Behavior of quasilocal mass under conformal transformations

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We show that in a generic scalar-tensor theory of gravity, the ''referenced'' quasilocal mass of a spatially bounded region in a classical solution is invariant under conformal transformations of the spacetime metric. We first extend the Brown-York quasilocal formalism to such theories to obtain the ''unreferenced'' quasilocal mass and prove it to be conformally invariant. However, this quantity is typically divergent. It is, therefore, essential to subtract from it a properly defined reference term to obtain a finite and physically meaningful quantity, namely, the referenced quasilocal mass. The appropriate reference term in this case is defined by generalizing the Hawking-Horowitz prescription, which was originally proposed for general relativity. For such a choice of reference term, the referenced quasilocal mass for a general spacetime solution is obtained. This expression is shown to be a conformal invariant provided the conformal factor is a monotonic function of the scalar field. We apply this expression to the case of static spherically symmetric solutions with arbitrary asymptotics to obtain the referenced quasilocal mass of such solutions. Finally, we demonstrate the conformal invariance of our quasilocal mass formula by applying it to specific cases of four-dimensional charged black hole spacetimes, of both the asymptotically flat and non-flat kinds, in conformally related theories. [S0556-2821(99)04402-1]

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I. INTRODUCTION

The lack of a generically meaningful notion of local energy density in general relativity is well known [1–3]. Essentially, this is due to the absence of an unambiguous prescription for decomposing the spacetime metric into ''background'' and ''dynamical'' components.[1] If such a prescription were available, then one could associate energy in general relativity with the dynamical component of the metric. In the past there have been attempts to define quasilocal energy using pseudotensor methods [1,2,5]. However, these approaches led to coordinate-dependent expressions, which lacked an unambiguous geometrical interpretation. Another way of defining quasilocal energy has been via the spinor constructions [6–9]. There are, however, several unresolved questions regarding this approach, a key issue being the lack of a rigorous proof of the Witten-Nester integral being a boundary value of the gravitational Hamiltonian [10].[2] Nevertheless, the total energy of an isolated system has been defined in terms of the behavior of the gravitational field at large distances from the system [11]. Moreover, Brown and York (BY) have introduced in Ref. [3] a way to define the quasilocal energy of a spatially bounded system in general relativity in terms of the total mean curvature of the boundary. Further, in spacetimes with a hypersurface-forming timelike Killing vector on the boundary of the system, it can be shown that there exists a conserved charge, which can be defined to be the quasilocal mass associated with the bounded region [12].

The past few years have seen a revival of interest in scalar-tensor theories of gravity, primarily string-inspired four-dimensional dilaton gravity, which has been shown to yield cosmological as well as charged black hole solutions (see Refs. [13–16] for reviews.) In particular, the spacetime structure (i.e., geodesics and singularities) of these black hole solutions and, also, those of the Brans-Dicke-Maxwell theory (in higher dimensions) [17,18] are known to have significant differences with respect to the Reissner-Nordström black holes. This prompts one to investigate the form of the classical laws of black hole mechanics and the ensuing picture of black hole thermodynamics in these theories. But the study of the thermodynamical laws entails knowledge of the energy and entropy associated with these spacetimes. Moreover, *equilibrium* thermodynamics of a black hole (specifically, in the case of an asymptotically flat solution) requires that it be put in a finite-sized ''box,'' just as one does in general relativity. Thus such a study requires knowledge of the quasilocal energy of these ''finite-sized'' systems.

Recently the BY formalism has been extended to the case of a generic scalar-tensor theory of gravity in spacetime dimensions greater than 2 [19,20]. Since solutions of two conformally related scalar-tensor theories will themselves be related by a conformal transformation, it is interesting to ask if the quasilocal masses of these solutions are also related. In the past, it has been suggested that the quasilocal mass is a

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[1]See, however, the field formulation of general relativity [4].

[2]For a more complete list of references on quasilocal energy, also see the ones cited in Ref. [3].

conformal invariant. The reason that is usually provided is that a conformal transformation is simply a local field reparametrization, which is supposed to leave physical quantities, such as the mass of a system, unchanged. In fact, in Ref. [20], Chan, Creighton, and Mann (CCM) argue that the quasilocal mass is indeed invariant under a conformal transformation.

As in general relativity, the (unreferenced) mass of a spacetime in a scalar-tensor theory of gravity is typically divergent unless one subtracts a (divergent) contribution from a suitable reference geometry to obtain a meaningful finite result [3]. Recently, Hawking and Horowitz (HH) gave a prescription in general relativity for obtaining an appropriate reference action based on the asymptotic behavior of the fields on a classical solution [21]. This reference action is subtracted from the original action of the theory to define what is called the physical action. The surface term that arises in the Hamiltonian associated with this action can be taken as the definition of ''total mass.'' This mass turns out to be finite and is termed as the physical mass. When evaluated on an asymptotically flat spacetime solution, the physical mass of the full spacetime coincides with the corresponding Arnowitt-Deser-Misner (ADM) expression.

Following HH, one could ask if a similar prescription can be formulated for scalar-tensor theories of gravity. If so, it would be interesting to see whether the total mass arising from such an action is conformally invariant. It has been claimed by CCM that the generalization of the HH prescription to scalar-tensor theories does *not* lead to a conformally invariant physical mass. They propose their own reference action for a general scalar-tensor theory of gravity and show that the associated physical mass of static, spherically symmetric (SSS) solutions is conformally invariant. However, unlike what happens in the HH prescription, the CCM reference action is not motivated by any boundary conditions on the fields that define the spacetime solution of interest.

In this paper, we first show that the HH prescription can be generalized to the case of scalar-tensor gravity. This reduces the arbitrariness in the choice of the reference action. More importantly, we prove that under certain conditions the resulting reference action leads to a conformally invariant referenced quasilocal mass. In the following, we will directly deal with only that conformal transformation that relates the scalar-tensor-gravity metric to that in the Einstein frame. Of course, our results can be readily extended to study the behavior of physical quantities under a conformal transformation relating one scalar-tensor theory to another. In Sec. II we derive the expression for the (unreferenced) quasilocal mass of a bounded region in $(D+1)$-dimensional spacetime solution of a scalar-tensor theory of gravity and prove it to be conformally invariant. In Sec. III we generalize the HH prescription to the case of scalar-tensor gravity. It is shown that for such a choice of the reference action, the referenced quasilocal mass is a conformal invariant provided the conformal factor is a monotonic function of the scalar field. Using this prescription, we give an expression for quasilocal mass of static spherically symmetric solutions (with arbitrary asymptotics). In Sec. IV, we demonstrate the conformal invariance of this quasilocal mass formula by applying it to
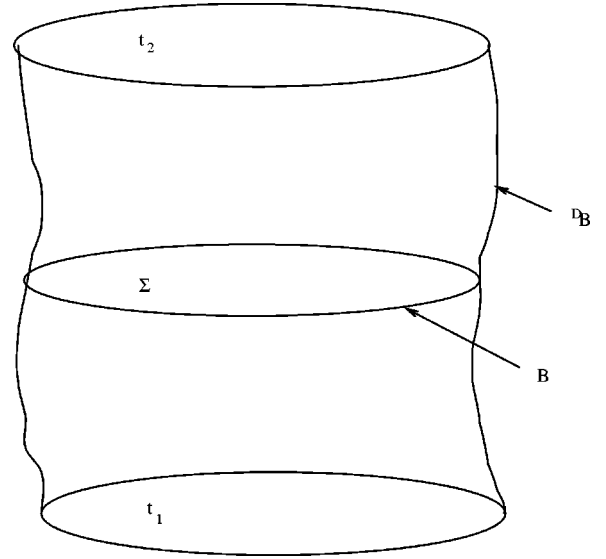

FIG. 1. A bounded spacetime region with boundary consisting of initial and final spatial hypersurfaces $t=t_1$ and $t=t_2$ and a D-dimensional surface ^{D}B. Here, ^{D}B itself is the time-evolution of the $(D-1)$-dimensional surface B, which is the boundary of an arbitrary spatial slice Σ.

specific cases of four-dimensional (4D) black hole spacetimes, of both the asymptotically flat and non-flat kinds, in conformally related theories. We briefly summarize and discuss our results in Sec. V. In the Appendix, we show how the standard prescription for determining the stress-energy pseudotensor in general relativity can be suitably adapted to find the quasilocal energy in scalar-tensor theories of gravity. Finally, we demonstrate the consistency of the results of the pseudotensor method with the quasilocal formalism of Brown and York as applied to scalar-tensor theories. Throughout this paper, we use the conventions of Misner, Thorne, and Wheeler [1] and work in ''geometrized units'' $c=1=G$.

II. QUASILOCAL MASS
UNDER CONFORMAL TRANSFORMATION

Consider a spatially bounded region of a $(D+1)$-dimensional spacetime that is a classical solution of a scalar-tensor theory of gravity, such as dilaton gravity or Brans-Dicke theory. In this section we extend the formalism of Brown and York [3] to derive an expression for the quasilocal energy of gravitational and matter fields associated with such regions. Subsequently, we will give an expression for the quasilocal mass.

The BY derivation of the quasilocal energy, as applied to a $(D+1)$-dimensional spacetime, can be summarized as follows. The system we consider is a D-dimensional spatial hypersurface Σ bounded by a $(D-1)$-dimensional spatial hypersurface B in a spacetime region that can be decomposed as a product of a D-dimensional hypersurface and a real line interval representing time (see Fig. 1). The time-evolution of the boundary B is the surface ^{D}B. One can then obtain a surface stress-tensor on ^{D}B by taking the functional

derivative of the action with respect to the D-dimensional metric on ^{D}B. The energy surface density is the projection of the surface stress-tensor normal to a family of spacelike surfaces such as B that foliate ^{D}B. The integral of the energy surface density over such a boundary B is the quasilocal energy associated with a spacelike hypersurface Σ whose *orthogonal* intersection with ^{D}B is the boundary B. Here we assume that there are no inner boundaries, such that the spatial hypersurfaces Σ are complete. In the case where horizons form, one simply evolves the spacetime inside as well as outside the horizon.

We follow the same notation as BY. The spacetime metric is $g_{\mu\nu}$ and n^μ is the outward pointing unit normal to the surface ^{D}B. The metric and extrinsic curvature of ^{D}B are denoted by $\gamma_{\mu\nu}$ and $\Theta_{\mu\nu}$, respectively, and they obey $n^\mu \gamma_{\mu\nu}=0$ and $n^\mu \Theta_{\mu\nu}=0$. Alternatively, $\gamma_{\mu\nu}$ and $\Theta_{\mu\nu}$ can be viewed as tensors on ^{D}B, denoted by γ_{ij} and Θ_{ij}, where i,j refer to coordinates in ^{D}B. Similarly, the metric and extrinsic curvature of Σ are given by the spacetime tensors $h_{\mu\nu}$ and $K_{\mu\nu}$, respectively. When viewed as tensors on Σ, they will be denoted by h_{ij} and K_{ij}. As in BY, here we will assume that the hypersurface foliation Σ is ''orthogonal'' to the surface ^{D}B in the sense that on the boundary ^{D}B, the future-pointing unit normal u^μ to the hypersurface Σ and the outward pointing spacelike unit normal n^μ to the surface ^{D}B satisfy $(u \cdot n)|_{^D\text{B}}=0$. This implies that the shift vector, V^i, normal to the boundary vanishes, i.e., $V^i n_i=0$.

A. Action

We study the following action for a scalar-tensor theory of gravity in a $(D+1)$-dimensional spacetime:

$$S[\bar{g}_{ab},\phi,\mathcal{F}]=\frac{1}{2\kappa}\int d^{(D+1)}x \sqrt{-\bar{g}} U(\phi)$$
$$\times [\bar{R}-W(\phi)(\bar{\nabla}\phi)^2-V(\phi)+X(\phi)\bar{L}_\text{m}],$$
$$(2.1)$$

where $\bar{g}_{ab}$ is the ''physical'' metric, ϕ is a scalar field, $\mathcal{F}$ represents matter fields, $\kappa \equiv 8\pi$, and U, V, W, and X are functions of ϕ. Also, $\bar{L}_\text{m}$ is the matter Lagrangian that includes a possible cosmological constant term. The overbar denotes the functional dependence of quantities on the physical metric $\bar{g}_{ab}$. Here we assume that $\bar{L}_\text{m}$ does not involve any derivatives of the metric. The dynamics of the scalar field is governed by its kinetic term, the effective potential term $V(\phi)$ and the non-minimal coupling to the scalar curvature, $\bar{R}$, described by $U(\phi)$. The effective potential term can be inclusive of an arbitrary additive constant which may occur as a Lagrange multiplier, an integration constant, or even a fall out of the renormalization procedure of the other matter fields described by $\bar{L}_\text{m}$. It is this constant that is responsible for the ''cosmological constant problem.'' One can consider the potential term as an effective term obtained by integrating over ''heavy'' degrees of freedom as long as one does not go beyond the leading semiclassical approximation for the scalar field ϕ.

The BY analysis is of course readily applicable in the ''Einstein'' frame, which is associated with the ''auxiliary'' metric

$$g_{ab} \equiv U^{2/(D-1)} \bar{g}_{ab},\qquad (2.2)$$

where $D>1$. In terms of the auxiliary metric, one can always cast action (2.1) as a sum of the Hilbert action, which is independent of the scalar and matter fields, and a functional S_f that depends on these fields:

$$S[g_{ab}]=\frac{1}{2\kappa}\int d^{(D+1)}x\sqrt{-g}R+S_f,\qquad (2.3)$$

where

$$S_f \equiv \frac{1}{2\kappa}\int d^{(D+1)}x$$
$$\times \sqrt{-g}\left[\frac{D}{D-1}[\nabla \ln U(\phi)]^2-W(\phi)(\nabla\phi)^2\right.$$
$$\left.-U^{2/(1-D)}[V(\phi)-X(\phi)L_\text{m}]\right].\qquad (2.4)$$

Above, L_m is a functional of the matter fields, their derivatives, the auxiliary metric g_{ab}, and the scalar field ϕ. In Eq. (2.3), we have ignored the surface term contributions for the present. The subscript f represents the scalar field ϕ and the matter fields. Note that S_f does not involve any derivatives of the metric. In the following, $(\bar{g},\bar{f})$ will denote a field configuration that is a solution to Eq. (2.1), whereas (g,f) is the conformally related solution in the theory (2.3). Although there is no bar over ϕ, note that ϕ is implicitly included in the configuration $(\bar{g},\bar{f})$.

B. Quasilocal energy and mass

We begin this section by briefly discussing the Hamilton-Jacobi analysis used by BY to evaluate the quasilocal energy of a spatially bounded region in Einstein gravity. We will later give the expression for the quasilocal energy in a generic scalar-tensor theory of gravity.

In general relativity, to make the action functionally differentiable under boundary conditions that fix the metric on the boundary, one appends appropriate surface terms to the Hilbert action. The resulting action in $D+1$ dimensions is

$$S^1=\frac{1}{2\kappa}\int d^{(D+1)}x\sqrt{-g}R+\frac{1}{\kappa}\int_{t'}^{t''}d^Dx\sqrt{h}K$$
$$-\frac{1}{\kappa}\int_{^D\text{B}}d^Dx\sqrt{-\gamma}\Theta+S_f,\qquad (2.5)$$

where $\int_{t'}^{t''}d^Dx$ represents the difference of the integral over a spatial three-surface $t=t''$ and that over a three-surface $t=t'$. Of course, the equations of motion obtained from the variation of the above action are unaffected by the addition

of an arbitrary function S^0 of fixed boundary data to S^1. Hence, the variation of such an action restricted to classical solutions gives

$\delta S_{\rm cl} = \{$terms involving variations in the matter fields$\}$

$$+ \int_{t'}^{t''} d^D x P_{\rm cl}^{ij} \delta h_{ij} + \int_{D_B} d^D x (\pi_{\rm cl}^{ij} - \pi_0^{ij}) \delta \gamma_{ij},$$
(2.6)

where P^{ij} and π^{ij} are, respectively, the momenta conjugate to h_{ij} and γ_{ij}, and

$$\pi_0 \equiv \frac{\delta S^0}{\delta \gamma_{ij}}.$$
(2.7)

Above, the subscript ''cl'' denotes the value of a quantity on a classical solution. As we discuss in detail later, different choices for S^0 arise by imposing different physical requirements on the quasilocal energy.

From Eqs. (2.6) and (2.7) given above, we obtain the following Hamilton-Jacobi equations:

$$P_{\rm cl}^{ij}|_{t''} = \frac{\delta S_{\rm cl}}{\delta h_{ij}(t'')},$$
(2.8)

$$(\pi_{\rm cl}^{ij} - \pi_0^{ij}) = \frac{\delta S_{\rm cl}}{\delta \gamma_{ij}}.$$
(2.9)

The quantity that is of interest to us is the surface stress-tensor for spacetime and the fields, which is given by

$$\tau^{ij} \equiv \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\rm cl}}{\delta \gamma_{ij}}.$$
(2.10)

Using Eq. (2.9), we obtain

$$\tau^{ij} = \frac{2}{\sqrt{-\gamma}} (\pi_{\rm cl}^{ij} - \pi_0^{ij}).$$
(2.11)

If u^i is the unit timelike normal to Σ on the boundary B, then the proper energy surface density ϵ is

$$\epsilon \equiv u_i u_j \tau^{ij} = -\frac{1}{\sqrt{\sigma}} \frac{\delta S_{\rm cl}}{\delta N},$$
(2.12)

where σ_{ij} is the metric on the boundary B. Above, we made use of the following identity:

$$\frac{\partial \gamma_{ij}}{\partial N} = -\frac{2 u_i u_j}{N}.$$
(2.13)

Equation (2.12) together with Eq. (2.9) can be used to show that that the energy surface density is related to the trace of the extrinsic curvature, $k_{\rm cl}$, of the boundary B embedded in a spatial hypersurface Σ (which in turn is embedded in a classical solution):

$$\epsilon = \frac{1}{\kappa} (k_{\rm cl} - k_0),$$
(2.14)

where k_0 is the trace of the extrinsic curvature of a surface that is *isometric* to B, but is embedded in a reference space.

Following BY, the extrinsic curvature of B as embedded in Σ is defined by

$$k_{\mu\nu} = -\sigma_\mu^\alpha D_\alpha n_\nu,$$
(2.15)

where D_α is the covariant derivative on Σ. Therefore, $k = \sigma^{\mu\nu} k_{\mu\nu}$. The quasilocal energy associated with all the fields on the spacelike hypersurface Σ with boundary B in this ''auxiliary'' spacetime is

$$E = \int_B d^{(D-1)} x \sqrt{\sigma} \epsilon$$

$$= - \int_B d^{(D-1)} \frac{\delta S_{\rm cl}}{\delta N}$$

$$= \frac{1}{\kappa} \int_B d^{(D-1)} x \sqrt{\sigma} (k_{\rm cl} - k_0).$$
(2.16)

In other words, E represents the proper quasilocal energy in the Einstein frame. The above expression is interpreted as energy because it is minus the change in the classical action due to a uniform, unit increase in the proper time along ^{D}B. Also, for a unit lapse and zero shift, it is equal to the Hamiltonian corresponding to the action (2.5), as evaluated on a classical solution. It is satisfying to note that this is a geometric expression independent of the coordinates on the quasilocal surface. However, it does depend on the choice of the quasilocal surface and also on the foliation of the spacetime by spacelike hypersurfaces.

When there is a timelike Killing vector field ξ^μ on the boundary ^{D}B, such that it is also hypersurface forming, one can define an associated conserved quasilocal mass for the bounded system [12,3]:

$$M = \int_B d^{(D-1)} x \sqrt{\sigma} N \epsilon,$$
(2.17)

where N is the lapse function related to ξ^μ by $\xi^\mu = N u^\mu$. Further, if $\xi \cdot u = -1$, then $N = 1$ and consequently the quasilocal mass is the same as the quasilocal energy (2.16). Unlike the quasilocal energy (2.16), the quasilocal mass is independent of any foliation of the bounded system.

We now ask, what is the analogous expression for the quasilocal energy or mass for a bounded spatial region of a spacetime solution of the scalar-tensor theory (2.1)? Note that under boundary conditions that fix the metric on the boundary, the appropriate surface action to be added to the action (2.1) is

$$S_{D\mathsf{B}}[\bar{g}_{ab},\phi] = \frac{1}{\kappa} \int_{t'}^{t''} d^D x \sqrt{-\bar{h}}\, U(\phi)\bar{K}$$

$$- \frac{1}{\kappa} \int_{D_\mathsf{B}} d^D x \sqrt{-\bar{\gamma}}\, U(\phi)\bar{\Theta}, \quad (2.18)$$

where $\bar{K}$ is the trace of the extrinsic curvature of a spatial hypersurface and $\bar{\Theta}$ that of the boundary $^D\mathsf{B}$ when embedded in a spacetime solution of action (2.1). It can be verified that under a conformal transformation (2.2), the action S^1 in Eq. (2.5) transforms exactly to $S[\bar{g}_{ab},\phi,\mathcal{F}]+S_{D\mathsf{B}}[\bar{g}_{ab},\phi]$, which is defined by Eqs. (2.1) and (2.18). As in the case of Einstein gravity discussed above, one can use the BY approach to derive the expression for quasilocal energy from the above surface action in a non-minimally coupled theory. Such a calculation was done by Creighton and Mann [19] for four-dimensional pure dilaton-gravity. A straightforward generalization of their derivation to the case of a $(D+1)$-dimensional scalar-tensor theory (2.1) including matter fields gives the quasilocal energy in such theories to be

$$\bar{E} = \int_\mathsf{B} d^{(D-1)} x \sqrt{\bar{\sigma}}\,\bar{\epsilon}$$

$$= \frac{1}{\kappa} \int_\mathsf{B} d^{(D-1)} x \sqrt{\bar{\sigma}}[U(\phi)\bar{k} - \bar{n}^i \partial_i U(\phi)] - \bar{E}_0.$$

$$(2.19)$$

In the next section we will consider appropriate reference actions S^0 and their respective contributions, $\bar{E}_0$, to the above expression. In the Appendix we give an alternative derivation of the above energy expression using the pseudo-tensor method.

Analogous to Eq. (2.17) one can also define the quasilocal mass in the scalar-tensor theory to be

$$\bar{M} = \int_\mathsf{B} d^{(D-1)} x \sqrt{\bar{\sigma}}\,\bar{N}\bar{\epsilon}$$

$$= \frac{1}{\kappa} \int_\mathsf{B} d^{(D-1)} x \sqrt{\bar{\sigma}}\,\bar{N}[U(\phi)\bar{k} - \bar{n}^i \partial_i U(\phi)] - \bar{M}_0,$$

$$(2.20)$$

where $\bar{M}_0$ is an appropriate reference term.

C. Conformal transformation

We now study how the quasilocal mass, $\bar{M}$, modulo the reference term $\bar{M}_0$, behaves under a conformal transformation. Equation (2.17) shows that this requires knowledge of how the total mean curvature k of the boundary B behaves under a conformal transformation. Let the physical metric $\bar{g}_{ab}$ of the scalar-tensor theory be related to the auxiliary metric g_{ab} by the conformal transformation

$$\bar{g}_{ab} \equiv \Omega^2 g_{ab}, \quad (2.21)$$

where Ω is generally a function of the spacetime coordinates. Comparing with Eq. (2.2), we find that

$$\Omega = U^{-1/(D-1)}. \quad (2.22)$$

Note that on-shell $U(\phi)$ will be determined through the equations of motion pertaining to the action (2.1).

Let us embed the $(D-1)$-dimensional spatial boundary B in each of the two conformally related spacetimes, assuming that such embeddings are feasible and unique. Then the unit timelike normal $\bar{u}^i$ in the physical spacetime is related to that in the auxiliary spacetime, u^i, as follows:

$$\bar{u}^i = \Omega^{-1} u^i. \quad (2.23)$$

Similarly, the outward pointing unit normals to the surface $^D\mathsf{B}$ in the two spacetimes are related as

$$\bar{n}^i = \Omega^{-1} n^i. \quad (2.24)$$

One can further show that the extrinsic and the total mean curvature of the boundary B, as embedded in these spacetimes, are related as follows:

$$\bar{k}_{ij} = \Omega[k_{ij} - \sigma_{ij} n^l \nabla_l(\ln \Omega)], \quad (2.25)$$

$$\bar{k} = \Omega^{-1}[k - (D-1)n^l \nabla_l(\ln \Omega)], \quad (2.26)$$

where n^l is the spacelike unit normal to the surface $^D\mathsf{B}$ (embedded in the auxiliary spacetime). Formally, we associate the covariant derivatives ∇_l and $\bar{\nabla}_l$ with metrics g_{ab} and $\bar{g}_{ab}$, respectively. In spacetime regions where Ω is non-singular, one can invert the above relation to obtain

$$k = \Omega\bar{k} + (D-1)\Omega\bar{n}^l\bar{\nabla}_l(\ln \Omega), \quad (2.27)$$

where k is the total mean curvature of the boundary B, as embedded in the auxiliary spacetime.

Equation (2.27) shows that under a conformal transformation the quasilocal mass defined in Eq. (2.17), modulo the reference term arising from k_0, transforms as follows:

$$\frac{1}{\kappa} \int_\mathsf{B} d^{(D-1)} x N \sqrt{\sigma} k$$

$$= \frac{1}{\kappa} \int_\mathsf{B} d^{(D-1)} x \bar{N} \sqrt{\bar{\sigma}}[U(\phi)\bar{k} - \bar{n}^i \partial_i U(\phi)],$$

$$(2.28)$$

where we have used Eq. (2.22). Applying the above identity to the mass expressions (2.17) and (2.20) proves that the quasilocal masses of conformally related spacetimes are the same, provided the reference term $\bar{M}_0$ is conformally invariant.

Consider the behavior of the timelike vector ξ^μ defined above Eq. (2.17). It is assumed to be Killing in a given frame, say, the Einstein frame. It is also a conformal invariant, i.e., $\bar{\xi}^\mu = \bar{N}\bar{u}^\mu = Nu^\mu = \xi^\mu$. However, it will not remain Killing in a general conformal transformation. Thus, al-

though the (unreferenced) quasilocal mass is a conformal invariant, its property of being a conserved charge in a given frame is not. However, if ξ^μ obeys $\xi^\mu \nabla_\mu \phi = 0$, then it continues to remain Killing in the conformally related frame. In such an event, the associated quasilocal mass remains a conserved charge in that frame too. Furthermore, the unreferenced quasilocal energy is not invariant under a conformal transformation. When Ω is independent of the coordinates on the quasilocal surface, it transforms as

$$\bar{E} - \bar{E}_0 = \Omega^{-1}(E - E_0) \qquad (2.29)$$

and, hence, is not a conformal invariant.

Finally, note that when we compare the quasilocal masses of conformally related spacetimes above, we assume that the boundary ^{D}B, which is taken to be embedded in a particular spacetime, is also embeddable in the conformally related spacetime. However, the embeddability of a hypersurface requires that the intrinsic and extrinsic geometry of the boundary obey the Gauss-Codazzi, Codazzi-Mainardi, and Ricci integrability conditions in both spacetimes separately. In general, not all of these integrability conditions are conformal invariants. Therefore, embeddability of a hypersurface in a spacetime does not guarantee its embeddability in a conformally related spacetime. Nevertheless, it can be shown that one can always embed a $(D-1)$-dimensional spacelike spherical boundary in $(D+1)$-dimensional SSS spacetime solutions, which are Ricci flat, and in spacetimes related through conformal transformations that preserve these spacetime properties [23,24].

III. REFERENCE ACTION AND QUASILOCAL MASS

The Brown-York definition of the quasilocal energy (2.16) associated with a spatially bounded region of a given spacetime solution is not unique. This is because an arbitrary functional S^0 of the boundary data can be added to the action without affecting the equations of motion. On the other hand, to get a well-defined (finite) expression for the quasilocal energy of spatially non-compact geometries, one is usually required to subtract the (divergent) contribution of some reference background. At the level of the action such a ''regularization'' is tantamount to the addition of a reference action S^0, which is a functional of appropriate background fields (g_0, f_0), to the original action S^1. For 4D Einstein gravity, BY prescribe the following reference action:

$$S^0 = -\int_{^3\text{B}} d^3x [N\sqrt{\sigma}(k/\kappa)|_0 + 2\sqrt{\sigma}V^a(\sigma_{ai}n_j P^{ij}/\sqrt{h})|_0], \qquad (3.1)$$

which is a linear functional of the lapse N and shift V^a. Above, ^{3}B is the time-evolution of a two-boundary B that is embedded in a fixed three-dimensional spacelike slice Σ of some fixed reference spacetime. Also, $k|_0$ and $(\sigma_{ai}n_j P^{ij}/\sqrt{h})|_0$ are arbitrary functions of the two-metric σ_{ab} on the boundary B, n^j is the unit normal to the 2-boundary B, and $\{h_{ij}, P^{ij}\}$ are the canonical 3-metric and the conjugate momentum on the three-dimensional spacelike slice Σ. Varying the lapse in the first term in Eq. (3.1) gives

the energy surface density, whereas varying the shift in the second term gives the momentum surface density in the reference spacetime [3]. Since we mainly discuss the application of Eq. (3.1) to evaluate the *proper* quasilocal mass or energy, which is obtained by the variation of the total action (on classical solutions) with respect to N, we will henceforth drop the last term in Eq. (3.1) from our consideration.

To calculate the quasilocal energy associated with regions of spacetime solutions of $(D+1)$-dimensional Einstein gravity, the appropriate generalization of the BY reference action is again given by Eq. (3.1), except that the integration is now over the boundary ^{D}B. The boundary ^{D}B itself is the time-evolution of the $(D-1)$-dimensional spatial boundary B. For asymptotically flat spacetimes an appropriate reference background might be vacuum flat spacetime.

However, for an arbitrary spacetime solution (e.g., spacetimes that are neither spatially closed nor asymptotically flat), a more well defined prescription for the choice of S^0 is required. Recently, one such prescription was given by Hawking and Horowitz in their quest to obtain the total mass of spacetimes with arbitrary asymptotic behavior in general relativity [21]. Their starting point is the ''physical'' action defined as

$$S_P(g,f) \equiv S(g,f) - S(g_0, f_0), \qquad (3.2)$$

where (g_0, f_0) are fields specifying a reference static background, which is a *solution* to the field equations. Therefore, the physical action of the reference background is zero. Given a solution (g,f), in order to determine a reference background, (g_0, f_0), HH fix a three-boundary (^{3}B) near infinity and require that (g,f) induce the same fields on this boundary as (g_0, f_0). The energy of a solution can be obtained from the physical Hamiltonian associated with S_P (for details, see Ref. [21]) and is similar to the BY quasilocal expression. For asymptotically flat spacetime solutions, the reference background is chosen to be flat space and the resulting energy expression agrees with the one obtained in the ADM formalism.

It is important to note that the HH prescription allows one to compute the total energy associated with a general time translation $t^\mu = Nu^\mu + V^\mu$. In a generic case, the resulting energy will have a shift-dependent contribution, such as the second term in Eq. (3.1). However, such a term vanishes when the spacetime is taken to approach a static background solution and the resulting expression (with $N=1$) is the same as the BY energy (2.16). Even if the spacetime is asymptotically non-static, this term will vanish when $V^a \sigma_{ab} = 0$. This happens, e.g., for cosmological solutions with the Robertson-Walker metric.

Building on the work of Brown and York, Chan, Creighton, and Mann [20,22] chose a particular reference action to compute the quasilocal masses of solutions in scalar-tensor theories. In the special case of SSS spacetimes, it has been shown by CCM that their choice leads to a conformally invariant referenced quasilocal mass. A second possibility of obtaining a reference action is to generalize the HH prescription to scalar-tensor theories. Such an attempt was also made by CCM [20]. However, they conclude that the mass formula

obtained using their generalization of the HH prescription is not conformally invariant. For details on this issue, we refer the reader to Ref. [20].

In this section, we extend the BY formalism to obtain the referenced quasilocal mass associated with bounded regions of spacetime solutions (with arbitrary asymptotic behavior) in scalar-tensor gravity. A relevant question in such an analysis is whether the reference action can be specified in a unique way. Finding an answer to this would in itself be an interesting pursuit and involves addressing issues of positivity of the mass or energy of such solutions as well as the stability of the corresponding reference solution. Here, we do not attempt to find if the reference action or solution can be uniquely specified at all. Below, after discussing the CCM analysis briefly, we present our alternative generalization of the HH prescription to scalar-tensor gravity. Although this does not select a unique reference action, nevertheless, invoking this prescription reduces the number of allowed reference actions. We prove that under certain conditions such a prescription does lead to a conformally invariant referenced quasilocal mass.

A. CCM prescription

For a non-minimally coupled action of the type (2.1), the reference action suggested by CCM is [20]

$$S^0 = -\int_{D_B} d^D x \bar{N} \sqrt{\bar{\sigma}} U(\phi) (\bar{k}_{\text{flat}}/\kappa), \qquad (3.3)$$

where $\bar{k}_{\text{flat}}$ is the trace of the extrinsic curvature of the $(D-1)$-boundary B embedded in a D-dimensional flat spatial slice. Consider the special case of an asymptotically flat SSS spacetime metric, as a solution in this theory:

$$d\bar{s}^2 = -\bar{N}^2(r)dt^2 + \frac{dr^2}{\bar{\lambda}^2(r)} + r^2 d\omega^2, \qquad (3.4)$$

where $\bar{N}$ and $\bar{\lambda}$ are functions of r only, and $d\omega^2$ is the line element on a unit $(D-1)$-sphere. Let us now make the following conformal transformation:

$$\tilde{g}_{ab} = \tilde{\Omega}^2 \bar{g}_{ab}, \quad \tilde{U} = \tilde{\Omega}^{(1-D)} U, \qquad (3.5)$$

where $U(\phi)$ is the scalar-field dependent coupling appearing in Eq. (3.3). Note that under this conformal transformation the functional form of S^0,

$$S^0 = -\int_{D_B} d^D x \tilde{N} \sqrt{\tilde{\sigma}} \tilde{U}(\phi)(\tilde{k}_{\text{flat}}/\kappa), \qquad (3.6)$$

remains unchanged provided we assume $\tilde{N} = \tilde{\Omega} \bar{N}$.

Let the metric (3.4) be related through this conformal transformation to the following SSS metric:

$$d\tilde{s}^2 = -\tilde{N}^2(r)dt^2 + \frac{dr^2}{\tilde{\lambda}^2(r)} + \tilde{\Omega}^2 r^2 d\omega^2, \qquad (3.7)$$

which is assumed to arise as a solution to another scalar-tensor theory that is related to action (2.1) by the conformal transformation (3.5). Above, $\tilde{N} = \tilde{\Omega} \bar{N}$ and $\tilde{\lambda} = \tilde{\Omega}^{-1} \bar{\lambda}$, where $\tilde{\Omega}$ is a function of r only.

For the special case of the spacetime solution (3.4), and with the choice of reference action (3.3), CCM argue that the quasilocal mass associated with the region inside a sphere of curvature radius r, which is embedded in spacetime (3.4), can be expressed as

$$\bar{M}(r) = \frac{\bar{N}(r)}{\kappa} \left(\frac{(D-1)\bar{A}_{D-1}(r)U(\phi)}{r} \right.$$
$$\left. -\bar{\lambda}(r)\frac{d}{dr}[\bar{A}_{D-1}(r)U(\phi)] \right). \qquad (3.8)$$

Above, $\bar{A}_{D-1}$ is the area of the boundary $(D-1)$-sphere of radius r given by

$$\bar{A}_n = \int_B d^n x \sqrt{\bar{\sigma}} = \frac{(4\pi)^{n/2}\Gamma(n/2)}{\Gamma(n)} r^n, \qquad (3.9)$$

$\bar{\sigma}_{ij}$ being the metric on B. Note that the first term in Eq. (3.8) is just the reference term

$$\bar{M}^0 = -\int_{D_B} d^{D-1} x \bar{N} \sqrt{\bar{\sigma}} U(\phi)(\bar{k}_{\text{flat}}/\kappa), \qquad (3.10)$$

whereas the second term arises from the quasilocal mass definition (2.20) on using the identity

$$\frac{1}{\kappa}\int_B d^{(D-1)} x \sqrt{\bar{\sigma}} \bar{N} \bar{k} = -\frac{\bar{N}(r)\bar{\lambda}(r)}{\kappa}\frac{d}{dr}\bar{A}_{(D-1)}(r), \qquad (3.11)$$

which holds for the SSS metric (3.4). We will call Eq. (3.8) the CCM mass expression. Similarly, CCM find that for the metric (3.7), the quasilocal mass is

$$\tilde{M}(r) = \frac{\tilde{N}(r)}{\kappa} \left(\frac{(D-1)\tilde{A}_{D-1}(r)\tilde{U}}{\tilde{\Omega} r} - \tilde{\lambda}(r)\frac{d}{dr}[\tilde{A}_{D-1}(r)\tilde{U}] \right), \qquad (3.12)$$

where $\tilde{A}_{(D-1)} = \tilde{\Omega}^{(D-1)}\bar{A}_{(D-1)}$.

Thus, the CCM mass $\bar{M}(r)$ defined in Eq. (3.8) is invariant under the conformal transformation (3.5), namely, $\bar{M}(r) = \tilde{M}(r)$. To be precise, each term in Eq. (3.8) is separately conformally invariant. Finally, let us emphasize that unlike in the HH prescription, in the CCM prescription one does not require the ''background'' fields appearing in the reference action (3.3) to constitute a solution of that action. Also, the choice of the CCM reference action is independent of the asymptotic behavior of the fields of the solution. [This is the reason why the referenced quasilocal mass (with $N = 1$) in this prescription differs from the Abbott-Deser definition of the total energy [25] when applied to asymptotically anti–de Sitter SSS spacetimes. One can, however, recover

this energy expression by generalizing the CCM reference action to the case of such spacetimes (for details, see Ref. [12]).]

B. Alternative prescription for reference action and quasilocal mass

First, we extend the applicability of the HH prescription to scalar-tensor gravity in order to obtain an appropriate reference action. Given a solution, $(\bar{g}, \bar{f})$, one chooses a reference background *solution*, $(\bar{g}_0, \bar{f}_0)$, by using the HH prescription as enunciated above in this section. Then the appropriate reference action is simply

$$[S[\bar{g}_{ab}, \phi, \mathcal{F}] + S_{D_B}[\bar{g}_{ab}, \phi]]_{\text{ref}}, \qquad (3.13)$$

where S and S_{D_B} are given by Eqs. (2.1) and (2.18), respectively, and $[\text{term}]_{\text{ref}}$ denotes the value of the term as evaluated on the reference solution. In general, the reference action can depend on the initial and final metrics $\bar{h}_{ij}(t')$ and $\bar{h}_{ij}(t'')$ through spatial boundary terms, namely, the first term on the right-hand side of Eq. (2.18). However, in the present calculation such contributions can be dropped since they do not affect the BY quasilocal mass.

Second, we address the following question: If the HH prescription is obeyed by a pair of solutions, $(\bar{g}, \bar{f})$ and $(\bar{g}_0, \bar{f}_0)$, for the boundary B in a given frame, then will it also be obeyed by conformally related fields in a conformally related frame? We answer this as follows. Note that the reference solution $(\bar{g}_0, \bar{f}_0)$ is conformally related to that in Einstein gravity, (g_0, f_0), by Eq. (2.21), where U is now a function of ϕ_0. Here, both $U(\phi_0)$ and the conformal factor Ω are positive-definite quantities. Thus, for a solution in scalar-tensor gravity, $(\bar{g}, \bar{f})$, if the lapse $\bar{N}$ and the fields $(\bar{\sigma}_{ab}, \phi)$ induced on the boundary B match with the lapse $\bar{N}_0$ and the fields $(\bar{\sigma}_{0ab}, \phi_0)$ at B in the reference spacetime, then for the conformally related configuration (g, f) in the Einstein frame, the lapse N and the field σ_{ab} at B will necessarily match with their reference spacetime counterparts N_0 and σ_{0ab} induced on the corresponding boundary. This holds provided Ω is a monotonic function of ϕ. To repeat, let

$$\bar{N}|_B = \bar{N}_0|_B, \quad \bar{\sigma}_{ab}|_B = \bar{\sigma}_{0ab}|_B, \quad \phi|_B = \phi_0|_B. \quad (3.14)$$

Then, using the above conditions, we can infer the following requirements on the Einstein frame fields:

$$N|_B = [\bar{N}\Omega^{-1}(\phi)]_B = [\bar{N}_0\Omega^{-1}(\phi_0)]_B = N_0|_B,$$

$$\sigma_{ab}|_B = [\bar{\sigma}_{ab}\Omega^{-2}(\phi)]_B = [\bar{\sigma}_{0ab}\Omega^{-2}(\phi_0)]_B = \sigma_{0ab}|_B.$$
$$(3.15)$$

This proves that, for such a conformal factor, if the HH prescription is obeyed in a given frame, say, the scalar-tensor frame, it will automatically be satisfied in the Einstein frame. It is easy to extend this proof to the case of any two conformally related frames.

A meaningful referenced quasilocal mass can now be defined. It is simply given by Eq. (2.20), where the reference term, $\bar{M}_0$, is obtained from the HH prescribed reference action (3.13) by a BY-type analysis (as described in Sec. II B), i.e.,

$$\bar{M}_0 = \frac{1}{\kappa} \int_B d^{(D-1)}x \sqrt{\bar{\sigma}_0} \bar{N}_0 [U(\phi_0)\bar{k}_0 - \bar{n}_0^i \partial_i U(\phi_0)], \qquad (3.16)$$

which is just the first term on the right-hand side of Eq. (2.20) as evaluated on the reference solution.

We now show that the referenced quasilocal mass so obtained is conformally invariant. In the previous section, we proved that the unreferenced quasilocal mass is a conformal invariant. What remains to be verified is that the reference term $\bar{M}_0$ is also invariant under the transformation $\bar{g}_{0ab} = \Omega(\phi_0)g_{0ab}$. This is easily done by applying the curvature-transformation identity (2.28) to the above expression for $\bar{M}_0$. This shows that $\bar{M}_0 = M_0$, which proves the conformal invariance of the reference term. Hence the referenced quasilocal mass is conformally invariant. (This, of course, presumes a monotonic Ω.)

We now illustrate this invariance explicitly for the case of SSS spacetimes. By applying the mass expressions (2.20) and (3.16) to the SSS metric (3.4), we obtain

$$\bar{M}(r) = \left[\frac{\bar{N}(r)}{\kappa} \bar{\lambda}(r) \frac{d}{dr} [\bar{A}_{D-1}(r)U(\phi)] \right]_{\text{cl}}^0, \quad (3.17)$$

where $\bar{A}$ is given in Eq. (3.9) and $[\text{term}]_{\text{cl}}^0$ is defined as the difference in the values of the term evaluated on the reference spacetime and on the spacetime solution whose mass we aim to compute. Note that in keeping with the HH prescription, we require that at the boundary, $r = r_B$, the SSS solution satisfy $\bar{N}(r_B) = \bar{N}_0(r_B)$, $\bar{\sigma}_{ab}(r_B) = \bar{\sigma}_{ab}(r_B)|_0$, and $U(\phi(r_B)) = U(\phi_0(r_B))$. To obtain the total mass of an asymptotically flat spacetime, one first evaluates $\bar{M}(r)$ for general r and then imposes the limit $r \to \infty$. In this limit, Eq. (3.17) yields the ADM mass when the reference solution is chosen to be flat. The referenced quasilocal mass defined in Eq. (3.17) is manifestly invariant under the conformal transformation (3.5) and, therefore, is the same as the expression obtained upon removing the overbars in that equation.

Alternatively, consider applying the mass expressions (2.20) and (3.16) to an SSS metric of the form (3.7), namely,

$$d\bar{s}^2 = -\bar{N}^2(r)dt^2 + \frac{dr^2}{\bar{\lambda}^2(r)} + \Omega^2 r^2 d\omega^2. \quad (3.18)$$

It is easy to verify that the resulting quasilocal mass expression is identical to Eq. (3.17). However, the area of B [as embedded in metrics of the type (3.18)] is now given as

$$\bar{A}_n = \int_B d^n x \sqrt{\bar{\sigma}} = \frac{(4\pi)^{n/2}\Gamma(n/2)}{\Gamma(n)} r^n \Omega^n. \quad (3.19)$$

Here too the referenced quasilocal mass (3.17) remains invariant under conformal transformations of the metric (3.18), provided the HH prescription is followed in determining the reference solution.

To summarize, we define the referenced quasilocal mass of a solution associated with a boundary **B** as the difference of its unreferenced quasilocal mass from that of a reference field configuration, which is also a solution of the theory and obeys the HH prescription. Under a conformal transformation, this pair of solutions has its "image" pair, which is comprised of two solutions in the conformally related frame; in that frame, the referenced quasilocal mass is again the difference of the unreferenced quasilocal masses of these two image solutions. To investigate the behavior of the referenced quasilocal mass under a conformal transformation, one must therefore study how the unreferenced quasilocal masses of these two solutions transform under the conformal map $\bar{g}_{ab} = \Omega(\phi) g_{ab}$ and $\bar{g}_{0ab} = \Omega(\phi_0) g_{0ab}$, *respectively*. Such a study reveals the conformal invariance of our referenced quasilocal mass (3.17).

We end this section by noting that, when applied to the case of asymptotically flat SSS spacetimes, there is a subtle but significant difference between the quasilocal mass definition (3.17), which we propose above, and the mass definition that CCM obtain by their generalization of the HH prescription [20], namely,

$$\bar{M}(r) = \left[\frac{\bar{N}(r)}{\kappa} [1 - \bar{\lambda}(r)] \frac{d}{dr} [\bar{A}_{D-1}(r) U(\phi)] \right]_{\text{cl}}. \tag{3.20}$$

Specifically, consider the case of SSS metrics of the form (3.4). Then, the above formula can be obtained from Eq. (3.17) in two steps. First, one sets $\bar{\lambda}(r)|_0 = 1$ in Eq. (3.17). This can always be done, for the reference spacetime solution in such a case is flat. Second, and more importantly, one *assumes* that

$$\frac{dU(\phi_{\text{cl}})}{dr} = \frac{dU(\phi_0)}{dr}, \tag{3.21}$$

at the boundary **B**. This, however, is an additional requirement over and above those included in the HH prescription. Consequently, Eq. (3.20) is different from Eq. (3.17), where condition (3.21) is not assumed. This is also the reason why Eq. (3.20), as opposed to Eq. (3.17), fails to be conformally invariant.

In the next section we apply our mass definition (3.17) to find the referenced quasilocal masses of charged black holes in 4D dilaton gravity and their conformally related cousins in 4D Einstein gravity.

IV. QUASILOCAL MASS IN SCALAR-TENSOR THEORIES OF GRAVITY: EXAMPLES

A. Asymptotically flat SSS spacetimes

Let us consider the charged black hole solutions of the four-dimensional dilaton gravity action (see Refs. [13,14] for reviews)

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-\bar{g}} e^{-2\phi} [\bar{R} + 4(\bar{\nabla}\phi)^2 - 2\Lambda - \bar{F}^2], \tag{4.1}$$

where $\bar{R}$ is the four-dimensional Ricci scalar, Λ is a cosmological constant and $\bar{F}_{\mu\nu}$ is the Maxwell field associated with a U(1) subgroup of $E_8 \times E_8$. In this subsection we will consider the case where $\Lambda = 0$. The magnetically charged black hole solution to the above action is [26,27]

$$d\bar{s}^2 = -\frac{e^{2\phi_\infty}(1 - 2me^{\phi_\infty}/r)}{1 - Q^2 e^{-\phi_\infty}/(mr)} dt^2 + \frac{dr^2}{(1 - 2me^{\phi_\infty}/r)[1 - Q^2 e^{-\phi_\infty}/(mr)]} + r^2 d\omega^2, \tag{4.2}$$

$$e^{-2\phi} = e^{-2\phi_\infty}\left(1 - \frac{Q^2 e^{-\phi_\infty}}{mr}\right) = U(\phi), \tag{4.3}$$

$$\bar{F} = Q \sin \theta d\theta \wedge d\phi, \tag{4.4}$$

where m and Q are classical hairs of the stringy black hole and ϕ_∞ is the asymptotic constant value of the dilaton. Above, m is also called the Schwarzschild mass of the spacetime and Q is the magnetic charge of the black hole. The strings couple to the above metric, $\bar{g}_{\mu\nu}$, as opposed to the one related through the conformal transformation $g_{\mu\nu} \equiv e^{-2\phi}\bar{g}_{\mu\nu}$, which casts the above action in the Hilbert form.

We will now demonstrate that the quasilocal mass of a spatial region enclosed inside the two-sphere of curvature radius r_B is conformally invariant. We first calculate the mass in the string frame. Since the spacetime (4.2) is asymptotically flat, we choose the reference metric to be flat:[3]

$$d\bar{s}_0^2 = -\bar{N}_0^2 dt^2 + dr^2 + r^2 d\omega^2, \tag{4.5}$$

where $\bar{N}_0$ is a constant. Note that the above metric is a solution of the action (4.1) with $\phi_0 = \text{const}$ and $\bar{F}_0 = 0$. A two-sphere boundary of curvature radius $r = r_B$ can be isometrically embedded in both the above spacetimes (4.2) and (4.5). For the lapse at the boundary to match in these spacetimes, we choose

$$\bar{N}_0 = e^{\phi_\infty}\left(1 - \frac{2me^{\phi_\infty}}{r_B}\right)^{1/2}\left(1 - \frac{Q^2 e^{-\phi_\infty}}{mr_B}\right)^{-1/2}. \tag{4.6}$$

For the remaining HH requirement to be satisfied, the value of ϕ induced at the boundary in these spacetimes should match. This implies that on the reference spacetime (4.5), one must have

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[3]See the discussion in Sec. V.

$$e^{-2\phi_0} = e^{-2\phi_\infty}\left(1 - \frac{Q^2 e^{-\phi_\infty}}{mr_B}\right) = U(\phi_0) \qquad (4.7)$$

everywhere. Using these expressions in Eq. (3.17), we find that the quasilocal mass is

$$\bar{M}(r_B) = e^{-\phi_\infty} r_B\left[-\left(1 - \frac{Q^2 e^{-\phi_\infty}}{mr_B}\right)\left(1 - \frac{2me^{\phi_\infty}}{r_B}\right)\right.$$
$$-\frac{e^{-\phi_\infty}Q^2}{2mr_B}\left(1 - \frac{2me^{\phi_\infty}}{r_B}\right)$$
$$\left.+ \sqrt{\left(1 - \frac{Q^2 e^{-\phi_\infty}}{mr_B}\right)\left(1 - \frac{2me^{\phi_\infty}}{r_B}\right)}\right]. \qquad (4.8)$$

In the limit $r_B \to \infty$, $\bar{M}(r_B) \to m$.

We next study the Einstein-frame solution that is conformally related to Eq. (4.2) through the conformal transformation (2.2), where

$$U = e^{-2\phi_\infty}\left(1 - \frac{Q^2 e^{-\phi_\infty}}{mr}\right). \qquad (4.9)$$

Thus, the Einstein metric is

$$ds^2 = -\left(1 - \frac{2me^{\phi_\infty}}{r}\right)dt^2 + e^{-2\phi_\infty}\left(1 - \frac{2me^{\phi_\infty}}{r}\right)^{-1}dr^2$$
$$+ e^{-2\phi_\infty}r^2\left(1 - \frac{Q^2 e^{-\phi_\infty}}{mr}\right)d\omega^2. \qquad (4.10)$$

Once again, since the above spacetime is asymptotically flat, we choose the reference metric to be flat:

$$ds_0^2 = -N_0^2 dt^2 + d\rho^2 + \rho^2 d\omega^2, \qquad (4.11)$$

where ρ is the radial coordinate and $N_0 = \text{const}$. In the above coordinates, a two-sphere (with t and ρ constant) embedded in this reference spacetime is not isometric with a two-sphere (with t and r constant) embedded in spacetime (4.10). However, they can be made isometric by defining ρ in terms of the curvature coordinate r as

$$\rho = r\left(1 - \frac{Q^2 e^{-\phi_\infty}}{mr_B}\right)^{1/2} e^{-\phi_\infty}. \qquad (4.12)$$

One can implement this coordinate transformation in either Eq. (4.10) or (4.11). Both choices yield the same mass expressions. We choose to apply it in Eq. (4.11). In these coordinates, the flat metric gets recast to

$$ds_0^2 = -N_0^2 dt^2 + e^{-2\phi_\infty}\left(1 - \frac{Q^2 e^{-\phi_\infty}}{mr_B}\right)dr^2$$
$$+ r^2\left(1 - \frac{Q^2 e^{-\phi_\infty}}{mr_B}\right)e^{-2\phi_\infty}d\omega^2. \qquad (4.13)$$

For matching the lapse on the boundary at $r = r_B$, we require

$$N_0 = \left(1 - \frac{2me^{\phi_\infty}}{r_B}\right)^{1/2} \qquad (4.14)$$

everywhere. Note that the flat metric (4.13) is indeed conformally related to the reference solution (4.5) in the string frame. By the application of Eq. (3.17), we find that the quasilocal mass turns out to be that given in Eq. (4.8). This shows that the quasilocal mass at any r is a conformal invariant.

We now consider electrically charged black hole solutions of the action (4.1). The associated metric, dilaton, and the non-vanishing Maxwell field tensor components are

$$d\bar{s}^2 = -\frac{e^{-\phi_\infty}[1 + (Q_e^2 - 2m^2 e^{2\phi_\infty})/(me^{\phi_\infty}r)]}{[1 + Q^2/(me^{\phi_\infty}r)]^2}dt^2$$
$$+ \frac{dr^2}{1 + (Q_e^2 - 2m^2 e^{2\phi_\infty})/(me^{\phi_\infty}r)} + r^2 d\omega^2, \qquad (4.15)$$

$$U(\phi) = e^{-2\phi} = \left(1 + \frac{Q_e^2 e^{-\phi_\infty}}{mr}\right), \qquad (4.16)$$

$$\bar{F}_{tr} = \frac{Q_e e^{4\phi}}{r^2}. \qquad (4.17)$$

Since this spacetime is asymptotically flat, we choose the reference solution to be flat with the metric (4.5), where the lapse is just $\sqrt{-\bar{g}_{tt}}$ in Eq. (4.15) evaluated at $r = r_B$.

By applying our prescription for finding the quasilocal mass (as we did in the case of the magnetically charged black holes) to this case, we find that, in the string frame,

$$\bar{M}(r_B) = e^{-\phi_\infty} r_B\left\{\bar{\lambda} - \bar{\lambda}^2 + \frac{Q_e^2 \bar{\lambda}^2}{2me^{\phi_\infty}r_B}\left(1 + \frac{Q_e^2}{me^{\phi_\infty}r_B}\right)^{-1}\right\}, \qquad (4.18)$$

where $\bar{\lambda}^{-2} \equiv \bar{g}_{rr}(r_B)$ in Eq. (4.15). Thus, the total mass of the spacetime is once again m.

On the other hand, in the Einstein frame the metric is

$$ds^2 = -e^{-\phi_\infty}\left(1 - \frac{2m^2 e^{2\phi_\infty}}{me^{\phi_\infty}r + Q_e^2}\right)dt^2$$
$$+ \left(1 - \frac{2m^2 e^{2\phi_\infty}}{me^{\phi_\infty}r + Q_e^2}\right)^{-1}dr^2$$
$$+ r^2\left(1 + \frac{Q_e^2}{me^{\phi_\infty}r}\right)d\omega^2, \qquad (4.19)$$

which is related to the string metric (4.15) via the conformal transformation (2.2), where U is given by Eq. (4.16). Since the above solution is asymptotically flat, the reference metric is chosen to be flat once again:

$$ds_0^2 = -N_0^2 dt^2 + \left(1 + \frac{Q_e^2}{me^{\phi_\infty}r_B}\right)dr^2 + r_B^2\left(1 + \frac{Q_e^2}{me^{\phi_\infty}r_B}\right)d\omega^2, \qquad (4.20)$$

where, as in Eq. (4.13), we use coordinates such that the 2-sphere boundary at $r=r_B$ is manifestly isometric with that in the spacetime (4.19). Also, ϕ_0 is defined so as to match with the solution (4.16) at the boundary 2-sphere at $r=r_B$:

$$e^{-2\phi_0}=U(\phi_0)|_{r=r_B}=1+Q_e^2/(me^{\phi_\infty}r_B). \quad (4.21)$$

This is then the value of ϕ_0 everywhere in the reference spacetime. Similarly the reference lapse N_0 is chosen to be $\sqrt{-g_{tt}}$ in Eq. (4.19) evaluated at $r=r_B$. Our prescription for evaluating the quasilocal mass then yields the same expression as found in Eq. (4.18) for the string frame, thus demonstrating its conformal invariance.

B. Asymptotically non-flat black holes

To demonstrate that our prescription yields a conformally invariant definition of quasilocal mass even for asymptotically non-flat solutions we consider a particular black hole solution of Chan, Horne, and Mann that arises from the following action [28]:

$$S=\frac{1}{2\kappa}\int d^4x\sqrt{-g}[R-2(\nabla\phi)^2-e^{-2\phi}F^2]. \quad (4.22)$$

The fields of the electrically charged black hole solution in this theory are

$$ds^2=-\frac{1}{\gamma^4}(r^2-4\gamma^2M)dt^2+\frac{4r^2}{r^2-4\gamma^2M}dr^2+r^2d\omega^2, \quad (4.23)$$

$$e^{-2\phi}=\frac{2Q^2}{r^2}, \quad (4.24)$$

$$F_{tr}=\frac{r}{2Q\gamma^2}, \quad (4.25)$$

where γ is a constant with dimensions of $\sqrt{r}$ and Q is the electric charge.

In this case, there is no unique way to choose the reference geometry. Here, we choose to compare the quasilocal mass of the above solution with respect to a geometry whose (non-flat) space part of the metric is determined by setting $M=0$ in Eq. (4.23). Thus, our reference geometry is

$$ds^2{}_0=-N_0^2dt^2+4dr^2+r^2d\omega^2. \quad (4.26)$$

For the 2-sphere boundary at $r=r_B$, the HH prescription dictates that

$$N_0=\frac{1}{\gamma^2}(r^2{}_B-4\gamma^2M)^{1/2} \quad (4.27)$$

be obeyed everywhere. Our prescription for the quasilocal mass then yields

$$M(r_B)=\frac{r_B^2}{2\gamma^2}\left[\sqrt{1-\frac{4\gamma^2M}{r^2{}_B}}-1+\frac{4\gamma^2M}{r^2{}_B}\right]. \quad (4.28)$$

As $r_B\rightarrow\infty$, $M(r_B)\rightarrow M$.

We next consider the string action conformally related to Eq. (4.22):

$$S=\frac{1}{2\kappa}\int d^4x\sqrt{-\bar{g}}e^{-2\phi}[\bar{R}+4(\bar{\nabla}\phi)^2-\bar{F}^2] \quad (4.29)$$

where the conformal factor is given by

$$\Omega=e^\phi=\frac{r}{\sqrt{2}Q}(=U^{-1/2}). \quad (4.30)$$

Therefore, the string metric conformally related to Eq. (4.23) is

$$d\bar{s}^2=-\frac{r^2}{2Q^2\gamma^4}(r^2-4\gamma^2M)dt^2+\frac{2r^2/Q^2}{1-4\gamma^2M/r^2}dr^2$$
$$+\frac{r^4}{2Q^2}d\omega^2. \quad (4.31)$$

The space part of the reference string metric is chosen by setting $M=0$ above, which gives the reference metric as

$$ds_0^2=-\bar{N}_0^2dt^2+\frac{2r^2}{Q^2}dr^2+\frac{r^4}{2Q^2}d\omega^2, \quad (4.32)$$

where the reference lapse is obtained by matching with that in Eq. (4.31) at the boundary $r=r_B$:

$$\bar{N}_0=\frac{r_B}{\sqrt{2}Q\gamma^2}(r^2{}_B-4\gamma^2M)^{1/2}. \quad (4.33)$$

With these prescribed choices for the reference fields, we find that the quasilocal mass is

$$\bar{M}(r_B)=\frac{r_B}{2\gamma^2}\left[1-\sqrt{\frac{1-4\gamma^2M}{r^2{}_B}}\right](r^2{}_B-4\gamma^2M)^{1/2}, \quad (4.34)$$

which is the same as that evaluated in the Einstein frame, namely, Eq. (4.28).

V. DISCUSSION

Naive expectations from classical field theory would suggest that physical quantities should remain invariant under a conformal transformation. However, when it comes to the behavior of the quasilocal mass under such a transformation, one must be cautious. This is because *a priori* it cannot be ruled out that in some frames the scalar field ϕ, which defines the conformal factor, itself contributes to the energy-momentum of the spacetime. In this paper we showed that, the preceding caveat notwithstanding, the unreferenced BY quasilocal mass is indeed conformally invariant.

However, to obtain the physical mass of a spacetime one is often required to subtract a reference term. At the level of the action, this is achieved by subtracting a reference action. Different choices of reference action will lead to different physical masses for the same classical solution. Moreover,

the reference term $\bar{M}_0$ arising from such actions and, consequently, the referenced quasilocal mass may not be conformally invariant.

In this paper, we attempted to reduce the arbitrariness in the choice of a reference action. We were motivated in this choice by a basic principle, in the form of the Hawking-Horowitz prescription, which requires the reference geometry to obey certain conditions. We proved that this prescription automatically gives rise to a conformally invariant referenced quasilocal mass if the conformal factor is monotonic in the scalar field.

We note, however, that the HH prescription does not attempt to specify a unique reference geometry, owing to which the referenced quasilocal mass is non-unique, albeit conformally invariant. It is only in some special cases that one can obtain a unique physical mass. Asymptotically flat spacetime solutions of general relativity belong to this category. There the positive energy theorem and the stability criterion for Minkowski spacetime ensure that under certain positivity conditions on the energy-momentum tensor, the total energy of such spacetimes is positive; it is zero only for the Minkowski spacetime. This selects the flat spacetime as a very special reference geometry for calculating the total energy and, in certain cases, the quasilocal mass and energy of regions in such spacetimes. The conformal invariance of the quasilocal mass implies that in conformally related spacetimes, which are asymptotically flat, the flat spacetime continues to be a special reference geometry.

In this vein, one may argue that if the positive energy theorem could be shown to hold for asymptotically non-flat cases, at least of a limited type such as the SSS spacetimes, then a corresponding special reference geometry may emerge, which could be used under the HH prescription to compute the referenced quasilocal mass in such spacetimes in some unique way. This and other related issues are currently under study [29].

ACKNOWLEDGMENTS

APPENDIX: THE STRESS-ENERGY PSEUDOTENSOR

In this section we present an alternative derivation of the quasilocal energy expression (2.19) using the pseudotensor method [5,2,1]. The outline of our approach, as applied to the scalar-tensor type theories, Eq. (2.1), is as follows. As is the case for the Brans-Dicke theory, even in our generalized non-minimally coupled theory, we make use of the Bianchi identity and the equations of motion to show that the covariant divergence of the matter stress-tensor vanishes. In a particular coordinate system, this is then shown to imply that the ordinary divergence of the sum of two quantities vanishes. One of the terms in this sum is the matter stress-tensor

in that coordinate frame, and the other term is then interpreted as the stress-energy contribution of the geometry (that is, of the gravitational as well as the Brans-Dicke type scalar field). Naturally, any statement made about the stress-energy content of the geometry using this approach will be frame-dependent. However, for spherically symmetric cases, a meaningful proper energy contained in the sphere S^D, with its origin at the center of spherical symmetry, can be defined using this method [1]. It is in this context that we present this alternative derivation of the quasilocal energy expression, Eq. (2.19).

We begin by finding the equations of motion for the theory described by Eq. (2.1) for the special case where $W(\phi) = X(\phi)$ and $U(\phi)W(\phi) = 1$. (Our results can be generalized in a straightforward manner to other cases not constrained by these conditions.) Requiring the action to be stationary under variations of the metric tensor and the field ϕ gives the equations of motion

$$
U(\phi)\left[\bar{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\bar{R}\right]
$$

$$
= -\frac{1}{2}[\bar{T}_m^{\mu\nu} + T_\phi^{\mu\nu} + 2U(\phi)^{;\mu;\nu} - 2\bar{g}^{\mu\nu}U(\phi)]_{;\lambda}^{;\lambda}],
$$

$$
\tag{A1}
$$

$$
\bar{g}^{\mu\nu}\phi_{;\mu;\nu} - 2\frac{\delta(UV)}{\delta\phi} - \bar{R}\frac{\delta U}{\delta\phi} = 0, \tag{A2}
$$

where we have used the notation $A_{,\mu} \equiv \partial A / \partial x^\mu$ and $A_{;\mu} \equiv \bar{\nabla}_\mu A$. Here, $\bar{T}_m^{\mu\nu}$ is the energy-momentum tensor of matter obtained by varying $\bar{L}_m$ with respect to $\bar{g}_{\mu\nu}$, and

$$
T_\phi^{\mu\nu} \equiv -2\partial^\mu\phi\partial^\nu\phi - \bar{g}^{\mu\nu}[-\partial^\lambda\phi\partial_\lambda\phi - U(\phi)V(\phi)]. \tag{A3}
$$

We have considered, for the present, a ϕ-independent $\bar{L}_m$. Since our discussion here is limited to the ''Brans-Dicke'' frame only, we shall henceforth drop the overbar.

It would be reasonable to demand that this theory conform to the equivalence principle. To demonstrate that this indeed holds i.e. $T_{m;\nu}^{\mu\nu} = 0$, we first note that the contracted Bianchi identity satisfied by the Einstein tensor gives

$$
[U(\phi)^{-1}(T_m^{\mu\nu} + t^{\mu\nu})]_{;\nu} = 0 \tag{A4}
$$

with

$$
t^{\mu\nu} \equiv T_\phi^{\mu\nu} + 2U(\phi)^{;\mu;\nu} - 2g^{\mu\nu}U(\phi)_{;\lambda}^{;\lambda}. \tag{A5}
$$

From the equation of motion (A1) and the Bianchi identity it follows that

$$
U(\phi)_{,\nu}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right] = -\frac{1}{2}[T_{m;\nu}^{\mu\nu} + t_{;\nu}^{\mu\nu}]. \tag{A6}
$$

The definition of the Riemann tensor yields the identity

$$
R_{\rho\alpha}U(\phi)^{;\rho} = U(\phi)_{;\lambda;\alpha}^{;\lambda} - U(\phi)_{;\alpha;\lambda}^{;\lambda}. \tag{A7}
$$

Taking the covariant derivative of Eq. (A3) and using the equation of motion (A2) for the scalar field ϕ, we obtain

$$T_{\phi;\nu}^{\mu\nu} = U(\phi)^{;\mu}R. \tag{A8}$$

Taking the covariant derivative of Eq. (A5) and using Eqs. (A7) and (A8) then gives the desirable vanishing covariant divergence of the matter stress energy tensor.

Our task now is to find the expression for a conserved stress-energy pseudotensor for the geometry. To achieve this we proceed by expressing the vanishing of the covariant divergence of the matter stress-energy tensor as

$$[\sqrt{-g}\,T_{m\mu}^{\nu}]_{,\nu} - \frac{1}{2}g_{\tau\beta,\mu}\sqrt{-g}\,T_m^{\tau\beta} = 0. \tag{A9}$$

To cast the left-hand side of the above equation into a total ordinary divergence one has to seek a representation of the second quantity in terms of a total ordinary divergence. This can be done as follows. First, we make use of the equation of motion (A6) to express the matter stress-energy tensor in terms of purely geometrical quantities, namely, the scalar field and metric-dependent quantities

$$T_m^{\tau\beta} = -t^{\tau\beta} - 2U(\phi)G^{\tau\beta}, \tag{A10}$$

where $G^{\tau\beta}$ is the Einstein tensor. (For simplicity, we have above dropped a possible contribution from a divergenceless term.) Second, note that the right-hand side of this expression is merely the functional derivative of

$$J \equiv 2\int d^{(D+1)}x\sqrt{-g}[U(\phi)R+L_{\phi}] \tag{A11}$$

under variations of the metric tensor, with boundary conditions that require the vanishing of the metric and its first derivatives on the boundary of a $(D+1)$-dimensional manifold over which this integral has been taken. For more general variations, one would get contributions from the surface integrals as well. Here, the variation of L_{ϕ} with respect to the metric tensor is taken to yield $T_{\phi}^{\mu\nu}$. We consider the standard decomposition of $\sqrt{-g}R$ into a pure divergence term and a simple expression involving only the metric and its first derivatives:

$$\sqrt{-g}R = \mathcal{U} + [\sqrt{-g}\,g^{\sigma\rho}\Gamma_{\sigma\alpha}^{\alpha}]_{,\rho} - [\sqrt{-g}\,g^{\sigma\rho}\Gamma_{\sigma\rho}^{\alpha}]_{,\alpha} \tag{A12}$$

with

$$\mathcal{U} \equiv \sqrt{-g}\,g^{\sigma\rho}[\Gamma_{\sigma\rho}^{\alpha}\Gamma_{\alpha\beta}^{\beta} - \Gamma_{\beta\rho}^{\alpha}\Gamma_{\alpha\sigma}^{\beta}]. \tag{A13}$$

It follows that the functional derivative of J with respect to the metric tensor is the same as that of

$$H \equiv \int d^{(D+1)}x[\mathcal{V} + \sqrt{-g}L_{\phi}], \tag{A14}$$

where

$$\mathcal{V} \equiv [U\mathcal{U} - \sqrt{-g}\,g^{\sigma\rho}\Gamma_{\sigma\alpha}^{\alpha}U_{,\rho} + \sqrt{-g}\,g^{\sigma\rho}\Gamma_{\sigma\rho}^{\alpha}U_{,\alpha}]. \tag{A15}$$

Comparing the variations of J and H with respect to the metric, we get

$$\sqrt{-g}\,UG_{\mu\nu} + \sqrt{-g}[U_{;\mu;\nu} - g_{\mu\nu}U_{;\alpha}^{;\alpha}] + \frac{1}{2}\sqrt{-g}\,T_{\phi\mu\nu}$$
$$= \frac{\partial(\mathcal{V}+\sqrt{-g}L_{\phi})}{\partial g^{\mu\nu}} - \left[\frac{\partial(\mathcal{V}+\sqrt{-g}L_{\phi})}{\partial g_{,\lambda}^{\mu\nu}}\right]_{,\lambda}, \tag{A16}$$

where we made use of Eqs. (A10) and (A5). Next, we define

$$\hat{\mathcal{V}} \equiv \mathcal{V} + \sqrt{-g}L_{\phi}. \tag{A17}$$

The expression for the ordinary derivative of $\hat{\mathcal{V}}$ and the field equation (A2) for ϕ enable us to express $g_{\tau\beta,\mu}[t^{\tau\beta} + 2U(\phi)G^{\tau\beta}]$ as a total divergence. Using Eq. (A10), we express Eq. (A9) as a vanishing total derivative

$$\left[\sqrt{-g}\,T_{m\mu}^{\nu} - \hat{\mathcal{V}}\delta_{\mu}^{\nu} - \frac{\partial\hat{\mathcal{V}}}{\partial g_{,\nu}^{\tau\beta}}g_{,\mu}^{\tau\beta} - \frac{\partial\hat{\mathcal{V}}}{\partial\phi_{,\nu}}\phi_{,\mu}\right]_{,\nu} = 0. \tag{A18}$$

For $\nu = 0$, the expression within the brackets integrated over a spacelike hypersurface is thus invariant under time translations for a distribution of matter with a compact support over the surface. This is the expression for the stress-energy pseudotensor that we seek. The quantity

$$P_{\mu} \equiv \frac{1}{2\kappa}\int_{\Sigma}d\Sigma\left[\sqrt{-g}\,T_{m\mu}^{0} - \hat{\mathcal{V}}\delta_{\mu}^{0} - \frac{\partial\hat{\mathcal{V}}}{\partial g_{,0}^{\tau\beta}}g_{,\mu}^{\tau\beta} - \frac{\partial\hat{\mathcal{V}}}{\partial\phi_{,0}}\phi_{,\mu}\right], \tag{A19}$$

evaluated on a constant-time spacelike hypersurface Σ, is thus conserved. This may be viewed as the generalization of the energy-momentum four-vector for a Brans-Dicke theory. As in general relativity, P_{μ} is not a generally covariant four-vector since $\mathcal{U}$ and $\mathcal{V}$ are not scalar densities. The intrinsic non-covariance of the energy-momentum density of the gravitational field has its origin in the intimate connection between geometry and the gravitational field. Had the expression been covariant, one could always have gone into a preferred (freely falling) frame to ensure vanishing of an arbitrary localized gravitational field.

The above form for the energy-momentum pseudotensor for the generalized Brans-Dicke theory can also be obtained by considering a variation of the coordinate system instead of the metric field. The analysis enables us to express the gravitational stress-energy pseudotensor in a very compact form, which is identical to the expression derived in the quasilocal formalism. To demonstrate this, we consider

$$H = \int \hat{\mathcal{V}}d^{(D+1)}x, \tag{A20}$$

where $\hat{\mathcal{V}}$ is a function of the metric, the scalar field ϕ, and their first derivatives. Its variation is

$$\delta\hat{\mathcal{V}}=\frac{\partial\hat{\mathcal{V}}}{\partial g^{\mu\nu}}\delta g^{\mu\nu}+\frac{\partial\hat{\mathcal{V}}}{\partial g_{\lambda}^{\mu\nu}}\delta g_{\lambda}^{\mu\nu}+\frac{\partial\hat{\mathcal{V}}}{\partial\phi}\delta\phi+\frac{\partial\hat{\mathcal{V}}}{\partial\phi_{,\lambda}}\delta\phi_{,\lambda}.$$
$$(A21)$$

Consider an infinitesimal change of coordinates of the form

$$\hat{x}^{\alpha}=x^{\alpha}+\epsilon\xi^{\alpha}. \qquad (A22)$$

Retaining terms up to the first order in ϵ, we get the following variations:

$$\frac{\partial x^{\alpha}}{\partial\hat{x}^{\lambda}}=\delta_{\lambda}^{\alpha}-\epsilon\frac{\partial\xi^{\alpha}}{\partial x^{\lambda}}+O(\epsilon^{2}), \qquad (A23)$$

$$\delta g^{\mu\nu}=\epsilon(\xi_{,\alpha}^{\mu}g^{\alpha\nu}+\xi_{,\alpha}^{\nu}g^{\alpha\mu}), \qquad (A24)$$

$$\delta g_{,\lambda}^{\mu\nu}=\epsilon(g_{,\lambda}^{\tau\nu}\xi_{,\tau}^{\mu}+g_{,\lambda}^{\mu\beta}\xi_{,\beta}^{\nu}-g_{,\alpha}^{\mu\nu}\xi_{,\lambda}^{\alpha}+g^{\tau\nu}\xi_{,\tau,\lambda}^{\mu}+g^{\tau\mu}\xi_{,\tau,\lambda}^{\nu}),$$
$$(A25)$$

$$\delta\sqrt{-g}=-\epsilon\sqrt{-g}\xi_{\alpha}^{\alpha}, \qquad (A26)$$

$$\delta\phi=0, \qquad (A27)$$

$$\delta(\phi_{,\lambda})=-\epsilon\phi_{,\alpha}\xi_{,\lambda}^{\alpha}. \qquad (A28)$$

A restriction to linear transformations enables one to get an elegant form for $\delta\hat{\mathcal{V}}$. The Christoffel symbols transform as tensors under such transformations and hence $\hat{\mathcal{V}}$ transforms as a scalar density. Thus

$$\delta\hat{\mathcal{V}}=\frac{\hat{\mathcal{V}}}{\sqrt{-g}}\delta\sqrt{-g}=-\epsilon\xi_{,\alpha}^{\alpha}\hat{\mathcal{V}}. \qquad (A29)$$

Substituting the variations (A24)–(A28) for an arbitrary linear coordinate transformation into Eq. (A21), and comparing the expression with Eq. (A29), we obtain the identity

$$\frac{\partial\hat{\mathcal{V}}}{\partial g^{\mu\nu}}g^{\alpha\nu}+\frac{\partial\hat{\mathcal{V}}}{\partial g_{,\lambda}^{\mu\nu}}g_{,\lambda}^{\alpha\nu}-\frac{1}{2}\frac{\partial\hat{\mathcal{V}}}{\partial g_{,\alpha}^{\beta\nu}}g_{,\mu}^{\beta\nu}-\frac{\partial\hat{\mathcal{V}}}{\partial\phi_{,\alpha}}\phi_{,\mu}=-\frac{1}{2}\hat{\mathcal{V}}g_{\mu}^{\alpha}.$$
$$(A30)$$

Although the above identity was derived for variations under linear coordinate transformations, one can verify that it holds quite generally [2]. The use of this identity yields a simple expression for the variation of $\hat{\mathcal{V}}$ under the general transformation (A22):

$$\delta\hat{\mathcal{V}}=-\epsilon\hat{\mathcal{V}}\xi_{,\alpha}^{\alpha}+2\epsilon\frac{\partial\hat{\mathcal{V}}}{\partial g_{,\lambda}^{\mu\nu}}\xi_{,\tau,\lambda}^{\mu}g^{\tau\nu}. \qquad (A31)$$

Under conditions where ξ and its derivatives are taken to vanish on the boundary, the variation of the metric tensor and its derivatives also vanish there. Under such boundary conditions, H has a vanishing variation, i.e.,

$$\delta H=\int_{\Sigma}\delta\left(\frac{\hat{\mathcal{V}}}{\sqrt{-g}}\right)\sqrt{-g}d^{(D+1)}x=0, \qquad (A32)$$

which, using the above identities, reduces to

$$\delta H=2\epsilon\int_{\Sigma}\frac{\partial\hat{\mathcal{V}}}{\partial g_{,\lambda}^{\mu\nu}}\xi_{,\tau,\lambda}^{\mu}g^{\tau\nu}d^{(D+1)}x=0. \qquad (A33)$$

This expression may be integrated by parts twice. Since δH vanishes for arbitrary ξ^{μ}, we obtain the following divergence law:

$$\left(\frac{\partial\hat{\mathcal{V}}}{\partial g_{,\lambda}^{\mu\nu}}g^{\tau\nu}\right)_{,\tau,\lambda}=0. \qquad (A34)$$

Thus

$$\sqrt{-g}F_{\mu}^{\tau}\equiv\left(\frac{\partial\hat{\mathcal{V}}}{\partial g_{,\lambda}^{\mu\nu}}g^{\tau\nu}\right)_{,\lambda} \qquad (A35)$$

defines a conserved quantity. Using the identity (A30) and the field equation (A1) gives

$$\sqrt{-g}F_{\mu}^{\tau}=-\sqrt{-g}T_{\mu}^{(m)\tau}-\frac{1}{2}\hat{\mathcal{V}}g_{\mu}^{\tau}+\frac{1}{2}\left(\frac{\partial\hat{\mathcal{V}}}{\partial g_{,\tau}^{\beta\nu}}g^{\beta\nu}\right)_{,\mu}$$
$$-\frac{1}{2}\frac{\partial\hat{\mathcal{V}}}{\partial\phi_{,\tau}}\phi_{,\mu}, \qquad (A36)$$

which is just the expression that we had obtained for the stress energy pseudotensor by the variation of the metric tensor earlier. The expression (A34) for a vanishing ordinary divergence implies that

$$P_{\mu}\equiv-\frac{1}{\kappa}\int_{V}\left(\frac{\partial\hat{\mathcal{V}}}{\partial g_{,\lambda}^{\mu\nu}}g^{0\nu}\right)_{,\lambda}dV \qquad (A37)$$

is a conserved quantity if V is the entire space at a given time. In the special case of a time independent metric, Gauss's theorem in D dimensions gives the energy momentum as surface integral over a $(D-1)$-dimensional surface:

$$P_{\mu}=-\frac{1}{\kappa}\int_{\Sigma}\left(\frac{\partial\hat{\mathcal{V}}}{\partial g_{,j}^{\mu\nu}}g^{0\nu}\right)d\Sigma_{j}. \qquad (A38)$$

This gives the interesting result that in the generalized Brans-Dicke theory, the generalized energy-momentum in a D-dimensional volume can be determined by the metric-tensor and its derivatives on the $(D-1)$-dimensional surface, the details of the field inside the volume being irrelevant.

We conclude this appendix by noting that in the curvature coordinates, the above expression (A38) evaluated for the SSS metric (3.4), tallies with the quasilocal energy expression (2.19) (with $\bar{E}_{0}$ set equal to zero). This can be seen from the definition of $\mathcal{V}$ given by Eqs. (A17) and (A15): The term $U\bar{k}$ in Eq. (2.19) yields the first term in the definition (A15) of $\mathcal{V}$, whereas the term $\bar{n}^{i}\partial_{i}U$ in Eq. (2.19) yields the second and third terms of $\mathcal{V}$.

[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[2] R. Adler, M. Bazin, and M. Schiffer, *An Introduction to General Relativity* (McGraw-Hill, Tokyo, 1975).

[3] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1407 (1993).

[4] L. P. Grishchuk, A. N. Petrov, and A. D. Popova, Commun. Math. Phys. **94**, 379 (1984); A. D. Popova and A. N. Petrov, Int. J. Mod. Phys. A **3**, 2651 (1988); A. N. Petrov, Class. Quantum Grav. **10**, 2663 (1993); A. N. Petrov and J. V. Narlikar, Found. Phys. **26**, 1201 (1996).

[5] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975).

[6] E. Witten, Commun. Math. Phys. **80**, 381 (1981).

[7] J. M. Nester, in *Asymptotic Behavior of Mass and Spacetime Geometry*, edited by F. Flaherty (Springer, Berlin, 1984).

[8] R. Penrose, Proc. R. Soc. London **A381**, 53 (1982).

[9] A. J. Dougan and L. J. Mason, Phys. Rev. Lett. **67**, 2119 (1991).

[10] L. J. Mason, Class. Quantum Grav. **6**, L7-13 (1989).

[11] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

[12] J. D. Brown, J. D. E. Creighton, and R. B. Mann, Phys. Rev. D **50**, 6394 (1994).

[13] G. T. Horowitz, in ''Proceedings of the 1992 Trieste Spring School on String Theory and Quantum Gravity,'' hep-th/9210119.

[14] J. Harvey and A. Strominger, in *Recent Directions in Particle Theory: From Superstrings and Black Holes to the Standard Model (TASI-92)*, Proceedings of the Theoretical Advanced Summer Institute, Boulder, Colorado, edited by J. Harvey and J. Polchinski (World Scientific, Singapore, 1993).

[15] G. Veneziano, in *String Gravity and Physics at the Planck Energy Scale*, edited by N. Sanchez and A. Zichichi (Kluwer Academic, Dordrecht, 1996).

[16] M. Gasperini, in *String Gravity and Physics at the Planck Energy Scale* [15].

[17] B. C. Xanthopoulos and T. Zannias, Phys. Rev. D **40**, 2564 (1989).

[18] R.-G. Cai and Y. S. Myung, Phys. Rev. D **56**, 3466 (1997).

[19] J. D. E. Creighton and R. B. Mann, Phys. Rev. D **52**, 4569 (1995).

[20] K. C. K. Chan, J. D. E. Creighton, and R. B. Mann, Phys. Rev. D **54**, 3892 (1996).

[21] S. W. Hawking and G. T. Horowitz, Class. Quantum Grav. **13**, 1487 (1996).

[22] K. C. K. Chan, ''Is mass conformally invariant?'', University of Waterloo Report No. WATPHY-TH-96-14, gr-qc/9701029.

[23] R. Capovilla, R. Cordero, and J. Guven, Mod. Phys. Lett. A **11**, 2755 (1996).

[24] T. J. Willmore, *Total Curvature in Riemannian Geometry* (Ellis Horwood Limited, 1982).

[25] L. Abbott and S. Deser, Nucl. Phys. **B195**, 76 (1982).

[26] G. Gibbons, Nucl. Phys. **B207**, 337 (1982); G. Gibbons and K. Maeda, *ibid.* **B298**, 741 (1988).

[27] D. Garfinkle, G. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991); **45**, 3888(E) (1992).

[28] K. C. K. Chan, J. H. Horne, and R. B. Mann, Nucl. Phys. **B447**, 441 (1995).

[29] S. Bose (work in progress).