Minimal length uncertainty principle and the trans-Planckian problem of black hole physics

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The minimal length uncertainty principle of Kemf, Mangano and Mann (KMM), as derived from a mutilated quantum commutator between coordinate and momentum, is applied to describe the modes and wave packets of Hawking particles evaporated from a black hole. The trans-Planckian problem is successfully confronted in that the Hawking particle no longer hugs the horizon at arbitrarily close distances. Rather the mode of Schwarzschild frequency ω deviates from the conventional trajectory when the coordinate r is given by $|r-2M|\simeq\beta_H\omega/2\pi$ in units of the nonlocal distance legislated into the uncertainty relation. Wave packets straddle the horizon and spread out to fill the whole nonlocal region. The charge carried by the packet (in the sense of the amount of ''stuff'' carried by the Klein-Gordon field) is not conserved in the non-local region and rapidly decreases to zero as time decreases. Read in the forward temporal direction, the non-local region thus is the seat of production of the Hawking particle and its partner. The KMM model was inspired by string theory for which the mutilated commutator has been proposed to describe an effective theory of high momentum scattering of zero mass modes. It is here interpreted in terms of dissipation which gives rise to the Hawking particle into a reservoir of other modes (of as yet unknown origin). On this basis it is conjectured that the Bekenstein-Hawking entropy finds its origin in the fluctuations of fields extending over the nonlocal region. [S0556-2821(99)05402-8]

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I. INTRODUCTION

More than two decades after its discovery, Hawking's theory of black hole evaporation [1] continues to be plagued by the ''trans-Planckian'' problem. Presumably, because of an insufficient treatment of the gravitational back reaction, the theory suffers from an overdose of localization. Outgoing photons hug the horizon [$r=2M$ in Planckian units at a distance $O(Me^{-\alpha M^2})$ with $\alpha=O(1)$]. This results in proper energies near the horizon $O(\omega e^{\alpha M^2})$ where ω is the observed energy at asymptotic distances; typically $\omega=O(M^{-1})$ and $M=O(10^{40})$ for a macroscopic black hole.

Considerable effort has gone into introducing the necessary ingredients of non-locality to cure this disease. Relevant to the present paper is the methodology [2] that has been brought to bear to exploit Unruh's model of the dumb hole [3]. In particular, we shall use Eddington-Finkelstein (EF) coordinates as introduced by Damour-Ruffini (DR) [4]. This turns out to be an efficient tool. Though interesting in itself, in that the dumb hole analogy shows the robustness of Hawking's radiation in resisting mutilation of the conventional theory, it is nevertheless inadequate. This is because in fluids, there is a cut-off in momentum as well as in energy. In adopting this to the black hole problem, as in Ref. [2], one cuts off the energy but not the momentum, elsewise one would lose the Hawking effect. There has, as yet, been no justification offered for this procedure.

In this paper, using the DR technique, we introduce an alternative mechanism of non-locality which we believe has some chance of being founded in the correct physics of the situation. This is the non-local commutator, the object of study of Kempf, Mangano and Mann (KMM) [5]. Such ''mutilated'' commutators have been proposed in the context of string theory (see Refs. [5] and [6] for a bibliography), but they may arise in a more general context wherein the mode relevant to a particular problem (such as the evaporating photon) interacts with ''reservoir'' modes in general. In this paper we do not enter into this fundamental question aside from some (superficial) concluding remarks. Rather we take the pragmatic point of view: assume KMM and see if it cures the transplanckian problem. And it does. Furthermore, in so doing the formalism suggests the origin of the Bekenstein-Hawking black hole entropy [7], as we shall point out at the end of the development. The main physical picture that emerges is that the reservoir, which is responsible for KMM non-locality, boils off a Hawking pair on either side of the horizon, in a region whose extension is ($\beta_H\omega$) units of the nonlocal length scale. Here $\beta_H=8\pi M$ is the inverse Hawking temperature.

II. KMM THEORY

We begin with a brief summary of KMM. The point of departure is (for one degree of freedom)

$$[\hat{P},\hat{Q}]=-i\left(1+\frac{\hat{P}^2}{\mu^2}\right). \qquad (1)$$

In what follows we adopt for momentum and coordinate the nondimensional variables $\hat{p}=\hat{P}/\mu$; $\hat{q}=\mu\hat{Q}$. The scale μ could be Planckian or it could involve some fractional power

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of M. All we require is $(M/m_{pl}^2)\gg\mu^{-1}$ in order to have a sufficiently large asymptotic region so that there exists a region $\mu^{-1}\ll(r-2M)\ll 2M$; from now on all lengths are in Planckian units.

Equation (1) implies a modified uncertainty relation

$$\Delta p\Delta q\geq\frac{1}{2}[1+\langle\hat{p}^2\rangle]=\frac{1}{2}[1+(\Delta p)^2+\langle\hat{p}\rangle^2] \tag{2}$$

where $\Delta p\equiv\langle(\Delta\hat{p})^2\rangle^{1/2},\Delta q\equiv\langle(\Delta\hat{q})^2\rangle^{1/2}$. Thus Δq has a minimal value (=unity) at $\langle\hat{p}\rangle=0$ and $\Delta p=1$. We note in passing the frame dependence. This has not been studied either by KMM or by us. Perhaps its elucidation will require a careful study of the underlying fundamental mechanisms behind Eq. (1). In keeping with this, our initial pragmatic exploration, we adopt the most natural assumption. As in Ref. [2], the frame is taken to be the rest frame of the black hole, expressed in ingoing EF coordinates (see Sec. III). Indeed these coordinates (v,r) have a geometrical meaning; $v=$Cte being the equations of light rays falling into the black hole, and r being an affine parameter on them.

The main thrust of KMM is the search for a Hilbert space formalism which is physically sensible. Thus one requires that all expectation values of $\hat{q}$ be real. The matter is subtle in that there are no eigenstates of $\hat{q}$ compatible with Eq. (2) whereas eigenstates of $\hat{p}$ do exist. So one works in p representation and requires that $\hat{q}$ be a symmetric operator. The scalar product of two state vectors, $|f\rangle$ and $|g\rangle$, is thus conveniently represented by

$$\begin{aligned}\langle f,g\rangle&=\int_{-\infty}^{\infty}f^*(p)g(p)\frac{dp}{1+p^2}\\&=\int_{-\pi/2}^{\pi/2}f^*(\tan\theta)g(\tan\theta)d\theta\end{aligned} \tag{3}$$

with $\hat{q}$ expressed in p-representation as

$$\begin{aligned}&\hat{q}=i(1+p^2)\partial_p=i\partial_\theta,\\&\theta\equiv\arctan p,\end{aligned} \tag{4}$$

since $\langle\hat{q}f|g\rangle=\langle f|\hat{q}g\rangle$ by integration by parts, provided the domain of $\hat{q}$ is the set of functions that vanish at the limits of integration $\pm\pi/2$. There are important ''self adjoint extensions'' of $\hat{q}$. Abbreviating $f(\tan\theta)=F(\theta)$, $\hat{q}$ becomes an essentially self adjoint operator on the domain [8] wherein

$$F(\pi/2)=e^{i\alpha}F(-\pi/2). \tag{5}$$

It then follows from Eqs. (4) and (5) that the position eigenfunctions in momentum space are $e^{iq_\alpha\theta}/\sqrt{\pi}$ where q_α are lattice points

$$q_\alpha=2k+\alpha/\pi,\quad k\ \text{integer} \tag{6}$$

and $0\leq\alpha<2\pi$. The choices of α define physically indistinguishable bases of the Hilbert space (corresponding to the inability to localize). Thus any problem governed by KMM dynamics is possessed of a U(1) symmetry corresponding to these equivalent choices.

To familiarize the reader with the consequences of this kind of effective quantum mechanics, consider the square well problem

$$\hat{p}^2\Psi=2mE\Psi\quad\text{with}\quad\Psi(0)=\Psi(L)=0. \tag{7}$$

One may then work in q-representation wherein

$$\hat{p}=\tan(-i\partial_q) \tag{8}$$

acting on the subspace of functions of wave lengths greater than 4 (in conformity with the domain of convergence of the power series of the tangent around zero [5]). It follows that Ψ is an eigenfunction of ∂_q^2 given by $C\sin(n\pi q/L)$ of eigenvalue $2mE=\tan^2(n\pi/L)$. The spectrum cuts off at $n=[L/2]$ (i.e., excluding wavelengths shorter than 4). The bracket $[x]$ symbolizes the integer part of x. Note that the problem is rather ill defined in that $x=L$ is not a physically legitimate concept in that some fuzziness is always required by Eqs. (1) and (2).

It is to be expected that radical effects of the like will emerge in the black hole problem once one tries to cram a mode too close to the origin. This is our motivation. In this we recommend [5] for a careful discussion of physical states, maximally localized states and the critical role of wave length 4.

III. APPLICATION TO THE BLACK HOLE PROBLEM

For the model of black hole evaporation we take a Schwarzchild black hole, work in EF coordinates and neglect the centrifugal barrier that sends low frequency (ω) outgoing s waves back into the singularity $[\omega\leq(\beta_H)^{-1}]$. These outgoing s waves for $\omega\geq(\beta_H)^{-1}$ are modes ψ of the form $e^{-i\omega v}\chi_\omega(r)$ where in the conventional theory one has [2]

$$(1-2M/r)\partial_r\psi=-2\partial_v\psi, \tag{9}$$

$$(1-2M/r)\partial_r\chi_\omega=2i\omega\chi_\omega, \tag{10}$$

taken together with the Unruh-Jacobson boundary condition reexpressed by DR in the form $p>0$.[1] The interesting physics encoded in Eq. (10) is near the horizon where it reduces to

$$x\partial_x\chi_\Omega(x)=ix\hat{p}\chi_\Omega(x)=4Mi\omega\chi_\Omega(x)\equiv i\Omega\chi_\Omega(x), \tag{11}$$

where χ_ω is relabeled as χ_Ω and $\Omega=\beta_H\omega/(2\pi)$ is the dimensionless frequency. We have identified $\hat{p}$ to $-i\partial_x$ in x-representation (where $x=r-2M$). Thus in Eq. (11) the units of length drops out of both sides. It will be reinstated subsequently. In order to have a complete set of states, so as

[1]Recall p is the energy near the horizon, ∂_x being light like. So $p>0$ is the restriction to positive energy modes near the horizon $(r-2M<2M)$ and these are the modes that give rise to steady state radiation [2].

to fulfill the boundary condition of continuity as the outgoing mode crosses the star's surface to emerge into the exterior Schwarzchild space, both signs of ω must occur in the linear combination of these positive energy ($p>0$) modes. One calls these the ''in'' modes, the basis of second quantization in the distant past. The ''out'' modes, those counted by the distant Schwarzschild observer in the future have a fixed sign of ω, since there the space is flat. It is the mixture of positive and negative ω which characterizes positive energy modes near the horizon that encodes Hawking evaporation. See Ref. [2] for explanations.

We now propose to adopt KMM for this problem and use Eq. (8). The ensuing equation is difficult, to say the least, so we go over to p-representation. In this we are (consciously) cavalier in that boundary terms may get in the way. They do and we shall have more to say on this problem in due course. From Eq. (11), we then have

$$(1+p^2)\partial_p(p\tilde{\chi}_\Omega(p))=-i\Omega\tilde{\chi}_\Omega(p) \tag{12}$$

or

$$\partial_\theta(\tan(\theta)\Phi_\Omega(\theta))=-i\Omega\Phi_\Omega(\theta) \tag{13}$$

where

$$\Phi_\Omega(\theta)=\tilde{\chi}_\Omega(\tan(\theta)).$$

The solution is

$$\Phi_\Omega(\theta)=A_\Omega[(\sin\theta)^{-i\Omega}\cot\theta]\Theta(\theta) \tag{14}$$

where we have applied the vacuum condition $p>0$, hence $\theta>0$. Taking the Fourier transform gives

$$\chi_\Omega(x)=A_\Omega\int_0^{\pi/2}e^{i\theta x}\Phi_\Omega(\theta)d\theta. \tag{15}$$

The constant A_Ω will be fixed subsequently. We repeat that x and p are nondimensionalised by the unit μ (i.e. $x=\mu X$ where all dimensionful quantities are in Planckian units). The integral (15) is feasible and we record the answer in terms of the beta-function and the hypergeometric ${}_2F_1$:

$$\begin{aligned}\chi_\omega(x)=&A_\Omega 2^{i\Omega}e^{\pi\Omega/2}\\&\times\Bigg\{B(i\Omega/2+x/2,-i\Omega)\frac{x}{x-i\Omega}\\&-\frac{2e^{i\pi x/2}e^{-\pi\Omega/2}}{(i\Omega+x+2)(i\Omega+x)}\\&\times{}_2F_1(i\Omega/2+x/2,1+i\Omega;i\Omega/2\\&+x/2+2;-1)\Bigg\}.\end{aligned} \tag{16}$$

But it is more informative to examine the properties of the integral (15) directly. The salient features are

(1) The function $\chi_\Omega(x)$ defined by Eq. (15) is not a solution of $x\hat{p}\chi_\Omega=\Omega\chi_\Omega$ [with $\hat{p}=\tan(-i\partial_x)$]. In performing the usual integration by parts one picks up a boundary contribution at $\theta=\pi/2$ (i.e. $p=\infty$), so that $(x\hat{p}-\Omega)\chi_\Omega(x)=A_\Omega e^{i\pi x/2}$, a term which oscillates on the scale of nonlocality. Recall here that x is an affine parameter on the outgoing geodesic near the horizon. Thus its average effect on this trajectory vanishes. We return to further discussion of this point after the other features are presented.

(2) Asymptotically ($x\gg\Omega$) one has $\chi_\Omega(x)\to A_\Omega x^{i\Omega}$ as is easily seen by carrying out steepest descents on Eq. (15) or using beta function properties. The width of the saddle at $\theta=\Omega/x$ is of order $O(\Omega/x^2)$ assuring the validity of the estimate. This tallies with the direct analysis of $\tan(-i\partial_x)\chi=(\Omega/x)\chi$ obtained by expanding $\tan(-i\partial_x)$ in powers of ∂_x. The constant A_Ω is then chosen to conform to the conventional norm of the Klein-Gordon current (see Ref. [2]). Following the argument of Jacobson [9], Hawking radiation then follows.

(3) The behavior of $\chi_\Omega(x)$ for $-\Omega<x<\Omega$ confronts the trans-Planckian problem neatly. At $x=\Omega$ the solution changes in character from the would be rapidly oscillating solution ($x^{i\Omega}$) to a slowly varying function. Indeed one calculates directly $\chi_\Omega(0)=(i/\Omega)A_\Omega$, and for example $\chi_\Omega(1)-\chi_\Omega(-1)=-(2/(\Omega+i))A_\Omega$; typically, $\Omega=O(1)$. This changeover of behavior is to be expected, since along the characteristic defined by the null outgoing geodesic the point $x=\Omega$ is where $p=1$ and this is where the important manifestation of nonlocality sets in, according to Eqs. (1) and (11). Any tight packet must then spread over a distance at least comparable to the unit of nonlocality. Hence all sense of hugging the horizon is lost. Indeed, in a sense we are really transcending the rules in this small region, in that Eqs. (1) and (2) are taken to define an effective theory which allows one to extrapolate down to the scale of nonlocality, but not beyond. [In this we are deliberately ambiguous because we are not sure whether the relevant scale is $x\simeq 1$ or $p\simeq 1$ or something else. Practically speaking, since $\Omega=O(1)$, the question is not of much importance, but conceptually it should be cleared up.]

(4) In the conventional theory Klein-Gordon current conservation is easily confirmed and is encoded in p representation through the integral

$$(2\pi)^{-1}\int_0^\infty(dp/p)p^{-i(\Omega-\Omega')}=\delta(\Omega-\Omega').$$

This integral now becomes [see Eq. (22) below]

$$(2\pi)^{-1}\int_0^{\pi/2}d\theta\cos(\theta)(\sin(\theta))^{-i(\Omega-\Omega')-1}$$

which is not a δ function. This nonorthogonality has profound repercussions on conservation theorems encountered in the evolution of wave packets. We report on this below.

Let us now return to what appears to be somewhat of a snag in these results, point [1] above. As mentioned the difficulty comes from the upper limit in Eq. (15), $p=\infty$ and this is really pushing the mechanism of non-locality well beyond what ought to be the range of its validity. Some regulator is therefore to be called upon. We mention two possibilities, each of which is not unattractive, but there are surely more. One can expect these will be revealed upon investigation of the fundamental theory.

The first of these is related to the ambiguity of a choice of origin, the equivalent representations labeled by α [see Eqs. (5) and (6)]. We have couched our (cavalier) treatment in terms of the continuum x [which would appear legitimate since $\chi_\Omega(p)\to 0$ as $p\to\infty$] but we have run into trouble because the operator $\hat{x}\hat{p}$ transforms $\chi_\Omega(p)$ into a function that does not vanish at the limits. Therefore it would seem mandatory to average $\hat{x}\hat{p}\chi_\Omega(p)$ over α. Now notice that in point [1], the term $A_\Omega e^{i(\pi/2)x}$ acts as a source term for the operator $\hat{x}\hat{p}-\Omega$. The average of this source over 4 units of nonlocality is zero. As we mentioned previously, states of wavelength 4 and smaller are to be excluded from physical states. Thus the oscillations $e^{i(\pi/2)x}$ of $(\hat{x}\hat{p}-\Omega)\chi_\Omega$ are ''unphysical.'' Only the average over at least 4 length units is meaningful and the average over 4 does vanish. Alternatively, one may project $(\hat{x}\hat{p}-\Omega)\chi_\Omega$ on the physical states $|\xi\rangle$ of KMM [5]. Since these packets have components of wavelength >4, this projection will vanish. In this sense the averaging process is mandatory, and indeed one must anticipate that some averaging of the modes $\chi_\Omega(x)$ is necessary to give the theory sense. Once more, at the present stage, it is difficult to give an exact recipe for averaging. Along theses lines, an alternative procedure is also possible, regularizing so as to exclude the component $p=\infty$ from $(\hat{x}\hat{p}-\Omega)\chi$. One may imagine that this will occur from an effective action which issues from the same fundamental theory from which the term p^2/μ^2 arises in the effective commutator and which is a manifestation of the same nonlocality. A very simple recipe is to add to the equation of motion an infinitesimal term, $i\epsilon p^2$, to give

$$(\hat{x}\hat{p}+i\epsilon\hat{p}^2)\chi_\Omega=\Omega\chi_\Omega\,. \tag{17}$$

At small p (large x in wave packets) the correction is negligible and at large p (small x) it takes care of the problem of the upper limit, so as to rid one of the $e^{i(\pi/2)x}$ oscillating source. One easily checks that χ_Ω now becomes $A_\Omega[(\sin\theta)^{i\Omega}/\tan\theta](\cos\theta)^{i\epsilon}$. The term $(\cos\theta)^{i\epsilon}$ is sufficient to eliminate the spurious source term (in the sense of distribution theory). Clearly the two methods are carrying similar messages; an average over a few units of nonlocality is necessary.

We now turn to our analysis of the evolution of wave packets. As a preliminary we first consider conservation of current and charge. This is carried out, following Noether, in conventional fashion in momentum representation. Using Eq. (13) one constructs the current (j^v,j^θ) where

$$j^v=\frac{1}{2}[\Psi^*(\theta)(\tan\theta\,\Xi(\theta))+(\tan\theta\,\Psi(\theta))^*\Xi(\theta)] \tag{18}$$

$$j^\theta=(\tan\theta\,\Psi(\theta))^*(\tan\theta\,\Xi(\theta)) \tag{19}$$

where Ψ and Ξ are solutions, whence $\partial_v j^v+\partial_\theta j^\theta=0$.

This conserved current, nevertheless, does not lead to a conserved charge owing to boundary terms in θ. The ''would be'' conserved charge is the diagonal element of the general form $\langle\boldsymbol{\Psi},\boldsymbol{\Xi}\rangle$ where

$$\langle\boldsymbol{\Psi},\boldsymbol{\Xi}\rangle=\int_0^{\pi/2}\Psi^*(\theta)(\tan\theta)\Xi(\theta)d\theta. \tag{20}$$

From Eq. (13) one then has

$$\frac{d}{dv}\langle\boldsymbol{\Psi},\boldsymbol{\Xi}\rangle=\Psi^*(\theta)\Xi(\theta)\tan^2\theta\big|_0^{\pi/2}. \tag{21}$$

As an example previously cited, the modes are not orthogonal

$$\langle\phi_\Omega,\phi_{\Omega'}\rangle=A^*_{\Omega'}A_\Omega\left[\pi\delta(\Omega-\Omega')-i\mathrm{P}\frac{e^{i(\Omega'-\Omega)}}{\Omega'-\Omega}\right] \tag{22}$$

where we have written $\phi_\Omega=e^{-i\Omega V}\Phi_\Omega(\theta)$ (and $V\equiv v/4M$).

The principal value term in Eq. (22) comes from the upper limit of the θ integral in Eq. (20), i.e. it is a high momentum effect. Unlike point (1) above, there is no argument on hand to eliminate this effect since there is no spatial averaging procedure available. In fact, any cut-off of the wavelength will yield such effects.

In x-representation the corresponding charge is

$$\langle\boldsymbol{\Psi},\boldsymbol{\Xi}\rangle=\frac{1}{2}\int_{-\infty}^{\infty}[\psi^*(x)\tan(-i\partial_x)\xi(x)-\xi(x)\tan(-i\partial_x)\psi^*(x)]dx \tag{23}$$

where $\psi(x)=\int_0^{\pi/2}e^{i\theta x}\Psi(\theta)d\theta$ and similarly for $\xi(x)$ and $\Xi(\theta)$. The limits on x are chosen to be $\pm\infty$ since we are interested in wave packets of finite extent. Such packets are centered asymptotically on $u=v-4M\ln|x|=$const i.e. in the region where the dominant components of p are $p<1$ (thus $\theta\ll\pi/2$). In this region, $\tan[(1/i)\,\partial_x]$ acts like $[(1/i)\,\partial_x]$ and conventional charge conservation obtains. When v is sufficiently early the center of the packet enters into the region of non-locality ($|x|<\Omega$ or $p>1$) and the above effects of charge nonconservation set in.

We have performed numerical computations on a packet of the form

$$\begin{aligned}\mathcal{F}(V,\theta)&=\int e^{-(\Omega-\Omega_H)^2/\sigma^2}\frac{A(\Omega_H)}{A(\Omega)}\phi_\Omega(V,\theta)d\Omega\\&=A(\Omega_H)e^{-i\Omega_H(V+\ln\sin\theta)}\cot\theta\, e^{-(\sigma^2/2)(V+\ln\sin\theta)^2}\end{aligned} \tag{24}$$

centered on frequency $\omega_H(=\Omega_H/4M)$. In configuration space it is asymptotically centered on the trajectory $u=$const (the value of the latter is irrelevant) where $u=v-4M\ln|x|$. We present in Fig. 1 the evolution of the

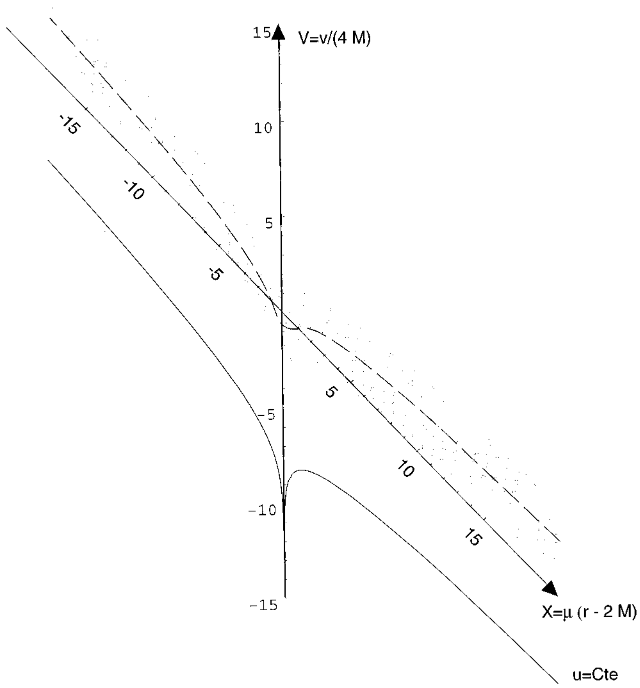

FIG. 1. The evolution of the wave packet of Eq. (24) is depicted. The parameters are $\Omega_H=1$, $\sigma=1$, and $V_H=0$. The coordinates are advanced Eddington-Finkelstein [V and x where $V=v/4M$ and $x=(r-2M)/\mu$]. Ingoing null geodesics are straight lines plotted on a 45-degree slant. The center of the packet is indicated by the dashed line. It is classical trajectory for $|x|>1$ (i.e. $u=$const; such an outgoing null geodesic is plotted on the bottom of the figure). The shaded region indicates the spread. For each x the spread in V is constant ($\simeq 4$) and this gives the large spread in x at $V=V_H(=0)$, the time about which the packet enters into the region of nonlocality.

packet in configuration space, given by $f(V,x)=\int_0^{\pi/2} e^{i\theta x}\mathcal{F}(V,\theta)d\theta$. We now discuss its qualitative features.

In the asymptotic region, the conventional nonspreading packet, centered on $u=$const, obtains. For $V<V_H(\equiv e^{\Omega_H})$ (i.e. for values of V which are earlier than that which corresponds to $x=\Omega_H$ along the asymptotic trajectory), the center of the trajectory deviates from $u=$const and bends in towards the horizon ($x=0$). It crosses the horizon and in the region $x<-\Omega_H$ becomes the classical trajectory of the Hawking ''partner'' which falls into the singularity. In the nonlocal region $|x|<\Omega_H$, the classical trajectory is not of quantitative significance since the packet spreads in x over the region of nonlocality.

These features are qualitatively analyzed as in Ref. [2] by a saddle point calculation of $f(V,x)$ or alternatively by the method of characteristics for the trajectory of the center of the packet. One finds for $p(V)$ the equation $p^2=e^{-2(V-V_H)}/(1-e^{-2(V-V_H)})$. Thus the conventional behavior obtains for $V>V_H$ whilst for $V<V_H$, p behaves like $\pm[V-V_H]^{1/2}$. It is to be noted that there is no extrapolation to the past for $V<V_H$ (i.e., the notion of a usual causally behaved trajectory stops). Correspondingly $x\simeq\Omega/p$ for $(V-V_H)\gg 1$ and $x\simeq\Omega/p^3$ for $V-V_H\simeq 0$. We again emphasize that this classical movement in the region $|x|<\Omega_H$ is without quantitative significance since the width in x in this region is $O(\Omega_H)$.

Most interesting is the evolution of the nonconserved charge. One can calculate explicitly from Eq. (23) with $\tilde{\psi}$ and $\tilde{\Xi}$ given by $\tilde{f}(V,x)$ [the Fourier transform of Eq. (24)]

$$Q(V)=2\pi|A(\Omega_H)|^2\int_0^{\pi/2} e^{-\sigma^2(V+\ln\sin\theta)^2}\cot\theta d\theta \quad (25)$$

$$=2\pi|A(\Omega_H)|^2\frac{\sqrt{\pi}}{2\sigma}[1+\mathrm{Erf}(\sigma V)] \quad (26)$$

where Erf is the error function and we have chosen $V_H=0$. Thus, for large V, $Q(V)$ is $[\mathrm{const}+O(e^{-\sigma^2V^2})]$ whereas for $V<0$, Q tends to zero like $e^{-\sigma^2V^2}$. The behavior changes around $V=0$ corresponding to a saddle in Eq. (25) (at $\ln\sin\theta^*=-V$) with θ^* near to $\pi/2$ i.e. $p\geqslant 1$. Thus the charge diminishes rapidly within the nonlocal region. Read forward in time, this means that the ''stuff'' in the packet is produced in this region.

IV. CONCLUDING REMARKS

From these various considerations one obtains an interesting interpretation wherein the region where nonlocality plays an active role: $(-\Omega_H<x<\Omega_H)$ is the zone of interaction of the Hawking photon mode with reservoir modes. Going forward in time, this region which straddles the horizon over a scale Ω_H in units of nonlocal length boils off a pair which for $|x|>\Omega_H$ looks like that of the conventional theory. It is ''solicited'' by the collapse, this latter being encoded in the boundary conditions characterizing the Unruh vacuum. Clearly, much more work is required in order to see just how this encoding is dynamically realized (perhaps combining Unruh's analysis of the collapsing shell with the present considerations or perhaps calling upon 't Hooft's scattering mechanism between incoming and outgoing degrees of freedom [11]).

It is highly significant that the theory based on the effective commutator (1) is nonunitary as well as noncausal in the usual sense of the words. Whereas the latter was to be anticipated at the outset, the former has arisen as a consequence. In the approach of this paper, curing the trans-Planckian problem is inevitably associated with nonunitarity in that the sector of Hilbert space describing the states of the light modes giving rise to Hawking radiation is not complete. This should come as no surprise in string theory since then the scattering matrix defined by the zero-mass sector is not unitary either. One will produce quanta of massy modes once energies and momenta are high enough. Apparently this effect is encoded in Eq. (1).

One may conjecture that the loss of Bekenstein-Hawking radiation entropy is due to this boiling off from the reservoir. It occurs in a ''zone of ignorance'' composed of Ω units of length where $\Omega=\beta_H\omega/(2\pi)$. Could it be that each unit corresponds to a volume 2π of ''ignorance'' due to the inability to localize in that region? The volume 2π is of course the group volume of the $U(1)$ of Kempf [10]. Much work is required to substantiate this conjecture, but it is remarkable that the length scale over which a given Hawking mode ''gets lost'' due to its interaction with the reservoir is Ω and not 1. And of course this is what incites one to make the conjecture.

Our approach then invites a series of problems and still further conjectures. First of all what is the physical interpretation of Eqs. (1) and (2). The problem comes in two parts, general and particular to the situation at hand.

As to the former, it it clear from the string theoretic model that the modification that sets in for the quantum description of the zero mass states of momentum $p>\mu$ (where $\mu \simeq \sqrt{\text{tension}}$) is related to the fact that such modes can dissipate into the reservoir of the higher mass modes of the string. The density of states of these latter is sufficient to create this dissipation. As Renaud Parentani has suggested to us, the situation is reminiscent of the Hagedorn temperature wherein energy that is poured into the zero mass modes gets redistributed into the higher mass modes. Thus increasing the energy of the former does not result in an increase of their temperature (their mean energy) once the energy becomes comparable with μ. Similarly were one to measure Q with zero mass modes with increasing precision one would be frustrated since increasing the energy of these modes would be of no avail. Hence a minimum of precision is reached which in fact is of the order of the inverse Hagedorn temperature.

Another interesting point in this same vein is the conjecture that there is a sort of quantum Nyquist theorem; see for example [12]. Dissipation of a given set of modes into a reservoir implies that such modes ''jiggle'' due to their recoil induced by the dissipating interactions. Then the thermal noise (the jiggle) is due to the vacuum fluctuations of the reservoir and the drag term will be the rate of dissipation. Equation (1) would then be viewed as a time dependent commutator averaged over the dissipation time and the reservoir states.

Accepting the reservoir interpretation, which seems inevitable, what then is the nature of the reservoir in our black hole problem? Does it concern vacuum fluctuations in general (usually swept under the rug due to the renormalizability of field theory relevant to particle physics, but highly relevant in production from horizons)? If so, then the problem would be related to current efforts to confront quantum gravity such as string or M theory. Or is the reservoir specific to the black hole (or horizons in general)? Serge Massar has raised the possibility that the reservoir modes are vibrations of the membrane of the so-called membrane paradigm.

It is far from clear that one can develop a satisfactory phenomenology without a complete understanding of the reservoir. If it is possible, the theory would occupy some middle ground between dynamics and thermodynamics prior to a full understanding which we all believe will require the correct theory of quantum gravity.

Note added.

After this paper was submitted two points came up which we now address.

(1) Our (self)criticism on the use of a momentum cut-off refers to Ref. [2]. It has been pointed to us that a recent work of Corley and Jacobson [13] avoids this difficulty by formulating the problem directly on a lattice free falling frame.

(2) The important question has been asked: what is the status of time reversal symmetry in our formulation? The growing of the modes to the future is not really a breakdown of this symmetry, but the reflection of a post selection assuming the existence of a Hawking photon in the far future. However the choice of the EF coordinate system to implement the modified commutation relation on the spacelike coordinate constitute a real breakdown of time reversal symmetry that needs enlightening. A satisfactory answer can only be given when we arrive at a fundamental understanding of Eq. (1). But if our conjectural statements given in Sec. IV are borne out, then the effective theory based on Eq. (1) would have an intrinsic violation of time reversal invariance, the origin of which would be in time averaging and average over states of the reservoir modes. We are presently engaged in the study of this point.

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