2D induced gravity from the canonically gauged WZNW system

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Starting from the Kac-Moody structure of the WZNW model for SL(2,R) and using the general canonical formalism, we formulate a gauge theory invariant under local SL(2,R)$\times$SL(2,R) and diffeomorphisms. This theory represents a gauge extension of the WZNW system, defined by a difference of two simple WZNW actions. By performing a partial gauge fixing and integrating out some dynamical variables, we prove that the resulting effective theory coincides with the induced gravity in 2D. The geometric properties of the induced gravity are obtained out of the gauge properties of the WZNW system with the help of the Dirac brackets formalism. [S0556-2821(99)06902-7]

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I. INTRODUCTION

The subject of two-dimensional (2D) gravity has twofold interest: first, it describes important dynamical aspects of string theory as an effective theory induced by quantum string fluctuations, and second, it represents a useful theoretical model for the realistic theory of gravity in four dimensions. Being closely related to the Weyl anomaly in string theory [1], the induced gravity features a deep analogy with the usual Wess-Zumino action in gauge theories, and represents its gravitational analogue [2]. The effective action for 2D gravity was originally calculated in the conformal gauge, where it has the form of Liouville theory [1,3]. Analyzing the dynamical structure of this theory in the light-cone gauge, Polyakov found an unexpected connection with $SL(2,R)$ current algebra [2]. The importance of this result has been confirmed by the existence of a canonical formulation of the theory in terms of gauge-independent variables, the $SL(2,R)$ currents [4,5].

Inspired by the above results, Polyakov studied the connection between the Wess-Zumino-Novikov-Witten (WZNW) model for SL(2,R) and the induced gravity in the *light-cone gauge*, trying to understand how the geometric structure of spacetime can be obtained out of the chiral SL(2,R) symmetry of the WZNW model [6] (see also [7]). A similar approach based on the *conformal gauge* showed that the related form of 2D induced gravity, Liouville theory, may be obtained from the SL(2,R) WZNW model by imposing certain conformally invariant constraints. A consistent approach to this reduction procedure has been formulated using a gauge extension of the WZNW model based on two gauge fields [8].

In the present paper we shall use the general canonical formalism to construct a gauge theory invariant under local SL(2,R)$\times$SL(2,R) transformations and diffeomorphisms, which represents a gauge extension of the WZNW system,

$$I(g_1,g_2)=I(g_1)-I(g_2),\quad g_1,g_2\in \mathrm{SL}(2,R),\tag{1.1}$$

defined by a difference of two simple WZNW actions for the SL(2,R) group; then, we shall show, by performing a suitable gauge fixing and integrating out some dynamical variables, that the resulting effective theory coincides with the induced gravity in 2D:

$$I_G(\phi,g_{\mu\nu})=\int d^2\xi\sqrt{-g}\times\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\frac{1}{2}\alpha\phi R+M(e^{2\phi/\alpha}-1)\right].\tag{1.2}$$

We are able to demonstrate this connection in a covariant way, fully respecting the *diffeomorphism invariance* of the induced gravity, generalizing thereby the results of Polyakov and others [6–8].

We are going to use the general canonical method of constructing gauge-invariant actions [9]. It is based on the fact that the Lagrangian equations of motions are equivalent to the Hamiltonian equations derived from the action

$$I(q,\pi,u)=\int d\xi(\pi_i\dot{q}^i-H_0-u^mG_m),\tag{1.3a}$$

where G_m are primary constraints and H_0 is the canonical Hamiltonian. If G_m are first class constraints, satisfying the Poisson brackets algebra

$$\{G_m,G_n\}=U_{mn}{}^rG_r,\quad \{G_m,H_0\}=V_m{}^rG_r,\tag{1.3b}$$

then the canonical action $I(q,\pi,u)$ is invariant under the following gauge transformations:

$$\delta F=\varepsilon^m\{F,G_m\},\quad F=F(q,\pi),$$
$$\delta u^m=\dot{\varepsilon}^m+u^r\varepsilon^sU_{sr}{}^m+\varepsilon^rV_r{}^m.\tag{1.4}$$

This paper represents not only an extension of the results obtained in a previous Letter [10], but also a significant simplification of the basic dynamical structure; it also gives a natural explanation of the gauge origin of the geometry of spacetime. The Hamiltonian approach presented here is in complete agreement with a recent Lagrangian analysis [11].

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We begin our exposition in Sec. II by recalling some basic facts about the WZNW model for SL(2,R). Then, we use the Hamiltonian formalism to derive the related SL(2,R) currents, by choosing $\tau=\xi^-$ and $\tau=\xi^+$ as the time variables. These currents are used to define the energy-momentum components $T_\pm$ as the first class constraints corresponding to diffeomorphisms. In Sec. III we study the problem of gauging the internal SL(2,R)×SL(2,R) symmetry of the WZNW theory by doubling the number of phase space variables. After defining a new set of currents $I_{\pm a}$, we apply the canonical gauge procedure to the set of first class constraints $G_m=(T_\pm,I_{\pm a})$, and obtain our basic model — canonically gauged action of the WZNW system (1.1). In Sec. IV we define a restriction of the theory based on a subset of first class constraints G_m, choose a set of gauge-fixing conditions that do not affect the diffeomorphism invariance, formulate the quantum action using the Becchi-Rouet-Stora-Tyutin (BRST) formalism, and finally integrate out some variables to obtain an effective theory that coincides with the induced gravity (1.2). In Sec. V we use the Dirac brackets to show how geometric properties of the induced gravity follow from gauge properties of the WZNW system, and Sec. VI is devoted to concluding remarks. Some technical details are presented in the Appendixes.

II. WZNW MODEL FOR SL(2,*R*) AND KAC-MOODY CURRENTS

Dynamical properties of the SL(2,R) WZNW model can be naturally analyzed in the Hamiltonian formalism based on $\tau=\xi^-$ or ξ^+ as the evolution parameters [12]. The related Kac-Moody (KM) structure of the model plays an essential role in the canonical formalism for constructing gauge-invariant theories.

WZNW action. The two-dimensional WZNW model is a field theory in which the basic field g is a mapping from Σ to G, Σ being a two-dimensional Riemannian spacetime and G being a semisimple Lie group. The model is defined by the action

$$I(g)=I_0+n\Gamma=\frac{1}{2}\kappa\int_\Sigma(*v,v)+\frac{1}{3}\kappa\int_M(v,v^2),$$
$$v=g^{-1}dg, \tag{2.1}$$

where the first term is the action of the nonlinear σ model, while the second one is the topological Wess-Zumino term, defined over a three-manifold M whose boundary is the spacetime: $\partial M=\Sigma$. Here, $\kappa=n\kappa_0$ (n is an integer and κ_0 a normalization constant), v is the Lie algebra valued one-form, $*v$ is the dual of v, and $(X,Y)=\frac{1}{2}\,\mathrm{Tr}\,(XY)$.

Now, in the case $G=$SL(2,R) one can use the fact that any element g of SL(2,R) in a neighborhood of identity admits the Gauss decomposition, defined in Appendix A in terms of the group coordinates $q^\alpha=(x,\varphi,y)$, and derive the following form of the WZNW action:

$$I=\kappa\int_\Sigma d^2\xi(\partial_+\varphi\partial_-\varphi+4\partial_+x\partial_-ye^{-\varphi}). \tag{2.2}$$

Kac-Moody currents. Following the investigations of 2D induced gravity [13,5] and the WZNW model [12], we shall use the Hamiltonian approach based on ξ^- and ξ^+ as the time variables, to derive a set of currents, satisfying an SL(2,R) KM algebra.

Let us first consider the choice $\tau=\xi^-$, $\sigma=\xi^+$. The definition of momenta $(\pi_x,\pi_\varphi,\pi_y)$, conjugate to the Lagrangian variables $q^\alpha=(x,\varphi,y)$ in Eq. (2.2), leads to a set of primary constraints, which we denote by $J_{-\alpha}=(J_{-x},J_{-\varphi},J_{-y})$. It is convenient to transform these constraints into the tangent space basis by writing $J_{-a}=\bar{E}^\alpha{}_aJ_{-\alpha}$, where $\bar{E}^\alpha{}_a$ are the vielbein components on the SL(2,R) manifold (Appendix A):

$$J_{-(+)}=\pi_x,$$
$$J_{-(0)}=x\pi_x+(\pi_\varphi-\kappa\varphi'),$$
$$J_{-(-)}=-x^2\pi_x-2x(\pi_\varphi-\kappa\varphi')-4\kappa x'+\pi_ye^\varphi, \tag{2.3}$$

where prime denotes the space (σ) derivative.

Now, we consider the second choice $\tau=\xi^+$, $\sigma=-\xi^-$ (the minus sign is adopted in order to preserve the orientation of the manifold). The primary constraints in the tangent space basis $J_{+a}=E^\alpha{}_aJ_{+\alpha}$ have the form

$$J_{+(+)}=y^2\pi_y+2y(\pi_\varphi+\kappa\varphi')-4\kappa y'-\pi_xe^\varphi,$$
$$J_{+(0)}=-y\pi_y-(\pi_\varphi+\kappa\varphi'),$$
$$J_{+(-)}=-\pi_y. \tag{2.4}$$

The Poisson brackets algebra of the primary constraints $J_{\mp a}$ defines an SL(2,R) KM algebra with central charge $c_\mp=\mp2\kappa$:

$$\{J_{\mp a},J_{\mp b}\}=f_{ab}{}^cJ_{\mp c}\delta\mp2\kappa\gamma_{ab}\delta'. \tag{2.5}$$

Diffeomorphisms. Now, we return to the usual formulation with $\tau=\xi^0$, and discuss how the KM structure of the WZNW model can be used to build the covariant extension (with respect to diffeomorphisms) of the WZNW model (2.2) [10] (see also Ref. [14]).

Using the above expressions for the KM currents, we can construct the related SL(2,R) invariant expressions, the components of the energy-momentum tensor:

$$T_-(q,\pi)=\frac{1}{4\kappa}\gamma^{ab}J_{-a}J_{-b}$$
$$=\frac{1}{4\kappa}[\pi_x\pi_ye^\varphi+(\pi_\varphi-\kappa\varphi')^2]-x'\pi_x,$$
$$T_+(q,\pi)=-\frac{1}{4\kappa}\gamma^{ab}J_{+a}J_{+b}$$
$$=-\frac{1}{4\kappa}[\pi_x\pi_ye^\varphi+(\pi_\varphi+\kappa\varphi')^2]-y'\pi_y. \tag{2.6}$$

These components satisfy two independent Virasoro algebras,

$$\{T_{\mp}(\sigma_1),T_{\mp}(\sigma_2)\}=-[T_{\mp}(\sigma_1)+T_{\mp}(\sigma_2)]\delta', \quad (2.7)$$

equivalent to the algebra of *diffeomorphisms* (see, e.g., Ref. [5]).

Using the general canonical formalism, we can construct a *covariant theory*, in which $H_0=0$, $G_m=(T_-,T_+)$. This is done by introducing the canonical Lagrangian

$$\mathcal{L}(q,\pi,h)=\pi_\alpha\dot{q}^\alpha-h^-T_- -h^+T_+ . \quad (2.8)$$

To see the usual content of this Lagrangian, one can eliminate the momentum variables with the help of the equations of motion. Then, after introducing new variables $(h^+,h^-)\to\tilde{g}^{\mu\nu}$, one obtains the covariant generalizations of the WZNW theory [10]. In the light-cone basis (Appendix B), the covariant Lagrangian has the form

$$\mathcal{L}(q,h)=\kappa\sqrt{-\hat{g}}[\hat{\partial}_+\varphi\hat{\partial}_-\varphi+4\hat{\partial}_+x\hat{\partial}_-ye^{-\varphi}]. \quad (2.9)$$

Note that the Lagrangian (2.9) is invariant under conformal rescalings of the metric.

III. GAUGING SL(2,*R*)×SL(2,*R*) AND THE WZNW SYSTEM

Now, we consider the possibility of gauging the *internal* SL(2,R)×SL(2,R) symmetry. One should observe that the currents $J_{\pm a}$ are not of the first class, since the related KM algebras have central charges $c_\pm=\pm 2\kappa$. We wish to find a set of generators satisfying two independent SL(2,R) algebras without central charges. To this end we double the number of dynamical variables, $q\to(q_1,q_2)$, $\pi\to(\pi_1,\pi_2)$, and introduce two sets of currents,

$$J^{(1)}_{\pm a}=J_{\pm a}(q_1,\pi_1), \quad J^{(2)}_{\pm a}=J_{\pm a}(q_2,\pi_2)|_{\kappa\to-\kappa}, \quad (3.1)$$

satisfying two SL(2,R) KM algebras with opposite central charges: $c^{(1)}_\pm=\pm 2\kappa$, $c^{(2)}_\pm=\mp 2\kappa$. Then, we introduce new currents

$$I_{\pm a}=J^{(1)}_{\pm a}+J^{(2)}_{\pm a}, \quad (3.2)$$

which are easily seen to satisfy two independent SL(2,R) algebras with *vanishing central charges*. The new currents are of the first class, and can be used to gauge the internal SL(2,R)×SL(2,R) symmetry.

In order to include the diffeomorphisms into this procedure, we introduce the energy-momentum components of two sectors, defined in terms of $J^{(1)}$ and $J^{(2)}$ as in Eq. (2.6),

$$T^{(1)}_\pm=T_\pm(q_1,\pi_1), \quad T^{(2)}_\pm=T_\pm(q_2,\pi_2)|_{\kappa\to-\kappa}. \quad (3.3)$$

The complete energy-momentum is defined by

$$T_\pm=T^{(1)}_\pm+T^{(2)}_\pm . \quad (3.4)$$

The Poisson brackets algebra between $I_{\pm a}$ and $T_\pm$ has the form

$$\{I_{\pm a}(\sigma_1),I_{\pm b}(\sigma_2)\}=f_{ab}{}^cI_{\mp c}(\sigma_2)\delta,$$
$$\{T_\pm(\sigma_1),I_{\pm a}(\sigma_2)\}=-I_{\pm a}(\sigma_1)\delta',$$
$$\{T_\pm(\sigma_1),T_\pm(\sigma_2)\}=-[T_\pm(\sigma_1)+T_\pm(\sigma_2)]\delta', \quad (3.5)$$

and represents two copies of the semi-direct product of the SL(2,R) and Virasoro algebras. The collection $(T_\pm,I_{\pm a})$ can be taken as a set of *first class constraints* in the general canonical construction based on Eqs. (1.3). Together with diffeomorphisms, we have here an additional SL(2,R)×SL(2,R) structure.

We display here the complete set of constraints, multipliers and gauge parameters:

$$G_m=T_-,T_+,I_{-a},I_{+a},$$
$$u^m=h^-,h^+,a^a_+,a^a_-,$$
$$\varepsilon^m=\varepsilon^-,\varepsilon^+,\eta^a_+,\eta^a_-.$$

Now, using $H_0=0$, $G_m=(T_-,T_+,I_{-a},I_{+a})$, one can construct the canonical Lagrangian

$$L(q_i,\pi_i,h)=\pi_{1\alpha}q^\alpha_1+\pi_{2\alpha}q^\alpha_2-h^-T_- -h^+T_+ -a^a_+I_{-a} -a^a_-I_{+a}, \quad (3.6)$$

representing a gauge theory invariant under both local SL(2,R)×SL(2,R) transformations and diffeomorphisms.

Using the general rule (1.4), one finds that the gauge transformations have the form

$$\delta h^\pm=(\partial_0+h^\mp\partial_1)\varepsilon^\pm-\varepsilon^\pm\partial_1h^\pm,$$
$$\delta a^c_\pm=(\partial_0+h^\mp\partial_1)\eta^c_\pm-f_{ab}{}^ca^a_\pm\eta^b_\pm-\varepsilon^\mp\partial_1a^c_\pm .$$
$$\delta q^\alpha_1=-\bar{E}^\alpha{}_a\eta^a_+-E^\alpha{}_a\eta^a_-+(\varepsilon^+J^\alpha_+-\varepsilon^-J^\alpha_-)/2\kappa, \quad (3.7)$$

while δq_2 is obtained by changing κ to $-\kappa$.

As before, we can eliminate the momenta $\pi_{1\alpha}$ and $\pi_{2\alpha}$ in order to clarify the usual Lagrangian content of the theory. The resulting Lagrangian describes the gauge extension of the WZNW system (1.1), in complete agreement with the results of Ref. [11].

IV. GAUGE EXTENSION OF THE WZNW SYSTEM AND INDUCED GRAVITY

In this section we shall show that 2D induced gravity can be obtained from the canonical gauge extension of the WZNW system, by (a) performing a suitable gauge fixing and (b) integrating out some dynamical variables in the functional integral.

A. Canonical $H_+\times H_-$ gauge theory

Let us consider a restriction of the canonical theory (3.6), defined by the following subset of first class constraints:

$$G'_m=(T_-,T_+,I_n), \quad I_n\equiv[I_{-(+)},I_{-(0)},I_{+(-)},I_{+(0)}], \tag{4.1}$$

representing a subalgebra of Eqs. (3.5). This restriction can be obtained from the full canonical theory (3.6) by imposing the following gauge conditions:

$$a_+^{(-)}=0, \quad a_-^{(+)}=0. \tag{4.2}$$

The restricted algebra describes diffeomorphisms combined with the internal symmetry

$$H=H_+\times H_-, \tag{4.3}$$

where H_+ and H_- are subgroups of SL(2,R) defined by the generators (t_+,t_0) and (t_0,t_-), respectively.

The canonical action of the restricted theory takes the form

$$\mathcal{L}(q_i,\pi_i,h)=\pi_{1\alpha}\dot{q}_1^\alpha+\pi_{2\alpha}\dot{q}_2^\alpha-h^-T_--h^+T_+-a^nI_n, \tag{4.4}$$

where $a^n\equiv[a_+^{(+)},a_+^{(0)},a_-^{(-)},a_-^{(0)}]$. Here, the energy-momentum components are given by Eq. (3.4), in conjunction with Eqs. (3.3) and (2.6), while the currents I_n are of the form

$$I_{-(+)}=\pi_{x_1}+\pi_{x_2},$$

$$I_{-(0)}=[x_1\pi_{x_1}+(\pi_{\varphi_1}-\kappa\varphi_1')]+[x_2\pi_{x_2}+(\pi_{\varphi_2}+\kappa\varphi_2')],$$

$$I_{+(-)}=-\pi_{y_1}-\pi_{y_2},$$

$$I_{+(0)}=[-y_1\pi_{y_1}-(\pi_{\varphi_1}+\kappa\varphi_1')]+[-y_2\pi_{y_2}-(\pi_{\varphi_2}-\kappa\varphi_2')].$$

It is clear that the canonical action (4.4) represents a gauge extension of the WZNW system (1.1). Indeed, by choosing the gauge fixing $a^n=0$, and eliminating the momenta $\pi_{1\alpha}$ and $\pi_{2\alpha}$, the action (4.4) reduces to the form

$$\mathcal{L}(q_1,q_2,h)=\mathcal{L}(q_1,h)-\mathcal{L}(q_2,h),$$

where $\mathcal{L}(q,h)$ is given by Eq. (2.9), representing the covariant extension of Eq. (1.1).

B. Effective theory in the canonical form

Quantum action. In order to demonstrate that the action (4.4) can be effectively reduced to the induced gravity (1.2), we begin by choosing the gauge conditions corresponding to the first class constraints I_n:

$$\Omega_n\equiv[\Omega_{-(+)},\Omega_{-(0)},\Omega_{+(-)},\Omega_{+(0)}],$$

$$\Omega_{\mp(\pm)}=J^{(1)}_{\mp(\pm)}-\mu_\mp=0, \quad \Omega_{\mp(0)}=J^{(2)}_{\mp(0)}-\lambda_\mp=0. \tag{4.5}$$

To impose these conditions on the functional integral, we use the BRST formalism and introduce a set of ghost fields (e^-,e^+,c^n), antighosts $\bar{c}^n$, and new multipliers b^n. While ghost fields correspond to gauge parameters, antighosts and multipliers are associated with the gauge conditions. Since the diffeomorphisms are not gauge fixed, the related antighosts and multipliers are not present in the formalism. The BRST transformation sX of a dynamical variable X, $X=(q_1,q_2,h^\pm)$, is obtained from the gauge transformation δX by replacing gauge parameters with ghosts; for the new fields we have $s\bar{c}^n=b^n$, $sb^n=0$, while sc^n is not needed here (sc^n follows from the nilpotency condition: $s^2X=0$).

Then, we introduce the gauge fermion $\Psi=\bar{c}^n\Omega_n$, and define the quantum action as

$$\mathcal{L}_Q=\mathcal{L}(q_i,\pi_i,h)+s\Psi=\mathcal{L}(q_i,\pi_i,h)+\mathcal{L}_{GF}+\mathcal{L}_{FP}. \tag{4.6}$$

The gauge-fixing and the Faddeev-Popov parts are given by

$$\mathcal{L}_{GF}=b^n\Omega_n, \quad \mathcal{L}_{FP}=-\bar{c}^n[s\Omega_n],$$

where

$$s\Omega_{\mp(\pm)}=-[e^\mp J^{(1)}_{\mp(\pm)}]'\mp c^{\mp(0)}J^{(1)}_{\mp(\pm)},$$

$$s\Omega_{\mp(0)}=-[e^\mp J^{(2)}_{\mp(0)}]'\pm c^{\mp(\pm)}J^{(2)}_{\mp(\pm)}\mp 2\kappa[c^{\mp(0)}]'.$$

Effective theory. Having derived the quantum action, we are now going to show that it can be effectively reduced to the induced gravity, by integrating out all the variables except φ_1,φ_2, and the related momenta. To simplify the exposition technically, we shall divide it into several smaller steps.

(a) The integration over $b^\pm$, a_+ and a_- transforms $\mathcal{L}_Q$ into the effective Lagrangian

$$\mathcal{L}_E(\varphi_i,\pi_{\varphi_i},h)=[\pi_{1\alpha}\dot{q}_1^\alpha+\pi_{2\alpha}\dot{q}_2^\alpha-h^-T_- -h^+T_++\mathcal{L}_{FP}]_{I=\Omega=0}.$$

It is now convenient to rewrite the relations $I_n=0$ and $\Omega_n=0$ in the form

$$\pi_{x_1}=\mu_-=-\pi_{x_2},$$

$$-\pi_{y_1}=\mu_+=\pi_{y_2},$$

$$x_1\pi_{x_1}+2K_{1-}=\lambda_-=-(x_2\pi_{x_2}+2K_{2+}),$$

$$-(y_1\pi_{y_1}+2K_{1+})=\lambda_+=y_2\pi_{y_2}+2K_{2-},$$

where $K_\pm=(\pi_\varphi\pm\kappa\varphi')/2$.

(b) The momentum variables π_{x_1},π_{y_1} and π_{x_2},π_{y_2} are constant, so that the related $\pi\dot{q}$ terms in the action can be ignored as total time derivatives.

(c) Also, the contribution of the Faddeev-Popov term is decoupled since the currents $J^{(1)}$ and $J^{(2)}$ are constant, so that the integration over ghosts and antighosts can be absorbed into the normalization of the functional integral.

(d) Finally, the expression for $T_{\mp}$, reduced to the surface $I=\Omega=0$, reads

$$\kappa\tilde{T}_{\mp}=[\pm(K_{1\mp})^2+2\kappa(K_{1\mp})']+[\mp(K_{2\pm})^2+2\kappa(K_{2\pm})'] \mp\frac{1}{4}\mu(e^{\varphi_1}-e^{\varphi_2}),$$

where $\mu=\mu_-\mu_+$, so that the effective theory in the canonical form is given by

$$\mathcal{L}_E(\varphi_i,\pi_{\varphi_i},h)=\pi_{\varphi_1}\dot{\varphi}_1+\pi_{\varphi_2}\dot{\varphi}_2-h^-\tilde{T}_--h^+\tilde{T}_+. \tag{4.7}$$

C. Transition to the induced gravity

In order to find out the usual dynamical content of the previous result, we shall eliminate the remaining momentum variables from Eq. (4.7) by using their equations of motion,

$$\pi_{\varphi_1}\pm\kappa\varphi_1=\sqrt{2}\kappa(\hat{\partial}_\pm\varphi_1+\hat{\omega}_--\hat{\omega}_+), \tag{4.8}$$

while π_{φ_2} is obtained by the replacement $\varphi_1\to\varphi_2$, $\kappa\to-\kappa$ ($\hat{\partial}_\pm$ and $\hat{\omega}_\pm$ are defined in Appendix B). The effective theory is described by the Lagrangian

$$\mathcal{L}_E(\varphi_1,\varphi_2,h)=\Lambda(\varphi_1,h)-\Lambda(\varphi_2,h),$$

$$\Lambda(\varphi,h)=\sqrt{-\hat{g}}[\kappa\hat{\partial}_+\varphi\hat{\partial}_-\varphi +2\kappa(\hat{\omega}_-\hat{\partial}_+\varphi-\hat{\omega}_+\hat{\partial}_-\varphi)+Me^{\varphi}], \tag{4.9}$$

where $M=\mu/2\kappa$. If we now change the variables according to

$$\phi=\sqrt{\kappa}(\varphi_1-\varphi_2),\quad 2F=\varphi_2,$$

the effective Lagrangian takes the final form

$$\mathcal{L}_E(\phi,F,h)=\sqrt{-\hat{g}}\{\hat{\partial}_+\phi\hat{\partial}_-\phi +2\sqrt{\kappa}[(\hat{\omega}_-+\hat{\partial}_-F)\hat{\partial}_+\phi-(\hat{\omega}_+-\hat{\partial}_+F)\hat{\partial}_-\phi] +Me^{2F}(e^{\phi/\sqrt{\kappa}}-1)\}.$$

The geometric meaning of this Lagrangian becomes more transparent if we use conformally rescaled metric $g_{\mu\nu}=e^{2F}\hat{g}_{\mu\nu}$ (Appendix B), whereupon the effective Lagrangian is easily seen to coincide with the induced gravity action (1.2):

$$\mathcal{L}_E(\phi,g_{\mu\nu})=\sqrt{-g}[\partial_+\phi\partial_-\phi+2\sqrt{\kappa}(\omega_-\partial_+\phi-\omega_+\partial_-\phi) +M(e^{\phi/\sqrt{\kappa}}-1)] =\sqrt{-g}[\partial_+\phi\partial_-\phi+\sqrt{\kappa}\phi R+M(e^{\phi/\sqrt{\kappa}}-1)]. \tag{4.10}$$

V. GEOMETRIC PROPERTIES FROM GAUGE TRANSFORMATIONS

In the process of constructing the induced gravity action from the gauged WZNW system, one expects the original *gauge* transformations of dynamical variables to go over into *geometric* transformations of the final, gravitational theory. It is straightforward to show that gauge transformations of canonical multipliers $h^\pm$ produce correct geometric transformations of the metric density $\tilde{g}^{\mu\nu}=\sqrt{-\hat{g}}\hat{g}^{\mu\nu}$ [10]. Complete interpretation of the induced gravity demands clarification of the nature of two additional fields $\sqrt{-g}$ and ϕ, given by

$$\sqrt{-g}=\frac{1}{2}(h^--h^+)e^{\varphi_2},\quad \phi=\sqrt{\kappa}(\varphi_1-\varphi_2). \tag{5.1}$$

We begin by noting that the transformation rule (3.7) describes the SL(2,R) gauge transformations, defined by parameters $\eta_\pm$ [11], and the $\varepsilon^\pm$ transformations, which we expect to be related to diffeomorphisms. In particular, the $\varepsilon^\pm$ transformation of φ_1 has the form

$$\delta_\varepsilon\varphi_1=-\frac{1}{2\kappa}[\varepsilon^+(\pi_{\varphi_1}+\kappa\varphi_1')-\varepsilon^-(\pi_{\varphi_1}-\kappa\varphi_1')], \tag{5.2}$$

while $\delta_\varepsilon\varphi_2$ is obtained by replacing $\kappa\to-\kappa$.

Now, let us go to the gauge-fixed, effective theory, expressed by Eq. (4.7). While the gauge transformations in the WZNW theory are defined using the Poisson brackets in Eqs. (1.4), the related transformation rules in the gauge fixed theory (induced gravity) should be calculated with the help of the *Dirac brackets*, determined by (I_n,Ω_n).

In order to check whether $\delta(\sqrt{-g})$ has the correct geometric form (B3), we replace the above transformation law for φ (i.e., φ_1 or φ_2) with the Dirac brackets expression

$$\delta^*_\varepsilon\varphi=\delta_\varepsilon\varphi-\partial_1(\varepsilon^-+\varepsilon^+), \tag{5.3}$$

where, after eliminating π_φ with the help of Eq. (4.8), $\delta_\varepsilon\varphi$ takes the form

$$\delta_\varepsilon\varphi=\frac{1}{\sqrt{2}}[-(\varepsilon^+\hat{\partial}_+\varphi-\varepsilon^-\hat{\partial}_-\varphi)+(\varepsilon^--\varepsilon^+)(\hat{\omega}_--\hat{\omega}_+)].$$

Comparing the expression (5.3) with Eq. (B4), one concludes that $\delta^*_\varepsilon\varphi$ yields the correct transformation law for $\sqrt{-g}$.

It is now easy to see that the variable ϕ behaves as a scalar field,

$$\delta^*_\varepsilon\phi=-\varepsilon\cdot\partial\phi, \tag{5.4}$$

in agreement with its geometric role.

The following relations characterize the geometric structure of the effective theory:

$$\{T_\pm^{(i)}(\sigma_1),T_\pm^{(i)}(\sigma_2)\}^* = -[T_\pm^{(i)}(\sigma_1)+T_\pm^{(i)}(\sigma_2)]\delta - c_\pm^{(i)}\delta''' \quad (i=1,2), \quad (5.5)$$

$$\{T_\pm(\sigma_1),T_\pm(\sigma_2)\}^* = -[T_\pm(\sigma_1)+T_\pm(\sigma_2)]\delta.$$

In the gauge-fixed theory, the energy-momentum components of the WZNW sectors 1 and 2 are not first class constraints, as opposed to the complete energy-momentum tensor.

VI. CONCLUDING REMARKS

In the present paper we used the canonical approach to elucidate how the induced gravity action, together with its geometric properties, can be obtained from the dynamical structure of the SL(2,R) WZNW system.

We first analyzed primary constraints of the SL(2,R) WZNW model (2.2), using the Hamiltonian formalism based on the choice of time $\tau=\xi^\pm$, which led us naturally to the SL(2,R) KM currents $J_{\pm a}$. These currents are basic objects in our canonical approach. They are used to construct the energy-momentum components that represent first class constraints corresponding to diffeomorphisms. Then, we defined the gauge extension of the WZNW system by introducing two sets of KM currents, $J_{\pm a}^{(1)}$ and $J_{\pm a}^{(2)}$, which are used to define the new first class constraints $I_{\pm a}=J_{\pm a}^{(1)}+J_{\pm a}^{(2)}$, satisfying an SL(2,$R$)×SL(2,$R$) algebra without central charge and the energy-momentum components $T_\pm$ corresponding to the whole WZNW system. The resulting theory is clearly gauge equivalent to the WZNW system (1.1), being its canonical gauge extension. As the main result of our analysis, we showed (a) by choosing a suitable gauge fixing and (b) integrating out some dynamical variables that this gauge theory reduces effectively to the induced gravity (1.2). Geometric properties of the gravitational theory are derived from gauge properties of the gauge-extended WZNW system, with the help of the Dirac brackets.

The results obtained here supplement those of the recent Lagrangian analysis [11], and improve our understanding of geometric properties of 2D spacetime in terms of the related gauge structure. They can be used to better understand singular solutions of the induced gravity in terms of globally regular solutions of the WZNW system, and clarify the nature of black holes [8,15].

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APPENDIX A: GEOMETRIC PROPERTIES OF $SL(2,R)$

In this appendix we outline some geometric properties of SL(2,R) [11].

Choosing the generators of SL(2,R) as $t_{(\pm)}=\frac{1}{2}(\sigma_1 \pm i\sigma_2), t_{(0)}=\frac{1}{2}\sigma_3$, where σ_k are the Pauli matrices, one can find the form of the related Lie algebra $[t_a,t_b]=f_{ab}{}^c t_c$, and evaluate the Cartan metric $\gamma_{ab}=(t_a,t_b)=\frac{1}{2}f_{ac}{}^d f_{bd}{}^c$.

Any element g of SL(2,R) in a neighborhood of identity can be parametrized by using the Gauss decomposition:

$$g=e^{xt_{(+)}}e^{\varphi t_{(0)}}e^{yt_{(-)}}=e^{-\varphi/2}\begin{pmatrix} e^\varphi+xy & x \\ y & 1\end{pmatrix},$$

where $q^\alpha=(x,\varphi,y)$ are group coordinates. Now, the Lie-algebra-valued one-form $v=g^{-1}dg=t_aE^a=t_aE^a{}_\alpha dq^\alpha$ defines the quantity $E^a{}_\alpha$, the vielbein on the group manifold. Similarly, the calculation of $\bar{v}=gdg^{-1}=t_a\bar{E}^a=t_a\bar{E}^a{}_\alpha dq^\alpha$ leads to $\bar{E}^a{}_\alpha$.

APPENDIX B: RIEMANNIAN STRUCTURE ON Σ

Here, we present some basic geometric features of two-dimensional spacetime Σ.

Light-cone basis. Starting from the interval on $\Sigma, ds^2 = g_{\mu\nu}d\xi^\mu d\xi^\nu$, we can solve the equation $ds^2=0$ for $h \equiv d\xi^1/d\xi^0$, $h^\pm=(-g_{01}\pm\sqrt{-g})/g_{11}$, and obtain

$$ds^2=2d\xi^+d\xi^-,$$
$$d\xi^\pm \equiv \sqrt{-g_{11}/2}\,(\mp h^\pm d\xi^0 \pm d\xi^1)=e^\pm{}_\mu d\xi^\mu.$$

If we introduce $-g_{11}=e^{2F}$, three independent components of the metric $g_{\mu\nu}$ can be expressed in terms of the new, light-cone variables (h^-,h^+,F). In particular,

$$\sqrt{-g}=e^{2F}\sqrt{-\hat{g}}, \quad \sqrt{-\hat{g}}\equiv\frac{1}{2}(h^- - h^+).$$

At each point of Σ the quantities

$$e^i{}_\mu=e^F\,\hat{e}^i{}_\mu, \quad \hat{e}^i{}_\mu\equiv\frac{1}{\sqrt{2}}\begin{pmatrix}-h^+ & 1\\ h^- & -1\end{pmatrix} \quad \text{(B1)}$$

$(i=+,-)$ define an orthonormal, light-cone basis of one-forms, $\theta^i=d\xi^i=e^i{}_\mu d\xi^\mu$. We also introduce the related basis of tangent vectors, $e_i\equiv\partial_i=e_i^\mu\partial_\mu$,

$$e_i^\mu=e^{-F}\,\hat{e}_i^\mu, \quad \hat{e}_i^\mu\equiv\frac{\sqrt{2}}{h^- - h^+}\begin{pmatrix}1 & h^-\\ 1 & h^+\end{pmatrix}. \quad \text{(B2)}$$

The metric η_{ij} in the tangent space has the light-cone form $\eta_{-+}=\eta_{+-}=1$, while $\hat{g}_{\mu\nu}$ and its inverse are defined in the usual way: $\hat{g}_{\mu\nu}=\hat{e}^i{}_\mu\hat{e}^j{}_\nu\eta_{ij}$, $\hat{g}^{\mu\nu}=\hat{e}_i^\mu\hat{e}_j^\nu\eta^{ij}$.

Diffeomorphisms. The standard transformation rule of the vielbein $e^i{}_\mu$ under the diffeomorphisms, $\xi^\mu\to\xi^\mu+\varepsilon^\mu(\xi)$, implies

$$\delta h^\pm=\partial_0\varepsilon^\pm+h^\pm\partial_1\varepsilon^\pm-\varepsilon^\pm\partial_1 h^\pm,$$

where $\varepsilon^\pm=\varepsilon^1-\varepsilon^0 h^\pm$. The usual transformation law for $\sqrt{-g}$,

$$\delta\sqrt{-g}=-\partial_\rho(\varepsilon^\rho\sqrt{-g}), \quad \text{(B3)}$$

in conjunction with $\sqrt{-g}=e^{2F}\sqrt{-\hat{g}}$, is equivalent to

$$\delta(2F)=-\partial_1(\varepsilon^+ +\varepsilon^-)+(\varepsilon^- -\varepsilon^+)\frac{\partial_1(h^- +h^+)}{h^- -h^+}$$

$$-\frac{1}{\sqrt{2}}(\varepsilon^+\hat{\partial}_+ -\varepsilon^-\hat{\partial}_-)2F. \tag{B4}$$

Connection and curvature. The Riemannian connection is defined by the first structural equation: $d\theta^i+\omega^i{}_j\wedge\theta^j=0$, where $\omega^i{}_j=\varepsilon^i{}_j\omega$. For the connection one-form $\omega=\omega_i\theta^i$ we find

$$\omega_\pm=e^{-F}(\hat{\omega}_\pm\mp\hat{\partial}_\pm F),\quad \hat{\omega}_\pm=\mp\frac{\sqrt{2}}{h^- -h^+}(h^\mp)'. \tag{B5}$$

The curvature is defined by the second structural equation: $d\omega^i{}_j=\frac{1}{2}R^i{}_{jkl}\,\theta^k\wedge\theta^l$, where we used $\omega^i{}_k\wedge\omega^k{}_j=0$. Since $d\omega=(\nabla_-\omega_+ -\nabla_+\omega_-)\theta^-\wedge\theta^+$, one finds

$$R=2R_{+-}=2(\nabla_-\omega_+ -\nabla_+\omega_-). \tag{B6}$$

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