Tolman wormholes violate the strong energy condition

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For an arbitrary Tolman wormhole, unconstrained by symmetry, we shall define the bounce in terms of a 3-dimensional edgeless achronal spacelike hypersurface of minimal volume (zero trace for the extrinsic curvature plus a ''flare-out'' condition). This enables us to severely constrain the geometry of spacetime at and near the bounce and to derive general theorems regarding violations of the energy conditions—theorems that do not involve geodesic averaging but nevertheless apply to situations much more general than the highly symmetric FRW-based subclass of Tolman wormholes. [For example, even under the mildest of hypotheses, the strong energy condition (SEC) must be violated.] Alternatively, one can dispense with the minimal volume condition and define a generic bounce entirely in terms of the motion of test particles (future-pointing timelike geodesics), by looking at the expansion of their timelike geodesic congruences. One re-confirms that the SEC must be violated at or near the bounce. In contrast, it is easy to arrange for *all* the other standard energy conditions to be satisfied. [S0556-2821(99)06802-2]

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I. INTRODUCTION

A so-called Tolman wormhole is formed if a collapsing universe somehow halts its contraction before encountering a big crunch singularity and then re-expands. Thus Tolman wormholes are prototypes for modeling the ''oscillating universe'' cosmologies that were in vogue in the 1930s [1,2]. In many cases, the precise nature of the ''bounce'' that was invoked to drive re-expansion was left unspecified (singular cusp? angular momentum barrier? analytic extension through the singularity?). In this article we shall explicitly assume that the ''bounce'' occurs at a moment when the geometry is non-singular and shall seek to extract as much generic information as possible about constraints that can then be placed on the bounce.

Specifically, we shall assume that the universe reaches a moment of minimum spatial volume, and call this minimum-volume edgeless achronal spacelike hypersurface the ''bounce.'' The Tolman wormhole will then be taken to be some suitable open region of spacetime surrounding this bounce. If we additionally assume rotational and translational symmetry, then the case of the corresponding bouncing Friedmann-Robertson-Walker (FRW) universe has already been considered in [3]. We shall use that Letter as guidance, but in this article wish to avoid unnecessary symmetry constraints, and so shall also seek guidance from recent analyses of generic traversable wormholes [4–6] and their throats [7–10].

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Tolman wormholes [3] and traversable wormholes [4–6] are rather different objects: the Tolman wormhole is intrinsically time dependent and involves a ''bounce'' for the entire universe; so the throat is a 3-dimensional spacelike hypersurface (timelike normal), whereas traversable wormholes are local objects [4–6] whose throats are $(2+1)$-dimensional timelike hypersurfaces (spacelike normals). Nevertheless, we shall see that many parts of the analysis can be naturally carried over from one case to the other.

The 1988 analysis of Morris and Thorne revitalized interest in *traversable* wormholes [4] when they were able to show that traversable wormholes were compatible with our current understanding of general relativity and semiclassical quantum gravity—but that there was a definite price to be paid—one had to admit violations of the null energy condition (NEC). More precisely, what Morris and Thorne showed was equivalent to the statement that for static spherically symmetric traversable wormholes there must be an open region surrounding the throat over which the NEC is violated [4–6]. For spherically symmetric homogeneous Tolman wormholes (bouncing FRW universes) the analogous statement is that there is an open temporal region surrounding the bounce on which the strong energy condition (SEC) must be violated [3]. (Traversable wormholes are cosmologically interesting in their own right [11,12], but we will not directly address that topic in this paper.)

To set up the analysis for a generic Tolman wormhole, we first have to define exactly what we mean by a such a wormhole—we find that there is a nice *geometrical* (not topological) characterization of the existence of, and location of, the ''bounce.'' This characterization is developed in terms of a hypersurface of minimal area, subject to a ''flare-out'' condition that generalizes that of [3]. With this definition in place, we can develop a number of theorems about SEC violations at or near the bounce. While SEC violations at or near the bounce are unavoidable, it is relatively easy to satisfy *all* the other standard energy conditions.

We develop a general analysis of energy condition violations in Tolman wormholes. (This analysis is based largely on [3,7–10]. For an analysis using similar techniques applied to static vacuum and electrovac black holes see Israel [13,14]. A related decomposition applied to the collapse problem is addressed in [15].) In view of the preceding discussion we want to get away from the notion that topology is the intrinsic defining feature of wormholes, either traversable or Tolman, and instead focus on the geometry of the wormhole throat and bounce. Our strategy is straightforward:

(1) Take any (3+1)-dimensional hypervolume, and look for a 3-dimensional edgeless achronal spacelike hypersurface of strictly minimal volume. Define such a surface, if it exists, to be the bounce of a Tolman wormhole. This generalizes the Morris-Thorne flare-out condition for static traversable wormholes to arbitrary Tolman wormholes.

(2) Use the Gauss-Codazzi and Gauss-Weingarten equations to decompose the (3+1)-dimensional spacetime curvature tensor in terms of the 3-dimensional curvature tensor of the bounce and the extrinsic curvature of the bounce as an embedded hypersurface in the (3+1)-dimensional geometry.

(3) Reassemble the pieces: Write the spacetime curvature in terms of the 3-curvature of the bounce and the extrinsic curvature of the bounce in (3+1) spacetime.

(4) Use the generalized flare-out condition to place constraints on the stress-energy tensor at and near the throat.

A somewhat different but complementary strategy which dispenses with the minimal volume condition in (1) is then presented which makes use instead of local properties of timelike geodesic congruences near the candidate bounce. For this we replace (1) by the following:

(1′) The bounce of a Tolman wormhole is a 3-dimensional spacelike hypersurface on which the expansion of a hypersurface orthogonal timelike geodesic congruence vanishes identically and for which the expansion is strictly positive to the immediate future of the bounce and strictly negative to the immediate past.

This latter characterization in terms of geodesic expansion is useful for when the volume of the hypersurface is ill-defined and is equivalent to the latter definition when the volume integral exists. *This version of the definition is also capable of dealing with situations where only a part of the universe is ''bouncing'' while the rest continues its collapse, or is already in its expanding phase.* One can deduce immediately the violation of the SEC in the neighborhood of the bounce without having to follow steps (2)–(4). However, the analysis implied by these additional steps is crucial for assessing the status of the other energy conditions [NEC, weak energy condition (WEC), dominant energy condition (DEC)] at and near the bounce.

II. DEFINITION OF A GENERIC BOUNCE

We define a bounce, Σ, to be an edgeless achronal 3-dimensional spacelike hypersurface of *minimal* volume. Compute the volume by taking

$$V(\Sigma)=\int \sqrt{{}^{(3)}g}\, d^3x. \tag{1}$$

Now use Gaussian normal coordinates, $x^i=(\tau;\vec{x}^i)$, wherein the hypersurface Σ is taken to lie at $\tau=0$, so that

$${}^{(3+1)}g_{\mu\nu}\, dx^\mu dx^\nu = -d\tau^2 + {}^{(3)}g_{ij}\, dx^i dx^j. \tag{2}$$

We do *not* demand that the manifold be globally of this form, but will remain satisfied with the knowledge that such a coordinate system exists and covers some open region surrounding the bounce. The variation in volume, obtained by pushing the hypersurface surface $\tau=0$ out to $\tau=\delta\tau(x)$, is given by the standard computation

$$\delta V(\Sigma)=\int \frac{\partial\sqrt{{}^{(3)}g}}{\partial\tau}\, \delta\tau(x) d^3x. \tag{3}$$

which implies

$$\delta V(\Sigma)=\int \sqrt{{}^{(3)}g}\frac{1}{2}\, g^{ij}\, \frac{\partial g_{ij}}{\partial\tau}\, \delta\tau(x) d^3x. \tag{4}$$

In Gaussian normal coordinates the extrinsic curvature is simply defined by

$$K_{ij}=-\frac{1}{2}\,\frac{\partial g_{ij}}{\partial\tau}. \tag{5}$$

[See [16], p. 552. In this section we use Misner-Thorne-Wheeler (MTW) sign conventions. The convention in [6], p. 156, is the opposite.] Thus

$$\delta V(\Sigma)=-\int \sqrt{{}^{(3)}g}\, \mathrm{tr}(K)\, \delta\tau(x) d^3x. \tag{6}$$

[We use the notation tr(X) to denote $g^{ij}X_{ij}$.] Since this is to vanish for arbitrary $\delta\tau(x)$, the condition that the area be *extremal* is simply $\mathrm{tr}(K)=0$. To force the volume to be *minimal* requires (at the very least) the additional constraint $\delta^2 V(\Sigma)\geqslant 0$. (We shall also consider higher-order constraints below.) But by explicit calculation

$$\delta^2 V(\Sigma)=-\int \sqrt{{}^{(3)}g}\left(\frac{\partial\ \mathrm{tr}(K)}{\partial\tau}-\mathrm{tr}(K)^2\right)\times\delta\tau(x)\,\delta\tau(x) d^3x. \tag{7}$$

Extremality $[\mathrm{tr}(K)=0]$ reduces this minimality constraint to

$$\delta^2 V(\Sigma)=-\int \sqrt{{}^{(3)}g}\left(\frac{\partial\ \mathrm{tr}(K)}{\partial\tau}\right)\delta\tau(x)\,\delta\tau(x) d^3x\geqslant 0. \tag{8}$$

Since this is to hold for arbitrary $\delta\tau(x)$, this implies that at the bounce we certainly require

$$\frac{\partial\ \mathrm{tr}(K)}{\partial\tau}\leqslant 0. \tag{9}$$

This is the simplest generalization of the ''flare-out'' condition for FRW-based Tolman wormholes to arbitrary Tolman wormholes [3]. This simple bounce condition can be rephrased as follows: We have as an identity that

$$\frac{\partial\ \mathrm{tr}(K)}{\partial \tau} = \mathrm{tr}\left(\frac{\partial K}{\partial \tau}\right) + 2\,\mathrm{tr}(K^2). \tag{10}$$

So minimality implies

$$\mathrm{tr}\left(\frac{\partial K}{\partial \tau}\right) + 2\,\mathrm{tr}(K^2) \leqslant 0. \tag{11}$$

We must now discuss some technical complications related to the fact that we eventually prefer to have a strong inequality ($<$) at or near the bounce, than to have a weak inequality ($\leqslant$) at the bounce itself. Similar technical complications arise when considering the Morris-Thorne static spherically symmetric wormhole [4] and the FRW-based Tolman wormholes of [3]. These technical issues are also the main stumbling block in setting up the analysis of generic traversable wormholes as carried out in [7–10]. Unfortunately the details are a little different for Tolman wormholes, and so we cannot simply copy the previous arguments.

To set the notation, let us consider some one-parameter set of deformations of the surface Σ specified by

$$\delta\tau(x) = \epsilon f(x). \tag{12}$$

This allows us to define a stratified collection of hypersurfaces Σ_ϵ by taking

$$\Sigma_\epsilon = \{\epsilon f(x), x^i\}. \tag{13}$$

We now ask that, for all $f(x)$, the volume of these sets of hypersurfaces $V[\Sigma_\epsilon]$ be a strict minimum at the bounce. This is equivalent to asserting that for every "direction" $f(x)$ timelike deformations of the bounce lead to strict increases in spatial volume. Now demanding that there be an open interval for which $V[\Sigma_\epsilon] > V[\Sigma_0]$ leads, by the fundamental theorem of calculus, to the existence of an open interval,

$$\exists \tilde{\epsilon} > 0 : \forall \epsilon \in (-\tilde{\epsilon}, 0) \cup (0, \tilde{\epsilon}), \quad \frac{d^2 V[\Sigma_\epsilon]}{d\epsilon^2} > 0. \tag{14}$$

This then implies, via Eq. (7),

$$\exists \tilde{\epsilon} > 0 : \forall \epsilon \in (-\tilde{\epsilon}, 0) \cup (0, \tilde{\epsilon}),$$

$$\int_{\Sigma_\epsilon} \sqrt{{}^{(3)}g}\, f^2(x) \left(\frac{\partial\ \mathrm{tr}(K)}{\partial \tau} - \mathrm{tr}(K)^2\right) d^3x < 0. \tag{15}$$

Since this integral is negative for all $f(x)$, there will be some (3+1)-dimensional open set $\mathcal{S}$ surrounding (but not necessarily including) the bounce Σ such that

$$\left(\frac{\partial\ \mathrm{tr}(K)}{\partial \tau} - \mathrm{tr}(K)^2\right) < 0. \tag{16}$$

But we also know that $\mathrm{tr}(K) = 0$ at the bounce itself. This allows us to apply the fundamental theorem of calculus a second time to derive the existence of a second open set $\tilde{\mathcal{S}}$ surrounding (but not necessarily including) the bounce Σ such that

$$\frac{\partial\ \mathrm{tr}(K)}{\partial \tau} < 0. \tag{17}$$

To see this note that Eq. (16) can be written as $dF(\tau)/d\tau - F(\tau)^2 < 0$ on $\tau \in (0, \tau_*)$ with $F(0) = 0$, from which we see that $F(\tau)$ must initially go negative. It is this final version of the bounce condition that will lead to the most general and powerful theorems.

These constraints on the extrinsic curvature lead to constraints on the spacetime geometry, and consequently constraints on the stress-energy tensor.

III. GEOMETRY AT AND NEAR A GENERIC BOUNCE

Using Gaussian normal coordinates in the region surrounding the bounce the Gauss-Codazzi and Gauss-Weingarten equations give

$${}^{(3+1)}R_{ijkl} = {}^{(3)}R_{ijkl} + (K_{ik}K_{jl} - K_{il}K_{jk}), \tag{18}$$

$${}^{(3+1)}R_{\tau ijk} = -(K_{ij|k} - K_{ik|j}), \tag{19}$$

$${}^{(3+1)}R_{\tau i \tau j} = \frac{\partial K_{ij}}{\partial \tau} + (K^2)_{ij}\,. \tag{20}$$

See [16], p. 514, Eqs. (21.75) and (21.76) and [16], p. 516, Eq. (21.82). Here the index τ refers to the temporal direction normal to the three-dimensional bounce. As usual, the vertical bar denotes a three-dimensional covariant derivative built out of the three-dimensional spatial metric.

These results hold both on the throat and in the region surrounding the throat: as long as the Gaussian normal coordinate system does not break down (such a breakdown being driven by the fact that the normal geodesics typically intersect after a certain distance).

Taking suitable contractions, and being careful *not* to use the extremality condition $\mathrm{tr}(K) = 0$, we find that, *at and near* the bounce,

$${}^{(3+1)}R_{ij} = {}^{(3)}R_{ij} - \left[\frac{\partial K_{ij}}{\partial \tau} + 2(K^2)_{ij} - \mathrm{tr}(K) K_{ij}\right], \tag{21}$$

$${}^{(3+1)}R_{\tau i} = \mathrm{tr}(K)_{|i} - K_{ij}{}^{|j}, \tag{22}$$

$${}^{(3+1)}R_{\tau\tau} = \mathrm{tr}\left(\frac{\partial K}{\partial \tau}\right) + \mathrm{tr}(K^2) = \frac{\partial \mathrm{tr}(K)}{\partial \tau} - \mathrm{tr}(K^2), \tag{23}$$

so that the Ricci scalar is

$${}^{(3+1)}R = {}^{(3)}R - \left[2\left(\frac{\partial\ \mathrm{tr}(K)}{\partial \tau} - \mathrm{tr}(K^2)\right) + \mathrm{tr}(K^2) - \mathrm{tr}(K)^2\right]. \tag{24}$$

To effect these contractions, we make use of the decomposition of the spacetime metric in terms of the bounce 3-metric and the set of three vectors e_i^μ tangent to the bounce and the four-vector n^ν normal to the bounce:

$$^{(3+1)}g^{\mu\nu} = -n^\mu n^\nu + e_i^\mu e_j^\nu \, {}^{(3)}g^{ij}. \tag{25}$$

(Note the minus sign in front of the $n^\mu n^\nu$ term.) For the spacetime Einstein tensor [cf. [16], p. 515, Eqs. (21.77) and (21.80) and [16], p. 552, Eqs. (21.162a)–(21.162c)],

$$^{(3+1)}G_{ij} = {}^{(3)}G_{ij} - \Bigg[\frac{\partial K_{ij}}{\partial \tau} - g_{ij} \frac{\partial \, \mathrm{tr}(K)}{\partial \tau} - \mathrm{tr}(K) K_{ij} + 2(K^2)_{ij} + \frac{1}{2} g_{ij} [\mathrm{tr}(K^2) + \mathrm{tr}(K)^2] \Bigg], \tag{26}$$

$$^{(3+1)}G_{\tau i} = \mathrm{tr}(K)_{|i} - K_{ij}{}^{|j} \tag{27}$$

$$^{(3+1)}G_{\tau\tau} = +\frac{1}{2} \, {}^{(3)}R - \frac{1}{2} [\mathrm{tr}(K^2) - \mathrm{tr}(K)^2]. \tag{28}$$

The calculations presented above are simply a matter of brute force index gymnastics—but we feel that there are times when explicit expressions of this type are useful.

IV. CONSTRAINTS ON THE STRESS-ENERGY TENSOR

A. First constraint: SEC violation

By using the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, the SEC applied to the stress-energy tensor is equivalent to the Ricci convergence condition [6]:

$$\forall \ \text{timelike} \ V^\mu: \quad R_{\mu\nu} V^\mu V^\nu > 0. \tag{29}$$

But by the simple flare-out condition (9) and Eq. (23), we see $^{(3+1)}R_{\tau\tau} \leq 0$. This implies that the SEC is either violated or on the verge of being violated at the throat. To really pin down SEC violation we must invoke the stricter inequality (17) to see that the SEC is definitely violated in some open region surrounding the bounce.

Equivalently, the spacetime Ricci tensor $^{(3+1)}R_{\mu\nu}$ has at least one negative definite eigenvalue (corresponding to a timelike eigenvector) everywhere in some open region surrounding the bounce. A similar result for Euclidean wormholes is quoted in [17] and the present analysis can of course be carried over to Euclidean signature with appropriate definitional changes.

B. Second constraint: Density

The energy density in the vicinity of the bounce is

$$\rho \equiv T_{\tau\tau} = \frac{1}{8\pi G} G_{\tau\tau} = \frac{1}{16\pi G} \, [{}^{(3)}R - \mathrm{tr}(K^2) + \mathrm{tr}(K)^2]. \tag{30}$$

The above is the generalization of the result that for a FRW-based Tolman wormhole [3]

$$\rho = \frac{3}{8\pi G} \left[\frac{k}{a^2} + \frac{\dot{a}^2}{a^2} \right]. \tag{31}$$

[With MTW conventions $^{(3)}R = 6/a^2$ for a three-sphere.] Since $\mathrm{tr}(K) = 0$ at the bounce, we see that, at the bounce itself,

$$\rho_{\text{bounce}} \leq \frac{1}{16\pi G} \, [{}^{(3)}R]. \tag{32}$$

Thus a *necessary* condition for the energy density to be positive at the bounce is that the bounce be a three-manifold of everywhere positive Ricci scalar.

C. Third constraint: Average pressure

Define an average pressure by

$$p \equiv \frac{1}{3} \, g^{ij} \, {}^{(3+1)}T_{ij} = \frac{1}{24\pi G} \, g^{ij} \, {}^{(3+1)}G_{ij}. \tag{33}$$

Then

$$p = \frac{1}{16\pi G} \Bigg[-\frac{1}{3} \, {}^{(3)}R + \frac{1}{3} [\mathrm{tr}(K^2) - \mathrm{tr}(K)^2] + \frac{4}{3} \left(\frac{\partial \, \mathrm{tr}(K)}{\partial \tau} - \mathrm{tr}(K^2) \right) \Bigg]. \tag{34}$$

The above is the generalization of the result that, for a FRW-based Tolman wormhole [3],

$$p = -\frac{1}{8\pi G} \left[\frac{k}{a^2} + \frac{\dot{a}^2}{a^2} + 2 \, \frac{\ddot{a}}{a} \right]. \tag{35}$$

Now at and near the bounce we can write the average pressure as

$$p = -\frac{1}{3} \, \rho + \frac{1}{12\pi G} \left[\frac{\partial \, \mathrm{tr}(K)}{\partial \tau} - \mathrm{tr}(K^2) \right]. \tag{36}$$

The term in square brackets is negative definite by Eq. (17); so there is an open region surrounding the bounce for which

$$p < -\frac{1}{3} \, \rho. \tag{37}$$

This is just the previously discussed SEC violation in another disguise, though it has the advantage of emphasizing the fact that positive densities near the bounce imply negative pressures near the bounce.

D. Fourth constraint: Energy conditions

Using the average pressure defined above, it is easy to prove that, even in the absence of any symmetries,

$$\text{NEC} \Rightarrow (\rho + p \geq 0), \tag{38}$$

$$\text{WEC} \Rightarrow (\rho \geq 0) \ \text{and} \ (\rho + p \geq 0), \tag{39}$$

$$\text{SEC} \Rightarrow (\rho + 3p \geqslant 0) \text{ and } (\rho + p \geqslant 0), \tag{40}$$

$$\text{DEC} \Rightarrow (\rho \geqslant 0) \text{ and } (\rho \pm p \geqslant 0). \tag{41}$$

Basic definitions of the energy conditions are given in [6,18]. It is important to note that in the case of a FRW universe these implications (⇒) are strengthened to equivalences (⇔) as discussed in [19–21].

To see how these relations are proved, focus as an example on the NEC, which states that for all null vectors $T_{\mu\nu}V^\mu V^\nu \geqslant 0$. Note that (up to arbitrary normalization) all null vectors can be written $V^\mu = (1;\beta^i)$ with $g_{ij}\beta^i\beta^j = 1$. Therefore, for all β^i we have

$$\rho + 2\ f_i\beta^i + T_{ij}\beta^i\beta^j \geqslant 0, \tag{42}$$

where the momentum flux is defined by $f_i = T_{\tau i}$. By averaging over the two null vectors $(1;\beta^i)$ and $(1;-\beta^i)$ this implies that, for all β^i,

$$\rho + T_{ij}\beta^i\beta^j \geqslant 0. \tag{43}$$

Finally average over three mutually perpendicular unit vectors β^i:

$$\rho + \frac{1}{3} T_{ij} g^{ij} \geqslant 0. \tag{44}$$

Equivalently,

$$\rho + p \geqslant 0. \tag{45}$$

The same logic can now be followed for the other pointwise energy conditions.

It therefore becomes interesting to use the Einstein equations to calculate $\rho \pm p$. We find

$$\rho + p = \frac{1}{16\pi G}\left[\frac{2}{3}\,{}^{(3)}R - \frac{2}{3}[\mathrm{tr}(K^2) - \mathrm{tr}(K)^2] + \frac{4}{3}\left(\frac{\partial\ \mathrm{tr}(K)}{\partial\tau} - \mathrm{tr}(K^2)\right)\right] \tag{46}$$

and

$$\rho - p = \frac{1}{16\pi G}\left[\frac{4}{3}\,{}^{(3)}R - \frac{4}{3}[\mathrm{tr}(K^2) - \mathrm{tr}(K)^2] - \frac{4}{3}\left(\frac{\partial\ \mathrm{tr}(K)}{\partial\tau} - \mathrm{tr}(K^2)\right)\right]. \tag{47}$$

We shall now show that there is an enormous class of spacetime geometries for which these two quantities are positive at and near the bounce. To see this, consider the following scaling argument: suppose we have some spacetime geometry which has a bounce and for which the bounce is a manifold of positive Ricci scalar. Now consider the class of geometries

$$g \to g_\epsilon : ds^2 = -dt^2 + \epsilon^2 g_{ij}\, dx^i dx^j.$$

For this class of geometries,

$${}^{(3)}R \to {}^{(3)}R_\epsilon = \frac{{}^{(3)}R}{\epsilon^2},$$

while on the other hand tr(K) and tr(K^2) are independent of ϵ. [$K_{ij} \to \epsilon^2 K_{ij}$ but $g^{-1} \to \epsilon^{-2} g^{-1}$, so $\mathrm{tr}(K) \to \mathrm{tr}(K)$.] Thus for ϵ sufficiently small the intrinsic curvature terms will always dominate over the extrinsic curvature terms and we can guarantee that the density [Eq. (30)] and Eqs. (46), (47) are all positive. Thus there is a large class of bounce geometries that are compatible with the NEC, WEC, and DEC. However bounce geometries must always violate SEC. This generalizes the result for FRW-based Tolman wormholes presented in [3]. Somewhat stronger statements can be made by looking at the explicit formulas for the components of the Einstein tensor:

$${}^{(3+1)}G_{ij}(\epsilon) = {}^{(3)}G_{ij}\epsilon^{-2} + O(\epsilon^2), \tag{48}$$

$${}^{(3+1)}G_{\tau i}(\epsilon) = O(1), \tag{49}$$

$${}^{(3+1)}G_{\tau\tau}(\epsilon) = +\frac{1}{2}\,{}^{(3)}R\,\epsilon^{-2} + O(1). \tag{50}$$

By choosing ϵ small enough we can guarantee that NEC, WEC, and DEC are satisfied, though SEC must always be violated.

V. GENERIC BOUNCES DEFINED USING TIMELIKE GEODESICS

The definition of a generic bounce starting from the volume integral in Eq. (1) is similar in spirit to and motivated by the definition of a generic wormhole throat developed in [7,8], but there are important differences we would like to underscore. First, of course, is the fact that a bounce is by definition an intrinsically time-dependent phenomenon, whereas wormholes may be either static or time-dependent. Second, whereas wormhole throats in spacetime are defined via two-dimensional spacelike hypersurfaces, the bounce is a three-dimensional spacelike hypersurface. The third, and perhaps the most important, difference stems from the fact that whereas wormhole throats are always closed (and thus have finite area) spatial hypersurfaces satisfying certain extremality and minimality properties, bounces may be spatially open (e.g., as in a FRW cosmology with flat or hyperbolic spatial sections) or closed (e.g., as in a FRW cosmology with closed spatial sections), depending on the type of cosmology being considered. In the latter case, the spatial volume integral is of course finite and well defined, but in the former case, it is not finite, and a definition of a generic bounce is called for which is not bound up with potentially infinite integrals, but which is nevertheless fully equivalent to the definition given earlier in this paper. That such a *local* pointwise definition of a generic bounce is possible is strongly suggested by the work in [9] and [10], which treated general dynamic wormholes on the basis of (null) geodesic congruences. The idea is simply to define what we mean by a bounce in terms of the local properties of timelike geodesic congruences in the neighborhood of the putative bounce. This is motivated by the very physical question which asks, how is the motion of test particles in the vicinity of a bounce affected by that bounce? The alternative definition is as follows: a bounce is a 3-dimensional spatial hypersurface such that the timelike geodesic congruence orthogonal to it vanishes on the hypersurface, is strictly expanding to the immediate future of the hypersurface, and is strictly contracting to the immediate past. This definition is capable of dealing with situations where only *part* of the universe is ''bouncing,'' while the rest either continues its collapse or is already in an expanding phase. The vanishing condition is equivalent to the minimality condition obtained in Sec. II and the contraction-expansion condition is none other than the Morris-Thorne ''flare-out'' condition generalized to bounces. Indeed, the mutual spreading out of a ''swarm'' of future-directed test particles in the immediate future of the bounce is what we mean by ''flare-out.'' As we will see, all these notions are pointwise. Our next task is to make them precise. In this section, we follow the same sign conventions and notation as used in [9,10] which are taken from Wald [22].

So consider a timelike geodesic congruence orthogonal to the spatial hypersurface Σ, to be conveniently located without loss of generality at $\tau=0$, and let ξ^a denote a tangent vector to a geodesic in this congruence; we can always arrange for all these tangents, parametrized by proper time τ, to have identical normalization:

$$\xi^a \xi_a = g_{ab}\xi^a \xi^b = -1, \tag{51}$$

where the spatial and spacetime metrics are related by

$${}^{(3)}g_{ab} = {}^{(3+1)}g_{ab} + \xi^a \xi^b. \tag{52}$$

Now define the tensor field

$$K_{ab} \equiv \nabla_b \xi_a; \tag{53}$$

by using the normalization condition and the fact that tangent vectors are parallel transported ($\xi^a \nabla_a \xi^b = 0$), one can easily show that this tensor is purely spatial, i.e., $\xi^a K_{ab} = \xi^b K_{ab} = 0$, and moreover is symmetric, $K_{ab} = K_{ba}$, because the congruence is hypersurface orthogonal. This tensor is in fact the extrinsic curvature of the hypersurface Σ and measures the ''degree of bending'' with respect to the embedding spacetime, as is well known. But it also contains useful information regarding the expansion θ and the traceless shear σ_{ab},

$$\theta = {}^{(3)}g_{ab} K^{ab} = \mathrm{tr}(K), \tag{54}$$

$$\sigma_{ab} = K_{(ab)} - \frac{1}{3}\, {}^{(3)}g_{ab}\theta, \tag{55}$$

of the timelike geodesic congruence normal to the hypersurface. The expansion θ measures the instantaneous ''spreading'' or divergence of nearby timelike geodesics while the symmetric shear tensor measures the ''slippage'' of nearby geodesics. The shear is a purely spatial tensor, which immediately implies that $\sigma^{ab}\sigma_{ab} \geqslant 0$ is always a positive semi-definite quantity.

The rates of change of the expansion and shear with respect to proper time (τ of the test particles) can be calculated, and in the case of the expansion, one obtains a simplified version of the celebrated Raychaudhuri equation [22]

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}\xi^a \xi^b, \tag{56}$$

where R_{ab} is the Ricci tensor of the full spacetime. This is independent of coordinate system. This is a simplified version because the additional contribution from the twist or anti-symmetric part of K_{ab} is absent here, since we are dealing with a hypersurface orthogonal congruence. Note that Eq. (23) is actually this special-case Raychaudhuri equation (56) in disguise [once we express Eq. (56) in terms of Gaussian coordinates and take into account the relative sign in the definitions for the extrinsic curvature used in this section, Eq. (53), and in Sec. II, Eq. (5)].

With these simple preliminaries out of the way, we can now give the (local) definition of what it means to be a generic bounce. A bounce is any three-dimensional spatial hypersurface on which the expansion of a hypersurface orthogonal timelike geodesic congruence vanishes identically,

$$\text{(i)} \quad \theta(0) = 0, \tag{57}$$

and for which the expansion is positive to the immediate future and negative to the immediate past:

$$\text{(ii)} \quad \exists \tilde{\tau}_+ > 0 : \forall \tau \in (0, \tilde{\tau}_+), \quad \theta(\tau) > 0, \tag{58}$$

$$\text{(iii)} \quad \exists \tilde{\tau}_- > 0 : \forall \tau \in (-\tilde{\tau}_-, 0), \quad \theta(\tau) < 0. \tag{59}$$

These three properties of the timelike geodesics capture the minimality and flare-out conditions of a bounce directly without needing to refer to the volume of the bounce. Indeed, a bunch of test particles traversing the bounce will initially have a cross section that first decreases in time, reaching a minimum at the throat, followed by a subsequent increase. A similar characterization was successfully employed recently in defining the general time-dependent wormhole throat [9,10], by means of null geodesics. By the fundamental theorem of calculus, conditions (i), (ii), and (iii) can be combined to imply

$$\exists \tilde{\tau}_0 > 0 : \forall \tau \in (-\tilde{\tau}_0, 0) \cup (0, \tilde{\tau}_0), \quad \frac{d\theta}{d\tau} > 0. \tag{60}$$

It will be noted that the Raychaudhuri equation (56) is independent of the underlying dynamics of the geometry: it is a statement only about (timelike) geodesics (test particles) in a particular geometry. If we *now* impose Einstein's equation (the geometrodynamics)

$$R_{ab} = 8\pi G\left(T_{ab} - \frac{1}{2} g_{ab} T\right), \tag{61}$$

and make use of the three conditions (i), (ii), and (iii) [in the form of Eq. (60)], then by the Raychaudhuri equation we must conclude that

$$\exists \tilde{\tau}_0 > 0 : \forall \tau \in (-\tilde{\tau}_0, 0) \cup (0, \tilde{\tau}_0),$$

$$\xi^a \xi^b \left(T_{ab} - \frac{1}{2} g_{ab} T \right) < 0. \tag{62}$$

That is, the SEC is strictly violated in an open region surrounding the bounce.

VI. DISCUSSION

One of the key results of traversable wormhole physics, perhaps *the* key result, is the unavoidable violations of the null energy condition at or near the throat [4–10]. In the case of a Tolman wormhole it is instead the strong energy condition that is violated at or near the bounce [3]. We have developed a number of general theorems that characterize the extent and generality of these SEC violations. An important point is that it is relatively easy to obtain SEC violations; they can be found already at the classical level and do not even require the standard appeal to quantum effects that is common in seeking to justify NEC violations [29].

There are a number of powerful constraints that can be placed on the stress-energy tensor at and near the bounce of a Tolman wormhole simply by invoking the minimality properties of the bounce. Depending on the precise form of the assumed flare-out condition, these constraints give the various energy condition violation theorems we are seeking. Even under the weakest assumptions they constrain the stress-energy tensor to at best be on the verge of violating the SEC.

In this article we have sought to give an overview of the energy condition violations that occur in generic Tolman wormholes. We point out that these violations of the energy conditions follow unavoidably from the definition of a Tolman wormhole (bounce) and the definition of the total stress-energy tensor via the Einstein equations. To show the generality of the energy condition violations, we have developed an analysis that is capable of dealing with Tolman wormholes of arbitrary symmetry. We have presented two complementary definitions of a bounce that agree where they overlap but are much more general than the FRW-based bounces considered in [3]. The present definitions work well in any spacetime and nicely capture the essence of the idea of what we would want to call a Tolman wormhole. We do not need to make any assumptions about the existence of any asymptotic regions; nor do we need to assume that the manifold is topologically non-trivial. It is important to realize that the essence of the definitions lies in the local geometrical structure of the bounce.

In the broader scheme of things, this article should be viewed as a contribution to the continuing debate as to whether the universe emerged from a mathematical singularity in the big bang or if something more subtle is going on. While there can be little doubt that the universe emerged from a hot dense fireball colloquially called the big bang, it is a big step from a hot dense fireball to a mathematical singularity. For many years it was believed that the Penrose [18] and Geroch [22] cosmological singularity theorems definitively proved the existence of a mathematical singularity, but these theorems are based on assuming the SEC. This article demonstrates that these theorems *cannot* be improved in the sense that we have exhibited a large class of Tolman wormholes that satisfy all energy conditions *except* the SEC. Furthermore, there is now a large body of evidence pointing to the fact that the SEC may not be the fundamental physical restriction it was once thought to be: there are many quite reasonable physical systems, even classical systems, that violate the SEC [3,23–28]. Likewise, gravitational vacuum polarization, although it is a small quantum effect, often violates the SEC (and other energy conditions) [29–33].

As discussed in [3] there are a number of singularity theorems provable within the ''eternal inflation'' paradigm [34–38], but these theorems obtain their results at the cost of making rather specific additional hypotheses and they are not in conflict with the results of the present paper.

Finally we should mention that a particularly large class of quite reasonably behaved Tolman wormholes is provided by the analytic continuation of Euclidean wormholes back to the Lorentzian signature [39].

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