$\langle T^\mu_\nu \rangle_{\text{ren}}$ of the quantized conformal fields in the Unruh state in Schwarzschild spacetime

Jerzy Matyjasek*

Institute of Physics, Maria Curie Skłodowska University, pl. Marii Curie Skłodowskiej 1, 20-031 Lublin, Poland

The renormalized expectation value of the stress-energy tensor of the conformally invariant massless fields in the Unruh state in Schwarzschild spacetime is constructed. It is achieved through solving the conservation equation in the optical space with a polynomial ansatz, utilizing the regularity conditions and expected asymptotic behavior of the stress-energy tensor in a physical metric, and performing a fit to the results of the numerical calculations. It is shown that the adopted method, which is based on the conformal transformations of the one-loop effective action, is complementary to that of Visser. [S0556-2821(99)01102-9]

I. INTRODUCTION

The expectation value of the stress-energy tensor of the conformally invariant massless fields in the Unruh state in the Schwarzschild geometry is known to possess some general features presented for the first time in the celebrated Christensen and Fulling paper [1]. The asymptotic behavior of tangential and radial components of $\langle T^\mu_\nu \rangle_{\text{ren}}$ and the regularity conditions on the future event horizon are quite restrictive, and allow construction of a class of approximate tensors. Further, the numerical data, such as the exact value of the $\langle T^\theta_\theta \rangle_{\text{ren}}$ on the future event horizon and the value of the luminosity may be used in the final determination of the model. It is very fortunate that, thanks to the excellent numerical analysis carried out in Refs. [2] and [3], we have detailed information concerning the overall character of the exact $\langle T^\mu_\nu \rangle_{\text{ren}}$ of the scalar field and consequently the validity of constructed approximations may by verified. Similar calculations has been carried out in the case of the conformal vector field.

The attempts to construct analytical or semianalytical approximations of $\langle T^\mu_\nu \rangle_{\text{ren}}$ in various ''vacuum'' states, see Refs. [4–19], are motivated, aside from self-evident curiosity, by the fact that they may be used as a source term of the semiclassical Einstein field equations [20–30] or give rise to the analyses of local and averaged energetic conditions and quantum inequalities [31–38]. The stress energy tensor is also useful in the thermodynamic calculations, see, for example, Refs. [20,25,27,29,39,40]. It should be noted that the back reaction calculations are limited to the $\langle T^\mu_\nu \rangle_{\text{ren}}$ evaluated in the Hartle-Hawking state.

Recently such semianalytic approximations of the stress tensor in the Unruh state have been presented [16–18]. In Refs. [16,17], where properties of the one-loop effective action under the conformal transformations have been used, the starting point is the assumption that in the ultrastatic companion of the Schwarzschild metric the tangential component of $\langle \tilde{T}^\mu_\nu \rangle_{\text{ren}}$ has a polynomial form

$$\langle T^\theta_\theta \rangle_{\text{ren}} = T p(s) \sum_{n=4}^{N} a_n x^n, \qquad (1)$$

where $x = 2M/r$ and $T = \pi^2 T_H^4/90$ $[T_H = 1/(8\pi M)]$. The numerical factor $p(s)$ depends on the spin of the field and is given in Table I. Such a choice is in accord with the asymptotic analyses of Christensen and Fulling. Further, integrating the conservation equation in the conformal space and making use of the regularity conditions in the physical one, a family of the stress tensors could be easily obtained. Taking $N=7$ as has been done in Refs. [16,17] leaves two free parameters which are to be determined from known value of the tangential pressure on the event horizon and the luminosity. It should be noted that it is, in a sense, a minimal choice. On the other hand, in Ref. [18] Visser chooses to work in the physical space and uses a decomposition of the stress tensor allowed by the covariant conservation equation. The tangential component of $\langle T^\mu_\nu \rangle_{\text{ren}}$ is also taken to be of the form (1) with $N=6$, this time however in Schwarzschild spacetime. The resulting model has three free parameters which are determined by performing global unconstrained fit to the numerical data. The tensors obtained in both models have a similar structure but differ in the numerical coefficients.

In this note we shall show that, although differently motivated, Visser's model may be obtained following the steps of Refs. [16,17] with $N=8$. The simplest $N=5$ models proposed by Vaz [10] and by Barrioz and Vaz [11] has no free parameters, it should be noted, however, that the latter uses different decomposition of the stress tensor in the conformal space. In our constructions scalar, spinor, and vector fields

TABLE I. The coefficients $a(s)$, $b(s)$, $c(s)$, and $p(s)$ for fields of helicity s. For $s=1$ the coefficient c is nonzero as predicted by point separation and ζ-function renormalization. Dimensional regularization gives $c(1)=0$.

s	0	$1/2$	1
$5760\pi^2 a(s)$	3	9	36
$5760\pi^2 b(s)$	-1	$-11/2$	-62
$5760\pi^2 c(s)$	0	0	-60
$p(s)$	1	$7/4$	2

*Email address:
matyjase@tytan.umcs.lublin.pl;jurek@iris.umcs.lublin.pl

are treated simultaneously. Presented method (with $n=1$) may be also employed in similar calculations in Hartle-Hawking state.

II. CONFORMAL TRANSFORMATIONS OF THE STRESS-ENERGY TENSOR

Transformational properties of the stress tensor of a conformally coupled massless field under the scaling transformation $\tilde{g}_{\mu\nu}=e^{-2\omega}g_{\mu\nu}$ are described by or may be obtained from the formulas derived by Page [4], Brown and Ottewill [5], Brown, Page, and Ottewill [7], and Dowker [41]. Under the conformal transformation the one loop effective action of the massless conformally invariant fields transforms according to the rule [5]

$$W_R[g_{\mu\nu}]=W_R[e^{-2\omega}g_{\mu\nu}]+aA[\omega;g]+bB[\omega,g]$$
$$+cC[\omega,g], \qquad (2)$$

where

$$A[\omega,g]=\int d^4x(-g)^{1/2}\left\{\left(\text{Riem}^2-2\text{Ricc}^2+\frac{1}{3}R^2\right)\omega\right.$$
$$\left.+\frac{2}{3}[R+3(\Box\omega-\kappa)](\Box\omega-\kappa)\right\}, \qquad (3)$$

$$B[\omega,g]=\int d^4x(-g)^{1/2}[(\text{Riem}^2-4\text{Ricc}^2+R^2)\omega$$
$$+4R_{\mu\nu}\omega^{;\mu}\omega^{;\nu}-2R\kappa+2\kappa^2-4\kappa\Box\omega], \qquad (4)$$

$$C[\omega,g]=\int d^4x(-g)^{1/2}\{[R+3(\Box R-\kappa)](\Box R-\kappa)\}, \qquad (5)$$

and $\kappa=\omega_{;\alpha}\omega^{;\alpha}$. We have distinguished quantities evaluated in the conformal space with a tilde. The spin-dependent coefficients a, b, and c are given in Table I.

Functionally differentiating (2) with respect to the metric tensor and restricting to the Ricci-flat spaces one has

$$\langle T^\mu_\nu\rangle_{\text{ren}}=\exp(-4\omega)\tilde{T}^\mu_\nu+a(s)A^\mu_\nu+b(s)B^\mu_\nu+c(s)C^\mu_\nu, \qquad (6)$$

where

$$A^{\mu\nu}=8R^{\alpha\mu\nu\beta}\omega_{;\alpha\beta}-\frac{4}{3}\kappa^{;\mu\nu}+2g^{\mu\nu}\left(2\omega^{;\alpha}\kappa_{;\alpha}+\kappa^2+\frac{2}{3}\Box\kappa\right)$$
$$-8\kappa^{;(\mu}\omega^{;\nu)}-8\omega^{;\mu}\omega^{;\nu}\kappa, \qquad (7)$$

$$B^{\mu\nu}=8R^{\alpha\mu\nu\beta}\omega_{;\alpha\beta}+8R^{\alpha\mu\nu\beta}\omega_{;\alpha}\omega_{;\beta}$$
$$-8\omega^{;\mu\alpha}\omega_{;\alpha}{}^{;\nu}-8\kappa^{;(\mu}\omega^{;\nu)}$$
$$-8\kappa\omega^{;\mu}\omega^{;\nu}+4g^{\mu\nu}\left(\omega_{;\alpha\beta}\omega^{;\alpha\beta}+\kappa_{;\alpha}\omega^{;\alpha}+\frac{1}{2}\kappa^2\right), \qquad (8)$$

and

$$C^{\mu\nu}=g^{\mu\nu}(2\Box\kappa+3\kappa^2+6\omega_{;\alpha}\kappa^{;\alpha})$$
$$-12\kappa\omega^{;\mu}\omega^{;\nu}-12\kappa^{;(\mu}\omega^{;\nu)}-2\kappa^{;\mu\nu}. \qquad (9)$$

The renormalized effective stress tensor has been defined by

$$\langle T^{\mu\nu}\rangle_{\text{ren}}=2(-g)^{-1/2}\frac{\delta W_R}{\delta g_{\mu\nu}}. \qquad (10)$$

We observe that since Eq. (6) is a general formula its meaning is clear: the better approximation of the stress tensor in the conformal space is constructed the better $\langle T^\mu_\nu\rangle_{\text{ren}}$ in the physical is obtained. There is nothing less general in working in the conformal space as long as the transformation formulas are correct. Moreover, although we heavily used scaling properties of the renormalized stress tensor the adopted method is neither the Page nor Brown and Ottewill approximation.

Taking the conformal factor in the form $\omega=1/2\,\ln|g_{00}|$, one obtains for A^μ_ν, B^μ_ν, and C^μ_ν the following formulas:

$$A^t_t=\frac{x^6(128-240x+113x^2)}{128M^4(1-x)^2}, \qquad (11)$$

$$A^r_r=\frac{x^6(32-96x+63x^2)}{384M^4(1-x)^2}, \qquad (12)$$

$$A^\theta_\theta=\frac{-x^6(64-120x+57x^2)}{384M^4(1-x)^2}, \qquad (13)$$

$$B^t_t=\frac{3x^6(48-96x+47x^2)}{128M^4(1-x)^2}, \qquad (14)$$

$$B^r_r=\frac{x^6(16-48x+33x^2)}{128M^4(1-x)^2}, \qquad (15)$$

$$B^\theta_\theta=\frac{-x^6(32-72x+39x^2)}{128M^4(1-x)^2}, \qquad (16)$$

$$C^t_t=\frac{3x^6(32-64x+33x^2)}{256M^4(1-x)^2}, \qquad (17)$$

$$C^r_r=\frac{-(8-9x)^2x^6}{256M^4(1-x)^2}, \qquad (18)$$

$$C^\theta_\theta=\frac{x^6(128-264x+135x^2)}{256M^4(1-x)^2}. \qquad (19)$$

The geometrical terms, i.e., the sum of the last three terms in the right hand side of Eq. (6) have, therefore, a simple form, proportional to simple polynomials in x multiplied by the second power of $1/g_{tt}$.

It is a well known property of the massless and conformally coupled fields that although on the classical level the

trace of the stress energy tensor identically vanish, after quantization it acquires nonzero value. Making use of the identity

$$\frac{\delta}{\delta\omega}S[e^{-2\omega}g_{\mu\nu}]|_{\omega=0}=-2g_{\sigma\tau}\frac{\delta}{\delta g_{\sigma\tau}}S[g_{\mu\nu}], \quad (20)$$

where S is a functional, one concludes that the trace anomaly is given by a general formula

$$\langle T^{\mu}_{\mu}\rangle_{\rm ren}=a(s)\left(\mathcal{H}+\frac{2}{3}\Box R\right)+b(s)\mathcal{G}+c(s)\Box R, \quad (21)$$

where

$$\mathcal{H}={\rm Riem}^{2}-2{\rm Ricc}^{2}+\frac{1}{3}R^{2} \quad (22)$$

and

$$\mathcal{G}={\rm Riem}^{2}-4{\rm Ricc}^{2}+R^{2}. \quad (23)$$

It is a very fortunate coincidence that for the scalar and spinor field $\langle \widetilde{T}^{\mu}_{\mu}\rangle_{\rm ren}$ in the optical metric is zero. However, choosing for the vector field the coefficient $c(1)$ as predicted by ζ—function renormalizaton requires some care. Indeed, in this case the trace of $\langle \widetilde{T}^{\mu}_{\nu}\rangle_{\rm ren}$ does not vanish and, of course, further calculations should reflect this fact.

The conservation equation for the line element of the conformal space

$$\widetilde{\nabla}_{\mu}\langle \widetilde{T}^{\mu}_{\nu}\rangle_{\rm ren}=0 \quad (24)$$

reduces to

$$\frac{\partial}{\partial x}\langle \widetilde{T}^{r}_{t}\rangle_{\rm ren}-\frac{2(1-2x)}{x(1-x)}\langle \widetilde{T}^{r}_{t}\rangle_{\rm ren}=0, \quad (25)$$

and

$$\frac{\partial}{\partial x}\langle \widetilde{T}^{r}_{r}\rangle_{\rm ren}-\frac{2-3x}{x(1-x)}\langle \widetilde{T}^{r}_{r}\rangle_{\rm ren}+\frac{2-3x}{x(1-x)}\langle \widetilde{T}^{\theta}_{\theta}\rangle_{\rm ren}=0. \quad (26)$$

Therefore one concludes that the stress tensor in the conformal space naturally splits into two parts

$$\langle \widetilde{T}^{\mu}_{\nu}\rangle_{\rm ren}=\mathcal{T}^{\mu}_{\nu}+\frac{9c}{8M^{4}}x^{6}(1-x)^{2}\delta^{\mu}_{0}\delta^{0}_{\nu}, \quad (27)$$

where $\mathcal{T}^{\mu}_{\nu}$ is a conserved traceless tensor in the conformal space. Such a decomposition is allowed by the covariant conservation equation in the ultrastatic space and its exact form is constructed from Eq. (21) evaluated in the optical space. Similar term has been introduced into the Page approximation by Zannias [6]. It should be noted that in a case of the dimensional renormalization the second term in the right hand side of Eq. (27) is absent.

III. THE MODEL

As has been said the idea of constructing $\langle T^{\mu}_{\nu}\rangle_{\rm ren}$ in Schwarzschild spacetime is to accept Eq. (1) and solve the conservation equation in the ultrastatic space. In order to reduce the number of unknown parameters we transform Eq. (1) to the physical space and make use of the regularity condition

$$\lim_{x\to 1}|\langle T^{\theta}_{\theta}\rangle_{\rm ren}|<\infty. \quad (28)$$

Inserting Eqs. (11)–(19) into Eq. (6) with appropriate numerical coefficients and making use of Eq. (28) yields

$$a_{7}=\frac{a}{48pt}-\frac{b}{32pt}+\frac{c}{32pt}-4a_{4}-3a_{5}-2a_{6} \quad (29)$$

and

$$a_{8}=-\frac{7a}{384pt}+\frac{7b}{128pt}+\frac{7c}{256t}-7+3a_{4}+2a_{5}+a_{6}, \quad (30)$$

where $t=M^{4}T$. After substitution of Eqs. (29) and (30) into Eq. (1) and performing integration of Eqs. (24) and (25) in the conformal space one has

$$\langle \widetilde{T}^{r}_{t}\rangle_{\rm ren}=-x^{2}(1-x)^{2}\frac{K}{4M^{4}}, \quad (31)$$

and the radial component of the stress tensor in the conformal space

$$\begin{aligned}
\langle \widetilde{T}^{r}_{r}\rangle_{\rm ren}=pT\Bigg[& \left(\left(\frac{a}{384pt}-\frac{b}{128pt}+\frac{c}{256pt}-d\right)x^{2}\right. \\
& -\left(\frac{a}{384pt}-\frac{b}{128pt}+\frac{c}{256pt}-d\right)x^{3}-a_{4}x^{4} \\
& +\left(\frac{2}{3}a_{4}-\frac{2}{3}a_{5}\right)x^{5}+\left(\frac{1}{3}a_{4}+\frac{5}{12}a_{5}-\frac{1}{2}a_{6}\right)x^{6} \\
& +\left(\frac{b}{40pt}-\frac{a}{120pt}-\frac{c}{80pt}-\frac{16}{5}+\frac{9}{5}a_{4}\right. \\
& \left.+\frac{29}{20}a_{5}+\frac{11}{10}a_{6}\right)x^{7}+\left(\frac{7}{640pt}-\frac{21b}{640pt}+\frac{21c}{1280pt}\right. \\
& \left.+\frac{21}{5}-\frac{9}{5}a_{4}-\frac{6}{5}a_{5}-\frac{3}{5}a_{6}\right)x^{8}\Bigg],
\end{aligned} \quad (32)$$

where K is an integration constant connected to the luminosity and d is another integration constant. Note that the leading behavior of Eq. (32) as $x\to 0$ is proportional to x^{2} as expected.

Now, on general grounds one expects that at large r the leading terms of $-\langle T^{t}_{t}\rangle_{\rm ren}$ and $\langle T^{r}_{r}\rangle_{\rm ren}$ should be equal to $\langle T^{t}_{r_{*}}\rangle_{\rm ren}$, where r_{*} is the Regge-Wheeler coordinate. Moreover, the Christensen-Fulling regularity conditions in the

Schwarzschild space, i.e., conditions for the regularity of the stress-energy tensor on the future event horizon,

$$|\langle T_{vv}\rangle_{\text{ren}}| < \infty, \tag{33}$$

$$|\langle T_t^t\rangle_{\text{ren}} + \langle T_r^r\rangle_{\text{ren}}| < \infty, \tag{34}$$

and

$$(1-x)^{-2}|\langle T_{uu}\rangle_{\text{ren}}| < \infty \tag{35}$$

as $x \to 1$, where

$$\langle T_{uu}\rangle_{\text{ren}} = \frac{1}{4}\left[(1-x)(\langle T_r^r\rangle_{\text{ren}} - \langle T_t^t\rangle_{\text{ren}}) + \frac{2K}{Mr^2}\right], \tag{36}$$

and

$$\langle T_{vv}\rangle_{\text{ren}} = \langle T_{uu}\rangle_{\text{ren}} - \frac{K}{M^2 r^2} \tag{37}$$

allow us to reduce the number of unknown parameters to three. These conditions together with Eq. (28) ensure that the stress-energy tensor measured in the local frames on the future event horizon will be finite.

The conditions (28) and (34) are already satisfied while the remaining ones after simple calculations give

$$K = pt\left(-\frac{14}{15}a_4 - \frac{17}{30}a_5 - \frac{1}{6}a_6\right) + \frac{11a}{60} + \frac{b}{5} - \frac{c}{10} \tag{38}$$

and

$$d = \frac{7}{30}a_4 + \frac{17}{120}a_5 + \frac{1}{20}a_6 - \frac{83a}{1920pt} - \frac{37b}{640pt} + \frac{37c}{1280pt}. \tag{39}$$

Returning to the physical space, after some algebra, one obtains the mean value of the energy momentum tensor of the quantized, massless, conformally invariant field in the Unruh state

$$\langle T_t^t\rangle_{\text{ren}} = \frac{pT}{1-x}\left\{\left(\frac{c}{40pt} - \frac{11a}{240pt} - \frac{b}{20pt} + \frac{7}{30}a_4 + \frac{17}{120}a_5 + \frac{a_6}{20}\right)x^2 - a_4 x^4 - \left(\frac{5}{3}a_4 + \frac{4}{3}a_5\right)x^5\right.$$
$$\left. + \left(\frac{a}{pt} + \frac{9b}{8pt} - \frac{3c}{4pt} - 2a_4 - \frac{7}{4}a_5 - \frac{3}{2}a_6\right)x^6 + \left(\frac{7c}{10pt} - \frac{109a}{120pt} - \frac{41b}{40pt} + \frac{21}{5}a_4 + \frac{14}{5}a_5 + \frac{7}{5}a_6\right)x^7\right\}, \tag{40}$$

$$\langle T_r^r\rangle_{\text{ren}} = \frac{pT}{1-x}\left\{\left(\frac{11a}{240pt} + \frac{b}{20pt} - \frac{c}{40pt} - \frac{7}{30}a_4 - \frac{17}{120}a_5 - \frac{a_6}{20}\right)x^2 - a_4 x^4 - \left(\frac{1}{3}a_4 + \frac{2}{3}a_5\right)x^5\right.$$
$$\left. + \left(\frac{a}{12pt} + \frac{b}{8pt} - \frac{c}{4p} - \frac{1}{4}a_5 - \frac{1}{2}a_6\right)x^6 + \left(\frac{3c}{10pt} - \frac{9a}{40pt} - \frac{9b}{40pt} + \frac{9}{5}a_4 + \frac{6}{5}a_5 + \frac{3}{5}a_6\right)x^7\right\}, \tag{41}$$

$$\langle T_\theta^\theta\rangle_{\text{ren}} = pT\left\{a_4 x^4 + (2a_4 + a_5)x^5 + \left(\frac{c}{4pt} - \frac{a}{6pt} - \frac{b}{4pt} + 3a_4 + 2a_5 + a_6\right)x^6\right\}, \tag{42}$$

and

$$\langle T_t^r\rangle_{\text{ren}} = -x^2 \frac{K}{4M^4}, \tag{43}$$

where K is given by

$$K = \frac{1}{60}(11a + 12b - 6c) - \frac{pT}{30}(28a_4 + 17a_5 - 6a_6). \tag{44}$$

Generalizations to greater N are obvious, however, it seems that the more complicated formulas are of little use. It should stressed that, by construction, obtained tensors satisfy all regularity and consistency requirements.

IV. DISCUSSION

In order to compare just obtained $\langle T_\nu^\mu\rangle_{\text{ren}}$ with those constructed by Visser in Ref. [18] first we introduce a new set of unknown parameters k_i defined as

$$a_4 = k_4, \tag{45}$$

$$a_5 = -2k_4 + k_5, \tag{46}$$

and

$$a_6 = k_4 - 2k_5 + k_6 + \frac{1}{pt}\left(\frac{a}{6} + \frac{b}{4} - \frac{c}{2}\right). \tag{47}$$

In terms of k_i the stress tensor becomes

$$\langle T^t_t \rangle_{\text{ren}} = \frac{pTx^2}{1-x}\left\{ \frac{k_5}{24} + \frac{k_6}{20} - \frac{3}{80pt}(a+b) - k_4 x^2 + \left(k_4 - \frac{4}{3}k_5\right)x^3 + \left[\frac{3}{4pt}(a+b) + \frac{5}{4}k_5 - \frac{3}{2}k_6\right]x^4 + \left[\frac{7}{5}k_6 - \frac{27}{40pt}(a+b)\right]x^5\right\},$$

$$(48)$$

$$\langle T^r_r \rangle_{\text{ren}} = \frac{pTx^2}{1-x}\left\{ \frac{3}{80pt}(a+b) - \frac{k_5}{24} - \frac{k_6}{20} - k_4 x^2 + \left(k_4 - \frac{2}{3}k_5\right)x^3 + \left(\frac{3}{4}k_5 - \frac{k_6}{2}\right)x^4 + \left[\frac{3}{5}k_6 - \frac{3}{40pt}(a+b)\right]x^5\right\}, \qquad (49)$$

and

$$\langle T^\theta_\theta \rangle_{\text{ren}} = pTx^4(k_4 + k_5 x + k_6 x^2). \qquad (50)$$

The net flux is described by Eq. (43), where the integration constant K is expressed in terms of k_i as

$$K = \frac{1}{60}[9a + 9b - 2(5k_5 + 6k_6)pt]. \qquad (51)$$

Eqs. (48)–(51) are equivalents of the $\langle T^\mu_\nu \rangle_{\text{ren}}$ constructed recently by Visser [18].

The parameters a_4, a_5, a_6 or equivalently k_4, k_5, k_6 are to be determined from the available numerical data. Such a procedure has proven to be very useful in constructing highly accurate analytical approximations to the exact stress tensor. In it simplest form one needs the horizon value of, say, $\langle T^\theta_\theta \rangle_{\text{ren}}$ the luminosity and one additional piece of information. When the results of detailed numerical calculations are

known one may perform unconstrained fit to the totality of the available data. In this approach known value of the luminosity is used by Eq. (51) as a consistency check rather than an input. This procedure has been adopted recently by Visser in the case of the scalar field.

Unfortunately, detailed calculations, even if executed, are rarely published, rather, the overall character of the stress tensor is presented graphically. However, in the static and spherically-symmetric geometries it is relatively easy to construct the asymptotic characteristics of $\langle T^\mu_\nu \rangle_{\text{ren}}$. Consequently, with $\langle T^\theta_\theta(1) \rangle_{\text{ren}}$ and the luminosity treated as input one may construct reasonable approximation. Indeed, taking into account a more restrictive hypothesis $N=7$, one concludes that the additional constraint

$$3a_4 + 2a_5 + a_6 - \frac{1}{384pt}\left(7a + 21b + \frac{21}{2}c\right) = 0 \qquad (52)$$

results, after some rearrangement, in the stress tensor

$$\langle T^t_t \rangle_{\text{ren}} = -\frac{x^2}{4M^4(1-x)}\left[K + \left(4h + 24K - \frac{119}{32}a - \frac{123}{32} + \frac{27}{64}c\right)x^2 - \left(4h + 56K - \frac{303}{32}a - \frac{339}{32}b + \frac{234}{64}c\right)x^3\right.$$
$$\left. + \left(30K - \frac{297}{32}a - \frac{357}{32}b + \frac{405}{64}c\right)x^4 + \left(\frac{113}{32}a + \frac{141}{32}b - \frac{189}{64}c\right)x^5\right], \qquad (53)$$

$$\langle T^r_r \rangle_{\text{ren}} = \frac{x^2}{4M^4(1-x)}\left[K - \left(4h + 24K - \frac{119}{32}a - \frac{123}{32}b + \frac{27}{64}c\right)x^2 + \left(4h + 40K - \frac{211}{32}a - \frac{321}{32}b + \frac{135}{64}c\right)x^3\right.$$
$$\left. - \left(18K - \frac{113}{32}a - \frac{141}{32}b + \frac{189}{64}c\right)x^4 - \left(\frac{21}{32}a + \frac{33}{32}b - \frac{81}{64}c\right)x^5\right], \qquad (54)$$

and

$$\langle T^\theta_\theta \rangle_{\text{ren}} = \frac{x^4}{M^4}\left[h + 6K - \frac{119}{128}a - \frac{123}{128}b - \frac{27}{256}c - \left(6K - \frac{69}{64}a - \frac{81}{64}b + \frac{81}{128}c\right)x - \left(\frac{19}{128}a + \frac{39}{128}b - \frac{135}{256}c\right)x^2\right], \qquad (55)$$

where $h = M^4 \langle T^\theta_\theta(2M) \rangle$. It is interesting to note that both $N=8$ and $N=7$ hypotheses yield the same structure of the renormalized stress tensor in the Unruh state. Substituting the spin-dependent coefficients taken from Table I for scalar and vector fields one easily obtains results presented in Refs. [16] and [17]. In a case of the conformal vector field the coefficient $c(1)$ has been taken as predicted by ζ-function

renormalization. Therefore, one can draw a conclusion that from the point of view of the applied method the only difference between the results of Refs. [16,17] and [18] is the choice of N in Eq. (1). On the other hand, Barrioz and Vaz [11] take $N=5$ (i.e., there are no free parameters left) and use more complicated decomposition of $\langle \tilde{T}^\mu_\nu \rangle_{\text{ren}}$ in the optical space

$$\langle \tilde{T}^{\mu}_{\nu} \rangle_{\mathrm{ren}} = \mathcal{T}^{\mu}_{\nu} + (\alpha \delta^{\mu}_0 \delta^0_{\nu} + \beta \delta^{\mu}_{\nu}) \langle \tilde{T}^{\sigma}_{\sigma} \rangle_{\mathrm{ren}}, \qquad (56)$$

where α and β are coefficients subjected to the obvious condition $\alpha + 4\beta = 1$.

V. CONCLUDING REMARKS

In this work our goal was to construct $\langle T^{\mu}_{\nu} \rangle_{\mathrm{ren}}$ in the Unruh state and to investigate how the choice of N in Eq. (1) affects the resulting stress-energy tensor. Although our analyses have been limited to $N \leq 8$ it seems that a three parameter family of the stress tensor is of sufficient generality. Since apparently the ambitious plan to construct the approximate stress tensor in the Unruh state using the polynomial ansatz and appropriate regularity conditions as the only available *a priori* information has failed it seems that the presented method (or the methods closely related) are the

only one which would give analytical formulas able to reconstruct the exact $\langle T^{\mu}_{\nu} \rangle_{\mathrm{ren}}$ to a high accuracy. Moreover, the price one should pay for such quality of the approximation is rather small: just two or three pieces of numeric data. As far as we know there are neither numerical calculations nor asymptotic analyses concerning the vacuum polarization effect of the conformally coupled massless spinor field in the Schwarzschild spacetime and consequently the stress-energy tensor cannot be determined completely. We expect however, that the general formulas supplemented by additional pieces of numerical data would give a good approximation of the exact stress tensor in this case also.

Finally, we remark that a similar method, with different asymptotics may be used in construction of $\langle T^{\mu}_{\nu} \rangle_{\mathrm{ren}}$ in the Hartle-Hawking state, specifically, the results of Refs. [14] may be rederived [42]. We intend to return to this group of problems in a separate publication.

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