the Hamiltonian is given by a volume integral over a hypersurface of constant time, plus surface integrals at the boundaries of the hypersurface. The volume integral vanishes if the constraints are satisfied; so the numerical value of the Hamiltonian comes from the surface terms. However, this does not mean that the energy and momentum reside on these boundaries. Rather it reflects that these are global quantities which cannot be localized. We shall argue that the same is true of entropy: it is a global property and cannot be localized as horizon states.

If the spacetime can be foliated by a family of surfaces of constant time, the Hamiltonian will indeed generate unitary transformations and there will be no gravitational entropy. However, if the topology of the Euclidean spacetime is nontrivial, it may not be possible to foliate it by surfaces that do not intersect each other and which agree with the usual Euclidean time at infinity. In this situation, the concept of unitary Hamiltonian evolution breaks down and mixed states with entropy will arise. We shall relate this entropy to the obstructions to foliation. It turns out that the entropy of a d-dimensional Euclidean spacetime ($d>2$) can be expressed in terms of bolts ($d-2$ dimensional fixed point sets of the time translation Killing vector) and Misner strings (Dirac strings in the Kaluza-Klein reduction with respect to the time translation Killing vector) by the universal formula

$$S=\frac{1}{4G}(\mathcal{A}_{\mathrm{bolts}}+\mathcal{A}_{\mathrm{MS}})-\beta H_{\mathrm{MS}}, \tag{1.1}$$

where G is the d dimensional Newton's constant, $\mathcal{A}_{\mathrm{bolts}}$ and $\mathcal{A}_{\mathrm{MS}}$ are respectively the $d-2$ volumes in the Einstein frame of the bolts and Misner strings and H_{MS} is the Hamiltonian surface term on the Misner strings. Where necessary, subtractions should be made for the same quantities in a reference background which acts as the vacuum for that sector of the theory.

The plan of this paper is as follows. In Sec. II we describe the Arnowitt-Deser-Misner (ADM) formalism and the expression for the Hamiltonian in terms of volume and surface integrals. In Sec. III we introduce thermal ensembles and give an expression for the action and entropy of Euclidean

*Email address: S.W.Hawking@damtp.cam.ac.uk
†Email address: C.J.Hunter@damtp.cam.ac.uk

metrics with a $U(1)$ isometry group. This is illustrated in Sec. IV by some examples. In Sec. V we draw some conclusions.

II. HAMILTONIAN

Let $\bar{\mathcal{M}}$ be a d-dimensional Riemannian manifold with metric $g_{\mu\nu}$ and covariant derivative ∇_μ, which has an imaginary time coordinate τ that foliates $\bar{\mathcal{M}}$ into non-singular hypersurfaces $\{\Sigma_\tau\}$ of constant τ. The metric and covariant derivative on Σ_τ are h_{ij} and D_i. If $\bar{\mathcal{M}}$ is non-compact, then it will have a boundary $\partial\bar{\mathcal{M}}$, which can include internal components as well as a boundary at infinity. The $d-2$ dimensional surfaces, $B_\tau=\partial\bar{\mathcal{M}}\cap\Sigma_\tau$, are the boundaries of the hypersurfaces Σ_τ and a foliation of $\partial\bar{\mathcal{M}}$. We will use Greek letters to denote indices on $\bar{\mathcal{M}}$ and Roman letters for indices on Σ_τ.

The Euclidean action for a gravitational field coupled to both a Maxwell and N general matter fields is

$$I=-\frac{1}{16\pi G}\int_{\mathcal{M}} d^d x\sqrt{g}[R-F^2+\mathcal{L}(g_{\mu\nu},\phi^A)]$$
$$-\frac{1}{8\pi G}\int_{\mathcal{M}} d^{d-1}x\sqrt{b}\Theta(b), \qquad (2.1)$$

where R is the Ricci scalar, $F_{\mu\nu}$ is the Maxwell field tensor, and $\mathcal{L}(g_{\mu\nu},\phi^A)$ is an arbitrary Lagrangian for the fields ϕ^A $(A=1,...,N)$, where any tensor indices for ϕ^A are suppressed. We assume that the $\mathcal{L}$ contains only first derivatives, and hence does not need an associated boundary term.

In order to perform the Hamiltonian decomposition of the action, we write the metric in ADM form [9]:

$$ds^2=N^2d\tau^2+h_{ij}(dx^i+N^id\tau)(dx^j+N^jd\tau). \qquad (2.2)$$

This defines the lapse function N, the shift vector N^i, and the induced metric on Σ_τ, h_{ij}. We can rewrite the action (see [10,11] for details) as

$$I=\int d\tau\left[\int_{\Sigma_\tau} d^{d-1}x\left(P^{ij}\dot{h}_{ij}+E^i\dot{A}_i+\sum_{A=1}^{N}\pi^A\dot{\phi}^A\right)+H\right], \qquad (2.3)$$

where P^{ij}, E^i and π^A are the momenta conjugate to the dynamical variables h_{ij}, A_i and ϕ^A respectively. The Hamiltonian, H, consists of a volume integral over Σ_τ and a boundary integral over B_τ.

The volume term is

$$H_c=\int_{\Sigma_\tau} d^{d-1}x\left[N\mathcal{H}+N^i\mathcal{H}_i+A_0(D_iE^i-\rho)+\sum_{A=1}^{M}\lambda^AC^A\right], \qquad (2.4)$$

where N, N^i, A_0 and λ^A are all Lagrange multipliers for the constraint terms $\mathcal{H}$, $\mathcal{H}_i$, $D_iE^i-\rho$ and C^A. The number of constraints, M, which arise from the matter Lagrangian depends on its exact form. ρ is the electromagnetic charge density. Since the constraints all vanish on metrics that satisfy the field equations, the volume term makes no contribution to the Hamiltonian when it is evaluated on a solution.

The boundary term is

$$H_b=-\frac{1}{8\pi G}\int_{B_\tau}\sqrt{\sigma}[Nk+u_i(K^{ij}-Kh^{ij})N_j$$
$$+2A_0F^{0i}u_i+f(N,N^i,h_{ij},\phi^A)], \qquad (2.5)$$

where $\sqrt{\sigma}$ is the area element of B_τ, k is the trace of the second fundamental form of B_τ as embedded in Σ_τ, u_i is the outward pointing unit normal to B_τ, K_{ij} is the second fundamental form of Σ_τ in $\bar{\mathcal{M}}$, and $f(N,N^i,h_{ij},\phi^A)$ is some function which depends on the form of the matter Lagrangian.

Generally the surface term will make both the action and the Hamiltonian infinite. In order to obtain a finite result, it is sensible to consider the difference between the action or Hamiltonian, and those of some reference background solution. We pick the background such that the solution approaches it at infinity sufficiently rapidly so that the difference in the action and Hamiltonian is well-defined and finite. This reference background acts as the vacuum for that sector of the quantum theory. It is normally taken to be flat space or anti–de Sitter space, but we will consider other possibilities. We will denote background quantities with a tilde, although in the interest of clarity, they will be omitted for most calculations.

III. THERMODYNAMIC ENSEMBLES

In order to discuss quantities like entropy, one defines the partition function for an ensemble with temperature $T=\beta^{-1}$, angular velocity Ω and electrostatic potential Φ as

$$\mathcal{Z}=\mathrm{Tr}\ e^{-\beta(E+\Omega\cdot J+\Phi Q)}=\int D[g]D[\phi]e^{-I[g,\phi]}, \qquad (3.1)$$

where the path integral is taken over all metrics and fields that agree with the reference background at infinity and are periodic under the combination of a Euclidean time translation β, a rotation through an angle $\beta\Omega$ and a gauge transformation $\beta\Phi$. The partition function includes factors for electric-type charges such as mass, angular momentum and electric charge, but not for magnetic-type charges such as nut charge and magnetic charge. This is because the boundary conditions of specifying the metric and gauge potential on a $d-1$ dimensional surface at infinity do not fix the electric-type charges. Each field configuration in the path integral therefore has to be weighted with the appropriate factor of the exponential of minus charge times the corresponding thermodynamic potential. Magnetic-type charges, on the other hand, are fixed by the boundary conditions and are the same for all field configurations in the path integral. It is therefore not necessary to include weighting factors for magnetic-type charges in the partition function.

The lowest order contribution to the partition function will be

$$\mathcal{Z}=\sum e^{-I}, \tag{3.2}$$

where I are the actions of Euclidean solutions with the given boundary conditions. The reference background, periodically identified, will always be one such solution and, by definition, it will have zero action. However, we shall be concerned in this paper with situations where there are additional Euclidean solutions with different topology which also have a $U(1)$ isometry group that agrees with the periodic identification at infinity. This includes not only black holes and p-branes, but also more general classes of solution, as we shall show in the next section.

In d dimensions the Killing vector $K=\partial/\partial\tau$ will have zeros on surfaces of even codimension which will be fixed points of the isometry group. The $d-2$ dimensional fixed points sets will play an important role. We shall generalize the terminology of [12–14] and call them bolts.

Let τ with period β be the parameter of the $U(1)$ isometry group. Then the metric can be written in the Kaluza-Klein form

$$ds^2=\exp\left[-\frac{4\sigma}{\sqrt{d-2}}\right](d\tau+\omega_i dx^i)^2 + \exp\left[\frac{4\sigma}{(d-3)\sqrt{d-2}}\right]\gamma_{ij}dx^i dx^j, \tag{3.3}$$

where σ, ω_i and γ_{ij} are fields on the space Ξ of orbits of the isometry group. Ξ would be singular at the fixed point, and so one has to leave them out and introduce $d-2$ boundaries to Ξ.

The coordinate τ can be changed by a Kaluza-Klein gauge transformation

$$\tau'=\tau+\lambda, \tag{3.4}$$

where λ is a function on Ξ. This changes the one-form ω by $d\lambda$ but leaves the field strength $F=d\omega$ unchanged. If the orbit space Ξ has non-trivial homology in dimension 2, then the two-form F can have non-zero integrals over two-cycles in Ξ. In this case, the one-form potential ω will have Dirac-like string singularities on surfaces of dimension $d-3$ in Ξ. The foliation of the spacetime by surfaces of constant τ will break down at the fixed points of the isometry. It will also break down on the string singularities of ω which we call Misner strings, after Charles Misner who first realized their nature in the Taub-NUT (Newman-Unti-Tamburino) solution [15]. Misner strings are surfaces of dimension $d-2$ in spacetime $\mathcal{M}$.

In order to do a Hamiltonian treatment using surfaces of constant τ, one has to cut out small neighborhoods of the fixed point sets and of any Misner strings leaving a manifold $\bar{\mathcal{M}}$. On $\bar{\mathcal{M}}$ one has the usual relation between the action and Hamiltonian:

$$I=\int d\tau\left[\int_{\Sigma_\tau} d^{d-1}x\left(P^{ij}\dot{h}ij+E^i\dot{A}_i+\sum_A \pi^A\dot{\phi}^A\right)+H\right]. \tag{3.5}$$

Because of the $U(1)$ isometry, the time derivatives will all be zero. Thus the action of $\bar{\mathcal{M}}$ will be

$$I(\bar{\mathcal{M}})=\beta H. \tag{3.6}$$

To get the action of the whole spacetime $\mathcal{M}$, one now has to put back the small neighborhoods of the fixed point sets and the Misner strings that were cut out. In the limit that the neighborhoods shrink to zero, their volume contributions to the action will be zero. However, the surface term associated with the Einstein-Hilbert action will give a contribution to the action of $\mathcal{M}$ of

$$I(\mathcal{M}-\bar{\mathcal{M}})=-\frac{1}{4G}(\mathcal{A}_{\text{bolts}}+\mathcal{A}_{\text{MS}}), \tag{3.7}$$

where $\mathcal{A}_{\text{bolts}}$ and $\mathcal{A}_{\text{MS}}$ are respectively the total area of the bolts and the Misner strings in the spacetime. The contribution of the Einstein-Hilbert term to the action from lower dimensional fixed points will be zero. The contribution at bolts and Misner strings from higher order curvature terms in the action will be small in the large area limit.

The Hamiltonian in Eq. (3.6) will come entirely from the surface terms. In a topologically trivial spacetime, the surfaces of τ will have boundaries only at infinity. However, in more complicated situations, the surfaces will also have boundaries at the fixed point sets and Misner strings. The Hamiltonian surface terms at the fixed points will be zero because the lapse and shift vanish there. On the other hand, although the lapse is zero, the shift will not vanish on a Misner string. Thus there will be a Hamiltonian surface term on a Misner string given by the shift times a component of the second fundamental form of the constant τ surfaces. The action of $\mathcal{M}$ is therefore

$$I(\mathcal{M})=\beta(H_\infty+H_{\text{MS}})-\frac{1}{4G}(\mathcal{A}_{\text{bolts}}+\mathcal{A}_{\text{MS}}). \tag{3.8}$$

On the other hand, by thermodynamics,

$$\log Z=S-\beta(E+\Omega\cdot J+\Phi Q). \tag{3.9}$$

But

$$H_\infty=E+\Omega\cdot J+\Phi Q, \tag{3.10}$$

and so

$$S=\frac{1}{4G}(\mathcal{A}_{\text{bolts}}+\mathcal{A}_{\text{MS}})-\beta H_{\text{MS}}. \tag{3.11}$$

The areas and Misner string Hamiltonian in Eq. (3.11) are to be understood as differences from the reference background.

In order for the thermodynamics to be sensible, it must be invariant under the gauge transformation (3.4) which rotates the imaginary time coordinate. Because the action (3.8) is gauge invariant, we see that the entropy will also be, provided that H_∞ is independent of the gauge. In the Appendix, we show that H_∞ is indeed gauge invariant, and hence the entropy is well-defined, for metrics satisfying asymptotically flat (AF), asymptotically locally flat (ALF) or asymptotically locally Euclidean (ALE) boundary conditions.

Previous expositions of gravitational entropy have not included ALF and ALE metrics. This is presumably because these metrics contain Misner strings, and hence do not obey the simple ''quarter-area law,'' but rather the more complicated expression (3.11).

IV. EXAMPLES

In this section we calculate the entropy of some four and five dimensional spacetimes. We set $G=1$. The first example considers the Taub-NUT and Taub-bolt metrics, which are ALF. We then move to solutions of Einstein-Maxwell theory, the Israel-Wilson metrics, and calculate the entropy in both the AF and ALF sectors. The Eguchi-Hanson instanton then provides us with an ALE example. Finally, we calculate the entropy of S^5 for two different $U(1)$ isometry groups, one with a bolt and the other with no fixed points but a Misner string, obtaining the same result both ways. The action calculations, reference backgrounds and matching conditions for Taub-NUT, Taub-bolt and Eguchi-Hanson are all presented in [14] and will not be repeated here.

A. Taub-NUT and Taub-bolt instantons

ALF solutions have a nut charge, or magnetic type mass, N, as well as the ordinary electric type mass, M. The nut charge is $\beta C_1/8\pi$, where C_1 is the first Chern number of the $U(1)$ bundle over the sphere at infinity, in the orbit space Ξ. If the Chern number is zero, then the boundary at infinity is $S^1\times S^2$ and the spacetime is AF. The black hole metrics are saddle points in the path integral for the partition function. They have a bolt on the horizon but no Misner strings, and hence Eq. (3.11) gives the usual result for the entropy. However, if the Chern number is nonzero, the boundary at infinity is a squashed S^3, and the metric cannot be analytically continued to a Lorentzian metric. Nevertheless, one can formally interpret the path integral over all metrics with these boundary conditions as giving the partition function for an ensemble with a fixed value of the nut charge or magnetic-type mass.

The simplest example of an ALF metric is the Taub-NUT instanton [16], given by the metric

$$\begin{aligned} ds^2 = V(r)(d\tau + 2N\cos\theta d\phi)^2 + \frac{1}{V(r)}dr^2 \\ + (r^2-N^2)(d\theta^2+\sin^2\theta d\phi^2), \end{aligned} \tag{4.1}$$

where $V(r)$ is

$$V_{TN}(r) = \frac{r-N}{r+N}. \tag{4.2}$$

In order to make the solution regular, we consider the region $r\geqslant N$ and let the period of τ be $8\pi N$. The metric has a nut at $r=N$, with a Misner string running along the z-axis from the nut out to infinity.

The Taub-bolt instanton [17] is also given by the metric (4.1). However, the function $V(r)$ is different:

$$V_{TB}(r) = \frac{(r-2N)(r-N/2)}{r^2-N^2}. \tag{4.3}$$

The solution is regular if we consider the region $r\geqslant 2N$ and let τ have period $\beta=8\pi N$. Asymptotically, the Taub-bolt instanton is also ALF. There is a bolt of area $12\pi N^2$ at $r=2N$ which is a source for a Misner string along the z-axis.

In order to calculate the Hamiltonian of the Taub-bolt instanton, we need to use a scaled Taub-NUT metric as the reference background. We can then calculate the Hamiltonian at infinity,

$$H_\infty = \frac{N}{4}, \tag{4.4}$$

and the contribution from the boundary around the Misner string,

$$H_{\mathrm{MS}} = -\frac{N}{8}. \tag{4.5}$$

The area of the Misner string is $-12\pi N^2$ (that is, the area of the Misner string is greater in the Taub-NUT background than in Taub-bolt). Combining the Hamiltonian, Misner string and bolt contributions yields an action and entropy of

$$I=\pi N^2 \quad \text{and} \quad S=\pi N^2. \tag{4.6}$$

It would be interesting to relate this entropy to the entropy of a conformal field theory defined on the boundary of the spacetime. This may be possible by considering Euclidean Taub-NUT anti–de Sitter and other spacetimes asymptotic to it. The boundary at infinity is a squashed three-sphere, and the squashing tends to a constant at infinity. One would then compare the entropy of asymptotically Taub-NUT anti–de Sitter spaces with the partition function of a conformal field theory on the squashed three sphere. Work on this is in progress [18].

B. Israel-Wilson metrics

The Euclidean Israel-Wilson family of metrics [19,20] are solutions of the Einstein-Maxwell equations with line element

$$ds^2 = \frac{1}{UW}(d\tau + \omega_i dx^i)^2 + UW\gamma_{ij}dx^i dx^j, \tag{4.7}$$

where γ_{ij} is a flat three-metric and U, W and ω_i are real-valued functions. The electromagnetic field strength is

$$F = \partial_i\Phi(d\tau+\omega_j dx^j)\wedge dx^i + UW\sqrt{\gamma}\epsilon_{ijk}\gamma^{kl}\partial_l\chi dx^i\wedge dx^j, \tag{4.8}$$

with complex potentials Φ and χ given by

$$\Phi=\frac{1}{2}\left\{\left(\frac{1}{U}-\frac{1}{W}\right)\cos\alpha+\left(\frac{1}{U}+\frac{1}{W}\right)i\sin\alpha\right\} \tag{4.9}$$

and

$$\chi=-\frac{1}{2}\left\{\left(\frac{1}{U}+\frac{1}{W}\right)\cos\alpha+\left(\frac{1}{U}-\frac{1}{W}\right)i\sin\alpha\right\}. \tag{4.10}$$

For F^2 to be real, we need to take Φ and χ to be either entirely real or purely imaginary. Taking them to be real, we obtain the magnetic solution

$$\Phi_{\rm mag}=\frac{1}{2}\left(\frac{1}{U}-\frac{1}{W}\right)\quad\text{and}\quad\chi_{\rm mag}=-\frac{1}{2}\left(\frac{1}{U}+\frac{1}{W}\right). \tag{4.11}$$

The dual of the magnetic solution is the electric one, with imaginary potentials. Calculating the square of the field strengths:

$$F^2_{\rm mag}=(DU^{-1})^2+(DW^{-1})^2=-F^2_{\rm elec}. \tag{4.12}$$

We consider only the magnetic solutions here. The action and entropy calculations for the electric case are similar.

U, W and ω_i are determined by the equations

$$D_iD^iU=0=D_iD^iW,$$

$$\frac{1}{\sqrt{\gamma}}\gamma_{ij}\epsilon^{jkl}\partial_k\omega_l=WD_iU-UD_iW, \tag{4.13}$$

where D_i is the covariant derivative for γ_{ij}. The solutions for U and W are simply three-dimensional harmonic functions, and we will take them to be of the form

$$U=1+\sum_{I=1}^{N}\frac{a_I}{|x-y_I|}\quad\text{and}\quad W=1+\sum_{J=1}^{M}\frac{b_J}{|x-z_J|}, \tag{4.14}$$

where y_I and z_J are called the mass and anti-mass points respectively, and comprise the fixed point set of ∂_τ. We assume that the points have positive mass, i.e., $a_I,b_J>0$.

There will generically be conical singularities in the metric at the mass and anti-mass points. In order to remove them we must apply the constraint equations

$$U(z_J)b_J=\frac{\beta}{4\pi}=W(y_I)a_I, \tag{4.15}$$

where β is the periodicity of τ. Note that these equations hold for each value of I and J; i.e., no summation is implied. While the resulting spacetime is non-singular, emanating from each fixed point there will be Misner string singularities in the metric and Dirac string singularities in the electromagnetic potential. These string singularities will end on either another fixed point or at infinity.

The Einstein-Maxwell action is

$$I=-\frac{1}{16\pi}\int_{\mathcal{M}}d^4x\sqrt{g}(R-F^2)-\frac{1}{8\pi}\int_{\partial\mathcal{M}}d^3x\sqrt{b}\Theta(b), \tag{4.16}$$

which we can divide up into a gravitational (Einstein-Hilbert) and an electromagnetic term, $I=I^{\rm EH}+I^{\rm EM}$.

Since the Ricci scalar, R, is zero, the gravitational contribution to the action is entirely from the the surface term at infinity,

$$I^{\rm EH}=-\frac{1}{8\pi}\int_{\partial\mathcal{M}}d^3x\sqrt{b}\Theta(b). \tag{4.17}$$

Substituting in the metric, we can evaluate this on a hypersurface of radius r,

$$I^{\rm EH}=-\beta r-\frac{\beta}{16\pi}\int_{\partial\Xi}d^2x\sqrt{\sigma}\,\frac{u^iD_i(UW)}{UW}, \tag{4.18}$$

where σ_{ij} is the metric induced on the boundary from γ_{ij}, and u^i is the unit normal to the boundary.

We can write the electromagnetic contribution to the action integral as

$$\begin{aligned}I^{\rm EM}&=\frac{1}{16\pi}\int_{\mathcal{M}}d^4x\sqrt{g}F^2\\&=\frac{\beta}{32\pi}\int_{\Xi}d^3x\sqrt{\gamma}\left[\frac{D_iD^iW}{U}+\frac{D_iD^iU}{W}\right]\\&\quad-\frac{\beta}{32\pi}\int_{\partial\Xi}d^2x\sqrt{\sigma}u^iD_i(UW)\left[\frac{1}{U^2}+\frac{1}{W^2}\right],\end{aligned} \tag{4.19}$$

where $\partial\Xi$ is the boundary of Ξ at infinity (since the internal boundaries about the fixed points will make no contribution). We can evaluate the volume integral by using the delta function behavior of the Laplacians of U and W,

$$\begin{aligned}I^{\rm EM}=&-\frac{\pi}{2}\left(\sum_{I=1}^{N}a_I^2+\sum_{J=1}^{M}b_J^2\right)\\&-\frac{\beta}{32\pi}\int_{\partial\Xi}d^2x\sqrt{\sigma}u^iD_i(UW)\left[\frac{1}{U^2}+\frac{1}{W^2}\right].\end{aligned} \tag{4.20}$$

Note that the sum is only over mass and anti-mass points which are not coincident.

Suppose that we consider metrics with an equal number of nuts and anti-nuts,

$$U=1+\sum_{I=1}^{N}\frac{a_I}{|x-y_I|}\quad\text{and}\quad V=1+\sum_{I=1}^{N}\frac{b_I}{|x-z_I|}. \tag{4.21}$$

Applying the constraint equations, we see that

$$\sum_{I=1}^{N}a_I=\sum_{I=1}^{N}b_I\equiv A. \tag{4.22}$$

Hence, the scalar functions asymptotically look like

$$U \sim 1+\frac{A}{r}+\mathcal{O}(r^{-2}) \quad \text{and} \quad W \sim 1+\frac{A}{r}+\mathcal{O}(r^{-2}), \tag{4.23}$$

while the vector potential vanishes,

$$\omega_i \sim \mathcal{O}(r^{-2}). \tag{4.24}$$

Thus, at large radius the metric is

$$ds^2 \sim \left(1-\frac{2A}{r}\right) d\tau^2+\left(1+\frac{2A}{r}\right) d\mathcal{E}_3^2, \tag{4.25}$$

so that the boundary at infinity is $S^1 \times S^2$, and the metric is AF.

The background is simply flat space which is scaled so that it matches the Israel-Wilson metric on a hypersurface of constant radius R,

$$d\tilde{s}^2=\left(1-\frac{2A}{R}\right) d\tau^2+\left(1+\frac{2A}{R}\right) d\mathcal{E}_3^2, \tag{4.26}$$

and has the same period for τ. There is no background electromagnetic field.

Using formula (4.18) for the gravitational contribution to the action, we obtain, after subtracting off the background term,

$$I^{\mathrm{EH}}=\frac{\beta}{2} A. \tag{4.27}$$

From Eq. (4.20) for the electromagnetic action we get

$$I^{\mathrm{EM}}=-\frac{\pi}{2} \sum_{I=1}^{N}\left(a_I^2+b_I^2\right)+\frac{\beta}{2} A. \tag{4.28}$$

Note that the constraint equations imply that I^{EM} is positive. The total action is therefore positive, and given by

$$I=\beta A-\frac{\pi}{2} \sum_{I=1}^{N}\left(a_I^2+b_I^2\right). \tag{4.29}$$

We can calculate the Hamiltonian by integrating Eq. (2.5) over the boundaries at infinity and around the Misner strings (note that in the background space there are no Misner strings). The gravitational contribution from infinity is

$$H_{\infty}=A, \tag{4.30}$$

while the electromagnetic contribution from infinity is zero, because there is no electric charge. On the boundary around the Misner strings, the Hamiltonian is

$$H_{\mathrm{MS}}=\frac{R}{4}-\frac{\pi}{2\beta} \sum_{I=1}^{N}\left(a_I^2+b_I^2\right), \tag{4.31}$$

where R is the total length of the Misner string. The area of the Misner strings is thus

$$\mathcal{A}=\beta R. \tag{4.32}$$

Hence we see that the entropy is

$$S=\frac{\pi}{2} \sum_{I=1}^{N}\left(a_I^2+b_I^2\right). \tag{4.33}$$

It is interesting to note that the $N=1$ case is in fact the charged Kerr metric subject to the constraint $\beta\Omega=2\pi$. This condition implies that, unlike the generic Kerr solution, the time translation orbits are closed. In a purely bosonic theory this means that the Kerr metric with $\beta\Omega=2\pi$ contributes to the partition function,

$$\mathcal{Z}=\operatorname{tr} e^{-\beta H}, \tag{4.34}$$

for a non-rotating ensemble. However, the partition function will now not contain the factor $\exp(-\beta\Omega\cdot J)$. This means that the entropy will be less than one-quarter the area of the horizon by $2\pi J$. The path integral for the partition function will also have saddle points at two Reissner-Nordström solutions, one extreme and the other non-extreme. Both will have the same magnetic charge. The non-extreme solution will have the same β while the extreme one can be identified with period β. The actions will obey

$$I_{\text{extreme}}>I_{\text{Kerr}}>I_{\text{non-extreme}}. \tag{4.35}$$

Thus, the non-extreme Reissner-Nordström solution will dominate the partition function.

The situation is different, however, if one takes fermions into account. In this case, the rotation through $\beta\Omega=2\pi$ changes the sign of the fermion fields. This is in addition to the normal reversal of fermions fields under time translation β. Thus, fermions in charged Kerr with $\beta\Omega=2\pi$ are periodic under the $U(1)$ time translation group at infinity, rather than anti-periodic as in the Reissner-Nordström solution. This means that the charged Kerr solution contributes to the ensemble with partition function

$$\mathcal{Z}=\operatorname{tr}(-1)^F e^{-\beta H}. \tag{4.36}$$

The extreme Reissner-Nordström solution identified with the same periodic spin structure also contributes to this partition function, but it will be dominated by the Kerr solution. On the other hand, the non-extreme Reissner-Nordström solution contributes to the normal thermal ensemble with partition function

$$\mathcal{Z}=\operatorname{tr} e^{-\beta H}. \tag{4.37}$$

If we take a solution with N nuts and M anti-nuts, where $K \equiv N-M>0$, then the metric asymptotically approaches

$$ds^2 \sim \left(1-\frac{A+B}{r}\right)\left[d\tau+(A-B)\cos\theta d\phi\right]^2 + \left(1+\frac{A+B}{r}\right)\left[dr^2+r^2 d\Omega_2^2\right], \tag{4.38}$$

where

$$A=\sum_{I=1}^{N} a_I \quad \text{and} \quad B=\sum_{J=1}^{M} b_J. \tag{4.39}$$

Applying the constraint equations, we see that

$$A-B=K\frac{\beta}{4\pi}, \tag{4.40}$$

where $K=M-N>0$. Thus, the boundary at infinity will have the topology of a lens space with K points identified, and hence the metric is ALF.

If we take Φ and χ to be real, then the Maxwell field will also be real, and will now have both electric and magnetic components. The choice of gauge is then quite important, as it affects how the electromagnetic Hamiltonian is split between the boundary at infinity and the boundary around the Misner strings. We can fix the gauge by requiring the potential to be non-singular on the boundary at infinity. Asymptotically, the field is

$$A_\mu dx^\mu \sim \left[A_\tau^\infty - \frac{A-B}{2r}\right]d\tau + \left[A_\phi^\infty + \frac{1}{2}(A+B)\right]\cos\,\theta d\phi, \tag{4.41}$$

where A_τ^∞ and A_ϕ^∞ are the gauge dependent terms that we have to fix. By writing the potential in terms of an orthonormal basis, we see that in order to avoid a singularity we must set

$$A_\tau^\infty = \frac{A+B}{2(A-B)} \quad \text{and} \quad A_\phi^\infty = 0. \tag{4.42}$$

We can take the background metric to be the multi-Taub-NUT metric [21] with K nuts. This will have the same boundary topology as the Israel-Wilson ALF solution, and has the asymptotic metric

$$ds^2 \sim \left(1-\frac{2NK}{r}\right)[d\tau + 2NK\,\cos\,\theta d\phi]^2 + \left(1+\frac{2NK}{r}\right)d\mathcal{E}_3^2, \tag{4.43}$$

where the periodicity of τ is $8\pi N$. By scaling the radial coordinate and defining the nut charge of each nut, N, appropriately, we can match this to the Israel-Wilson ALF metric on a hypersurface of constant radius R. The metric is then

$$ds^2 \sim \left(1-\frac{2B}{R}-\frac{A-B}{r}\right)[d\tau + (A-B)\cos\,\theta d\phi]^2 + \left(1+\frac{2B}{R}+\frac{A-B}{r}\right)d\mathcal{E}_3^2, \tag{4.44}$$

where the periodicity of τ is β.

Calculating the action, we find that the Einstein-Hilbert contribution is

$$I^{\mathrm{EH}} = \frac{\beta}{4}(A+B) - \frac{\beta^2}{16\pi}K, \tag{4.45}$$

while the electromagnetic contribution is

$$I^{\mathrm{EM}} = -\frac{\pi}{2}\left[\sum_{I=1}^{N} a_I^2 + \sum_{J=1}^{M} b_J^2\right] + \frac{\beta}{4}(A+B). \tag{4.46}$$

Hence the total action is

$$I = \frac{\beta}{2}(A+B) - \frac{\beta^2}{16\pi}K - \frac{\pi}{2}\left[\sum_{I=1}^{N} a_I^2 + \sum_{J=1}^{M} b_J^2\right], \tag{4.47}$$

which is always positive.

If we calculate the Hamiltonian at infinity, we get

$$H_\infty = \frac{3}{4}(A+B) - \frac{\beta}{8\pi}K, \tag{4.48}$$

while the contribution from the Misner string is

$$H_{\mathrm{MS}} = -\frac{\pi}{2\beta}\left[\sum_{I=1}^{N} a_I^2 + \sum_{J=1}^{M} b_J^2\right] - \frac{A+B}{4} + \frac{\beta}{16\pi}K. \tag{4.49}$$

Since the net area of the Misner string is zero, the entropy is simply given by the negative of the Misner string Hamiltonian,

$$S = \frac{\pi}{2}\left[\sum_I a_I^2 + \sum_J b_J^2\right] + \frac{\beta}{4}(A+B) - \frac{\beta^2}{16\pi}K. \tag{4.50}$$

This formula has some strange consequences. Consider the case of a single nut and no anti-nuts. Then the solution is the Taub-NUT instanton with an anti-self-dual Maxwell field on it. Being self-dual, the Maxwell field has a zero energy-momentum tensor and hence does not affect the geometry, which is therefore just that of the reference background. Yet according to Eq. (4.50), the entropy is $\beta^2/32\pi$. This entropy can be traced to the fact that although A_μ is everywhere regular, the ADM Hamiltonian decomposition introduces a non-zero Hamiltonian surface term on the Misner string. This may indicate that intrinsic entropy is not restricted to gravity, but can be possessed by gauge fields as well. An alternative viewpoint would be that the reference background should be multi-Taub-NUT with its self-dual Maxwell field. This would change the entropy (4.50) to

$$S = \frac{\pi}{2}\left[\sum_I a_I^2 + \sum_J b_J^2\right] + \frac{\beta}{4}(A+B) - \frac{3\beta^2}{32\pi}K. \tag{4.51}$$

C. Eguchi-Hanson metric

A non-compact instanton which is a limiting case of the Taub-NUT solution is the Eguchi-Hanson metric [22],

$$ds^2 = \left(1-\frac{N^4}{r^4}\right)\left(\frac{r}{8N}\right)^2(d\tau + 4N\,\cos\,\theta d\phi)^2 + \left(1-\frac{N^4}{r^4}\right)^{-1}dr^2 + \frac{1}{4}r^2 d\Omega^2. \tag{4.52}$$

The instanton is regular if we consider the region $r \geqslant N$, and let τ have period $8\pi N$. The boundary at infinity is $S^3/\mathcal{Z}_2$ and hence the metric is ALE. There is a bolt of area πN^2 at $r = N$, which gives rise to a Misner string along the z-axis.

To calculate the Hamiltonian for the Eguchi-Hanson metric we use as a reference background an orbifold obtained by identifying Euclidean flat space mod $\mathcal{Z}_2$. This has a nut at the orbifold point at the origin, with a Misner string lying along the z-axis. The Hamiltonian at infinity vanishes,

$$H_\infty = 0, \tag{4.53}$$

as does the Hamiltonian on the Misner string,

$$H_{\rm MS} = 0. \tag{4.54}$$

We then find that the area of Misner string, when the area of the background string has been subtracted, is simply minus the area of the bolt. Hence the action and entropy are both zero,

$$I = 0 \quad \text{and} \quad S = 0. \tag{4.55}$$

This is what one would expect, because the Eguchi-Hanson metric has the the same supersymmetry as its reference background. It is only when the solution has less supersymmetry than the background that there is entropy.

D. Five-sphere

Finally, to show that the expression we propose for the entropy, Eq. (3.11), can be applied in more than four dimensions, consider a five-sphere of radius R,

$$ds^2 = R^2(d\chi^2 + \sin^2\ \chi\{d\eta^2 + \sin^2\ \eta[d\psi^2 + \sin^2\ \psi(d\theta^2 + \sin^2\ \theta d\phi^2)]\}). \tag{4.56}$$

This can be regarded as a solution of a five-dimensional theory with cosmological constant $\Lambda = 6/R^2$. If we consider dimensional reduction with respect to the $U(1)$ isometry ∂_ϕ, then the fixed point set is a three sphere of radius R. There are no Misner strings; so our formula gives an entropy equal to the area of the bolt,

$$S = \frac{\pi^2 R^3}{2G}. \tag{4.57}$$

However, one can choose a different $U(1)$ isometry, whose orbits are the Hopf fibration of the five-sphere. In this case, we want to write the metric as

$$ds^2 = (d\tau + \omega_i dx^i)^2 + \frac{R^2}{4}\left[d\sigma^2 + \sin^2\frac{\sigma}{2}\left(\sigma_1^2 + \sigma_2^2 + \cos^2\frac{\sigma}{2}\sigma_3^2\right)\right], \tag{4.58}$$

where

$$\omega = \frac{R}{2}\left(-\cos^2\frac{\sigma}{2}\sigma_3 + \cos\ \theta d\phi\right), \tag{4.59}$$

the periodicity of τ is $2\pi R$, the range of σ and θ is $[0,\pi]$ and the periodicities of ψ and ϕ are 4π and 2π respectively. The isometry ∂_τ has no fixed points. So the usual connection between entropy and fixed points does not apply. The orbit space of the Hopf fibration is $\mathcal{C}P^2$ with the Kaluza-Klein two-form, $F = d\omega$, equal to the harmonic two-form on $\mathcal{C}P^2$. The one-form potential, ω, has a Dirac string on the two-surface in the orbit space given by $\theta = 0,\pi$. When promoted to the full spacetime, this becomes a three-dimensional Misner string of area

$$\mathcal{A} = 4\pi^2 R^3. \tag{4.60}$$

Calculating the Hamiltonian surface term on the Misner string, we find

$$H_{\rm MS} = \frac{\pi R^2}{4G}. \tag{4.61}$$

Hence, we see that the entropy is

$$S = \frac{\mathcal{A}}{4G} - \beta H_{\rm MS} = \frac{\pi R^2}{2G}. \tag{4.62}$$

While this example is rather trivial, it does demonstrate that the entropy formula (3.11) can be extended to higher dimensions.

V. CONCLUSIONS

There are three conclusions that can be drawn from this work. The first is that gravitational entropy just depends on the Einstein-Hilbert action. It does not require supersymmetry, string theory, or p-branes. Indeed, one can define entropy for the Taub-bolt solution which does not admit a spin structure, at least of the ordinary kind. The second conclusion is that entropy is a global quantity, like energy or angular momentum, and should not be localized on the horizon. The various attempts to identify the microstates responsible for black hole entropy are in fact constructions of dual theories that reside in separate spacetimes. The third conclusion is that entropy arises from a failure to foliate the Euclidean regime with a family of time surfaces. In these situations the Hamiltonian will not give a unitary evolution in time. This raises the possibility of the loss of information and quantum coherence.

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APPENDIX: GAUGE INVARIANCE OF H_∞

We are interested in making gauge transformations which shift the Euclidean time coordinate:

$$d\hat{\tau}=d\tau-2\lambda_{,i}dx^i. \tag{A1}$$

Under this transformation the Hamiltonian variables transform as

$$\hat{N}^2=\rho N^2, \tag{A2}$$

$$\hat{N}_i=N_i+2(N^2+N_kN^k)\lambda_{,i}, \tag{A3}$$

$$\hat{N}^i=\rho(1+2\lambda_{,k}N^k)N^i+2\rho N^2\lambda^{,i}, \tag{A4}$$

$$\hat{h}_{ij}=h_{ij}+2N_{(i}\lambda_{,j)}+4(N^2+N_kN^k)\lambda_{,i}\lambda_{,j}, \tag{A5}$$

$$\hat{h}^{ij}=h^{ij}+\rho[2\lambda^2N^iN^j-4N^2\lambda^{,i}\lambda^{,j} -2(1+2\lambda_{,k}N^k)N^{(i}\lambda^{,j)}], \tag{A6}$$

where

$$\rho=\frac{1}{2\lambda^2N^2+(1+2\lambda_{,k}N^k)^2}, \tag{A7}$$

and $\lambda^2=\lambda_{,i}\lambda^{,i}$. Indices for terms with a caret are raised and lowered with $\hat{h}_{ij}$, while those without are raised and lowered by h_{ij}. The total Hamiltonian is not invariant under such a transformation. However, the Hamiltonian contribution at infinity will be shown to be invariant for AF, ALF and ALE metrics.

The general asymptotic form of the AF metric is

$$ds^2\sim\left(1-\frac{2M}{r}\right)d\tau^2-\left(1+\frac{2M}{r}\right)[dr^2+r^2d\Omega_2^2]. \tag{A8}$$

We can apply a general gauge transformation (A1) to this, where we asymptotically expand λ as

$$\lambda\sim\lambda_0+\frac{\lambda_1}{r}+\mathcal{O}(r^{-2}). \tag{A9}$$

If we calculate the Hamiltonian after applying this gauge transformation, we find that

$$\hat{H}_\infty=-r+M. \tag{A10}$$

In order calculate the background value, we need to scale flat space so that the metrics agree of a surface of constant radius R. The metric is

$$d\tilde{s}^2=\left(1-\frac{2M}{R}\right)d\tau^2+\left(1-\frac{2M}{R}\right)[dr^2+r^2d\Omega_2^2]. \tag{A11}$$

Applying the gauge transformation and then calculating the Hamiltonian yields

$$\hat{\tilde{H}}_\infty=-r. \tag{A12}$$

Thus we see that the physical Hamiltonian is

$$\hat{H}_\infty=M, \tag{A13}$$

which is gauge invariant.

We now want to consider the value of the Hamiltonian at infinity for ALF spaces. The general asymptotic form of the ALF metric is

$$ds^2\sim\left(1-\frac{2M}{r}\right)(d\tau+2aN\cos\theta d\phi)^2 -\left(1-\frac{2M}{r}\right)[dr^2+r^2d\Omega_2^2]. \tag{A14}$$

If we calculate the Hamiltonian after applying a gauge transformation, then we find that, identical to the AF case,

$$\hat{H}_\infty=-r+M. \tag{A15}$$

In order calculate the background value, we need the matched ALF background metric,

$$d\tilde{s}^2=\left(1-\frac{2N}{r}-\frac{2(M-N)}{R}\right)(d\tau+2aN\cos\theta d\phi)^2 +\left(1-\frac{2N}{r}+\frac{2(M-N)}{R}\right)[dr^2+r^2d\Omega_2^2], \tag{A16}$$

which has the gauge independent Hamiltonian

$$\hat{\tilde{H}}_\infty=-r+N. \tag{A17}$$

Thus we see that the physical Hamiltonian is gauge invariant,

$$\hat{H}_\infty=M-N. \tag{A18}$$

The general asymptotic form of the ALE metric is

$$ds^2=\left(1+\frac{M}{r^4}\right)d\mathcal{E}_4^2+\mathcal{O}(r^{-5}). \tag{A19}$$

We note that the asymptotic background metric is simply the $M=0$ case of the general metric, and hence the physical Hamiltonian is

$$H_\infty=H(M)-H(0). \tag{A20}$$

If we calculate the Hamiltonian after applying the gauge transformation, then we get a very complicated function of M, R and λ. However, if we differentiate with respect to M, we find that

$$\frac{\partial \hat{H}_\infty}{\partial M} = \mathcal{O}(r^{-2}). \tag{A21}$$

Thus, the background subtraction will cancel the Hamiltonian up to $\mathcal{O}(r^{-2})$, and hence

$$\hat{H}_\infty = 0, \tag{A22}$$

which is obviously gauge invariant.

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