Stringy mass squared splittings reexamined

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An empirically successful mass formula derived long ago from hadronic string models is not explained by symmetries of the QCD Lagrangian. Consider that $M_\rho^2 - M_\pi^2 = M_{K^*}^2 - M_K^2$ to a few percent accuracy. We derive this and other equal spacing relations using chiral symmetry and several assumptions about the chiral transformation properties of the hadronic mass-squared matrix. We discuss the possible significance of these assumptions in the context of QCD. [S0556-2821(99)06001-4]

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I. INTRODUCTION

Consider the meson mass-squared splittings listed in Table I [1]. Why are these splittings equal to within a few percent? The near equality of the $D-D^*$ and $B-B^*$ mass-squared splittings is well understood as a consequence of heavy quark symmetry. However, the equality of all other meson pairs is puzzling since there is no obvious QCD symmetry which relates them. The near equality of the pairs could be accidental. There are rigorous mass formulas for heavy hadrons that also happen to work well for light hadrons, providing a similar mystery [2]. However, the equal spacing relations of Table 1 are special in that they are predicted by hadronic string models made consistent with chiral symmetry constraints [3]. The fact that the mass-squared splittings can be understood on the basis of regularity, albeit regularity whose link with QCD is not clear, suggests that they are not accidental. In the modern view, a demonstration of these relations will be convincing only if there is a symmetry argument based in QCD. The only post-sixties attempt to understand these relations that we are aware of is in modern string theory [4].

In this paper we show that these relations can be understood using $SU(N)_L \times SU(N)_R$ with $N=2,3$ together with an ansatz for the chiral transformation properties of the hadronic mass-squared matrix. This ansatz completely determines the reducible chiral representations filled out by mesons. The basic results for the chiral representations have been found by Weinberg using algebraic sum rules [5,6]. What is new here is the inclusion of explicit chiral symmetry breaking effects in the mass-squared matrix, the extension of the basic multiplets to three flavors of quarks and a discussion of the chiral representations of the heavy mesons. Our derivation of the equal spacing relations is related to the old string derivation; the constraints implied by the full chiral algebra and the mass-squared matrix ansatz are equivalent to standard assumptions of Regge asymptotic behavior in pion-hadron scattering [5], which are automatically incorporated in hadronic string models. What significance does the ansatz have from the point of view of QCD? We will offer a speculative interpretation of the ansatz in the context of lattice QCD. We will argue that the mass-squared matrix constraints are equivalent to a Z_2 symmetry which is a manifestation of chiral anomalies in a gauge theory with an intrinsic cutoff.

This paper is organized as follows. In Sec. II we discuss our basic assumptions and using these assumptions find the chiral representation filled out by the Goldstone bosons. We show that this representation implies equal spacing relations for the Goldstone bosons and their chiral partners. This is the most important section of the paper. Although Sec. II C presents material that is well known [5–7], it is essential for what follows. In Sec. III we consider the two-flavor chiral representation involving η and compare with Sec. II where η is a Goldstone boson of three-flavor QCD. Consistency of the two- and three-flavor multiplets unambiguously determines members of the 0^{++} scalar octet and requires octet-singlet mixing. In Section IV we consider the chiral representations of the $I=\frac{1}{2}$ mesons. We find the kaon representations and compare with the results of Sec. II where the kaons are in the Goldstone boson representation of three-flavor QCD. This enables us to express the kaon mass-squared splittings in terms of an $I=\frac{1}{2}$ matrix element, which we conjecture to be universal. We then construct the chiral representations of the heavy mesons consistent with heavy quark symmetry, and show that the heavy meson mass-squared splittings are related to the kaon splittings by the universal $I=\frac{1}{2}$ matrix element. We discuss the baryons briefly in Sec. V and provide an independent test of the universality conjecture. In Sec. VI we discuss the relation between our derivation of the equal spacing relations and the

TABLE I. Lowest lying mesons of a given character. Masses are central values from the particle data group [1], and we have defined $\alpha' \equiv 0.88\ \mathrm{GeV}^{-2}$ so as to normalize the ρ-π splitting to 0.50. The η_8 and ϕ_8 masses are defined by the Gell-Mann–Okubo formulas (see below).

A^*-A	$\alpha'(M_{A^*}^2 - M_A^2)$
$\rho-\pi$	0.50
K^*-K	0.49
$\phi_8-\eta_8$	0.48
D^*-D	0.48
B^*-B	0.43

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old hadronic string derivation. Finally, we give an interpretation of our ansatz for the hadronic mass-squared matrix in the context of lattice QCD. We summarize and conclude in Sec. VII.

II. THE GOLDSTONE QUARTET

A. Mended chiral symmetry

In a theory where a symmetry G is spontaneously broken, G is not a symmetry of the vacuum, but G is still a symmetry of the theory. The noninvariance of the vacuum makes it difficult to recover consequences of the symmetry in the broken phase. However, those consequences are there and they are important. Fortunately, in special Lorentz frames the QCD vacuum can be invariant with respect to chiral symmetry even when chiral symmetry is spontaneously broken [5,8]. The infinite momentum frame provides an intuitive picture of how this can occur. If a system is boosted to infinite momentum these is a sense in which the vacuum decouples and is thereby rendered irrelevant [8]. In the infinite momentum frame helicity is conserved and so, for each helicity, hadrons can be classified in representations of the full chiral algebra in the broken phase [8,5]. Since degeneracies in the broken phase are uncommon, the chiral representations are generally reducible, and so do allow nontrivial splitting of states. The symmetry constraints on the hadronic mass-squared matrix will be the focus of this paper. The chiral representations also constrain amplitudes for pion emission and absorption as discussed in many places [5–7,9]. Throughout this paper it is assumed that we are working in Lorentz frames in which helicity is conserved.

Consider QCD with two massless flavors. There is an anomaly free $SU(2)_L\times SU(2)_R$ symmetry which is assumed to be spontaneously broken to $SU(2)_V$ (isospin). We assume that the spectrum is confined. Hadrons fall into isospin multiplets. For each helicity, hadrons also fill out representations of SU(2)×SU(2) [5]. Since all mesons have zero helicity states, we will consider only zero helicity in this paper.

Meson states carry isospin 0 or 1 and therefore transform as combinations of (2,2), (3,1), (1,3), and (1,1) irreducible representations of SU(2)×SU(2). Charge conjugation leaves (2,2) and (1,1) unchanged and interchanges (1,3) and (3,1). Physical meson states have definite charge conjugation and isospin and therefore are linear combinations of the isovectors $|2,2\rangle_a$, $\{|1,3\rangle_a-|3,1\rangle_a\}/\sqrt{2}\equiv|V\rangle_a$, and $\{|1,3\rangle_a+|3,1\rangle_a\}/\sqrt{2}\equiv|A\rangle_a$, and the isoscalars $|2,2\rangle_4$ and $|1,1\rangle$. Roman subscripts are isospin indices. Only $|V\rangle_a$ changes sign under charge conjugation.

B. An ansatz for the mass-squared matrix

In helicity conserving Lorentz frames the hadronic mass-squared matrix $\hat{M}^2$ is the natural object to study [5,8]. We assume that the mass-squared matrix can be written as

$$\hat{M}^2=\hat{M}_0^2+\hat{M}^2_{\langle\overline{q}q\rangle}. \tag{1}$$

This is the statement that all mass-squared splittings between hadrons in a given chiral multiplet transform like the chiral order parameter $\langle\overline{q}q\rangle$; i.e., like (2,2) with respect to SU(2)×SU(2). $\hat{M}_0^2$ transforms like a chiral singlet. The only products of the allowed irreducible representations that contain (2,2) are $(2,2)\otimes(3,1)$, $(2,2)\otimes(1,3)$, and $(2,2)\otimes(1,1)$. It is clear that a chiral representation with mass-squared splittings must be reducible. All states in an irreducible representation must be degenerate.

The symmetry decomposition of the mass-squared matrix of Eq. (1) and the allowed representations allow arbitrarily complicated chiral representations for each helicity [5]. In Ref. [6] Weinberg showed that the further constraint that the two parts of the mass-squared matrix commute,

$$[\hat{M}_0^2,\hat{M}^2_{\langle\overline{q}q\rangle}]=0, \tag{2}$$

completely determines the chiral representations and the mixing angles of the reducible representations filled out by the mesons. This constraint is equivalent to a special superconvergent sum rule in pion-hadron scattering [5] and will be discussed in detail in Sec. VI.

In this paper we will assume that Eqs. (1) and (2) are satisfied by the hadronic mass-squared matrix. They are the fundamental assumptions made in this paper. In Sec. VI we will discuss how the mass-squared matrix constraints are related to assumptions of Regge asymptotic behavior and we will suggest a possible interpretation from the point of view of QCD.

C. The Goldstone multiplet

The assumption that $\hat{M}_0^2$ and $\hat{M}^2_{\langle\overline{q}q\rangle}$ commute determines the reducible chiral representations filled out by the light mesons [5–7]. There are two reducible chiral representations consistent with the ansatz for the mass-squared matrix.[1]

(a) $(2,2)\otimes(3,1)$ and $(2,2)\otimes(1,3)$:

$$|\mathrm{I}\rangle_a=\frac{1}{\sqrt{2}}\{|2,2\rangle_a-|A\rangle_a\},\quad M^2_{\mathrm{I}}=\mu^2-\delta,$$

$$|\mathrm{II}\rangle_a=\frac{1}{\sqrt{2}}\{|2,2\rangle_a+|A\rangle_a\},\quad M^2_{\mathrm{II}}=\mu^2+\delta, \tag{3}$$

$$|\mathrm{III}\rangle=|2,2\rangle_4,\quad |\mathrm{IV}\rangle_a=|V\rangle_a,\quad M^2_{\mathrm{III}}=M^2_{\mathrm{IV}}=\mu^2.$$

States $|\mathrm{I}\rangle$, $|\mathrm{II}\rangle$, and $|\mathrm{III}\rangle$ have charge conjugation sign $\pm\epsilon$, $|\mathrm{IV}\rangle$ has sign $\mp\epsilon$. In the right column we exhibit the mass relations implied by the representation content. The lowest lying member of this quartet must be an isovector.

(b) $(2,2)\otimes(1,1)$

[1] In what follows, μ^2 and δ represent generic elements of the mass-squared matrix which transform as (1,1) and (2,2), respectively; that is, $\mu^2\in\hat{M}_0^2$ and $\delta\in\hat{M}^2_{\langle\overline{q}q\rangle}$.

$$|\mathrm{I}\rangle=\frac{1}{\sqrt{2}}\{|2,2\rangle_4-|1,1\rangle\},\quad M_{\mathrm{I}}^2=\mu^2-\delta,$$

$$|\mathrm{II}\rangle=\frac{1}{\sqrt{2}}\{|2,2\rangle_4+|1,1\rangle\},\quad M_{\mathrm{II}}^2=\mu^2+\delta, \tag{4}$$

$$|\mathrm{III}\rangle_a=|2,2\rangle_a,\quad M_{\mathrm{III}}^2=\mu^2.$$

These states have the same charge conjugation sign. The lowest lying member of this triplet must be an isoscalar. We will see examples of both (a) and (b) below.

Since the pion must be the lowest lying member of its chiral representation, the pion must be in a representation of type (a). In the case of zero-helicity normality, $\Pi\equiv P(-1)^J$, is conserved, where P is intrinsic parity and J is spin. Since π has $\Pi=-1$, it follows that $\Pi_{\mathrm{II}}=-1,\Pi_{\mathrm{III}}=\Pi_{\mathrm{IV}}=1$. Since ΠG, where G is G parity, commutes with the chiral algebra, the zero-helicity mesons fall into distinct sectors labeled by ΠG [5]. The pion has $\Pi G=+1$ as do all states in its chiral representation. The quantum number assignments are discussed in detail in Ref. [5]. Following Ref. [5], we will assume that the pion is joined in this representation by a scalar ϵ $(\Pi=+1)$, and the zero-helicity components of ρ $(\Pi=+1)$ and a_1 $(\Pi=-1)$. We identify $|\mathrm{I}\rangle_a=|\pi\rangle_a$, $|\mathrm{II}\rangle_a=|a_1\rangle_a^{(0)}$, $|\mathrm{III}\rangle=|\epsilon\rangle$, and $|\mathrm{IV}\rangle_a=|\rho\rangle_a^{(0)}$. The superscript denotes helicity. The chiral representation of the pion is then

$$|\pi\rangle_a=\frac{1}{\sqrt{2}}\{|2,2\rangle_a-|A\rangle_a\},$$

$$|a_1\rangle_a^{(0)}=\frac{1}{\sqrt{2}}\{|2,2\rangle_a+|A\rangle_a\}, \tag{5}$$

$$|\epsilon\rangle=|2,2\rangle_4,\quad |\rho\rangle_a^{(0)}=|V\rangle_a.$$

Sandwiching $\hat{M}^2$ between the states of Eq. (5) gives

$$M_\pi^2=M_0^2-M_{\langle\bar{q}q\rangle}^2,$$

$$M_{a_1}^2=M_0^2+M_{\langle\bar{q}q\rangle}^2, \tag{6}$$

$$M_\rho^2=M_\epsilon^2=M_0^2,$$

where we have defined the matrix elements

$$\langle 2,2|\hat{M}_0^2|2,2\rangle=\langle A|\hat{M}_0^2|A\rangle=\langle V|\hat{M}_0^2|V\rangle\equiv M_0^2, \tag{7a}$$

$$\langle 2,2|\hat{M}_{\langle\bar{q}q\rangle}^2|A\rangle\equiv M_{\langle\bar{q}q\rangle}^2. \tag{7b}$$

It is useful to assign spurion transformation properties to the mass-squared matrix elements. By definition, the mass-squared matrix elements transform as $\hat{M}_0^2\to\hat{M}_0^2$ and $\hat{M}_{\langle\bar{q}q\rangle}^2\to L\hat{M}_{\langle\bar{q}q\rangle}^2R^\dagger$ with respect to $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$. Of course in the chiral limit Goldstone's theorem demands $M_0^2=M_{\langle\bar{q}q\rangle}^2$ and it follows that $M_{a_1}^2=2M_\rho^2=2M_\epsilon^2$.

In summary, *a priori*, the chiral representations filled out by light mesons in the broken phase are unknown. Irreducible representations imply degeneracies that are uncommon in the hadron spectrum. On the other hand, reducible representations allow mass-squared splittings and yet can be arbitrarily complicated. However, a reasonable ansatz for the mass-squared matrix reduces this *a priori* infinite number of reducible chiral representations to two representations with fixed mixing angles, which all light mesons with nontrivial mass-squared splittings must fall into. The pion falls into the unique reducible representation that allows massless isovector states [6,7].

D. Explicit chiral symmetry breaking

Assuming two degenerate flavors, to leading order in chiral perturbation theory $M_\pi^2=2Bm_q$, where $m_q=m_u=m_d$ and $B\equiv-\langle\bar{q}q\rangle/2f_\pi^2$ with $\langle\bar{q}q\rangle=\langle\bar{u}u+\bar{d}d\rangle$ [10]. We assign B and m_q the $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ spurion transformation properties

$$\Gamma\to L\Gamma R^\dagger, \tag{8}$$

where $\Gamma=m_q$ or B. These parameters therefore transform as (2,2). The product Bm_q transforms like the quark mass operator in the QCD Lagrangian and so should be an invariant if spurion transformation properties have been properly assigned. In effect, since $(2,2)\otimes(2,2)$ contains the singlet but not (2,2), $Bm_q\in\hat{M}_0^2$. Explicit chiral symmetry breaking effects are accounted for through the substitution $M_0^2\to M_0^2+2Bm_q$ in Eq. (6). With $M_0^2=M_{\langle\bar{q}q\rangle}^2\equiv\Delta$ we then have

$$M_\pi^2=2Bm_q,$$

$$M_{a_1}^2=2Bm_q+2\Delta, \tag{9}$$

$$M_\rho^2=M_\epsilon^2=2Bm_q+\Delta,$$

from which follows the equal-spacing relation

$$M_\rho^2-M_\pi^2=M_{a_1}^2-M_\rho^2=\Delta. \tag{10}$$

Note that these mass-squared splittings are independent of $O(m_q)$ explicit chiral symmetry breaking effects. This is so because the quark mass contribution to the mass-squared matrix at leading order in chiral perturbation theory is contained in $\hat{M}_0^2$ whereas $\Delta\in\hat{M}_{\langle\bar{q}q\rangle}^2$, a consequence of the structure of the mass-squared matrix implied by the ansatz of Sec. II B. This famous mass-squared relation was originally obtained using spectral function sum rules [11]. We will compare this relation to experiment in the next section.

E. Extension to SU(3)

Consider QCD with three flavors. Mesons states transform as combinations of $(\bar{3},3)$, $(3,\bar{3})$, (8,1), (1,8), and (1,1) irreducible representations of $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$. We now have the combinations $\{|1,8)-|8,1)\}/\sqrt{2}\equiv|V)$, $\{|1,8)+|8,1)\}/\sqrt{2}\equiv|A)$, $\{|3,\bar{3})_8-|\bar{3},3)_8\}/\sqrt{2}\equiv|Y)_8$, and $\{|3,\bar{3})_8$

$+|\overline{3})_8\}/\sqrt{2}\equiv|X)_8$. The subscripts signify that we are singling out the octet components of the $(\overline{3},3)\oplus(3,\overline{3})$ representations. Only $|V)$ changes sign with respect to charge conjugation. The symmetry breaking mass-squared matrix $\hat{M}^2_{\langle\bar{q}q\rangle}$ transforms as $(\overline{3},3)\oplus(3,\overline{3})$ with respect to $SU(3)_L\times SU(3)_R$. Identical arguments as used above give the chiral representations consistent with the ansatz for the mass-squared matrix. The Goldstone boson representation is [6]

$$|\mathrm{I}\rangle=\frac{1}{\sqrt{2}}\{|X)_8-|A)\},$$

$$|\mathrm{II}\rangle=\frac{1}{\sqrt{2}}\{|X)_8+|A)\}, \tag{11}$$

$$|\mathrm{III})=|Y)_8, \quad |\mathrm{IV})=|V),$$

where it is understood that we have kept only the zero-helicity states. We identify $|\mathrm{I})=|\mathcal{P})$, $|\mathrm{II})=|\mathcal{A})$, $|\mathrm{III})=|\mathcal{S})$, and $|\mathrm{IV})=|\mathcal{V})$, where $\mathcal{P}$ is the 0^{-+} Goldstone octet, $\mathcal{A}$ is the 1^{++} axialvector octet, $\mathcal{S}$ is the 0^{++} scalar octet, and $\mathcal{V}$ is the 1^{--} vector octet. Labeling by isospin as $\{1,\frac{1}{2},0\}$ we have $\mathcal{P}=\{\pi,K,\eta_8\}$, $\mathcal{V}=\{\rho,K^*,\phi_8\}$, $\mathcal{S}=\{?,K_0^*,?\}$ and $\mathcal{A}=\{a_1,K_{1A},f_8\}$ [1]. Several comments are in order. The question marks refer to slots that are not unambiguously filled by observed particles.[2] The assignments are consistent with the two-flavor chiral representation of Eq. (5) if we identify ϵ_8 as the $I=0$ member of the scalar octet. The physical ϵ of the previous sections is then a mixture of ϵ_8 with an SU(3) singlet. We postpone further discussion of the scalars to the next section. Generally, the $I=0$ members of the octets mix with SU(3) singlets when there is explicit symmetry breaking. Specifically, η_8 mixes with the singlet η_0 to give η and η', ϕ_8 mixes with the singlet ϕ_0 to give ω and ϕ and f_8 mixes with the singlet f_0 to give $f_1(1285)$ and $f_1(1510)$ [1]. We will further discuss octet-singlet mixing below. Following the particle data group we treat K_{1A} as an equal mixture of $K_1(1270)$ and $K_1(1400)$ [1].

In extending to three flavors we assume that $m_u=m_d\equiv m\neq m_s$. It is straightforward to generalize our results for the mass-squared matrix. The mass-squared matrix elements are replaced with the column vectors

$$\hat{M}^2_{\mathcal{P}}=(M^2_\pi \quad M^2_K \quad M^2_\eta)^T,$$

$$\hat{M}^2_{\mathcal{V}}=(M^2_\rho \quad M^2_{K^*} \quad M^2_{\phi_8})^T, \tag{12}$$

$$\hat{M}^2_{\mathcal{A}}=(M^2_{a_1} \quad M^2_{K_{1A}} \quad M^2_{f_8})^T,$$

as is the quark mass:

$$\hat{m}_q=[m \;\; (m+m_s)/2 \;\; (m+2m_s)/3]^T, \tag{13}$$

whose elements are inferred from leading order chiral perturbation theory [10]. We then have the generalization of Eq. (9) to three flavors:

$$\hat{M}^2_{\mathcal{P}}=2B\hat{m}_q,$$

$$\hat{M}^2_{\mathcal{V}}=2B\hat{m}_q+\Delta\mathbf{1}, \tag{14}$$

$$\hat{M}^2_{\mathcal{A}}=2B\hat{m}_q+2\Delta\mathbf{1},$$

where **1** is the unit vector. The Gell-Mann–Okubo formulas follow trivially from Eq. (14):

$$3M^2_{\eta_8}+M^2_\pi=4M^2_K, \tag{15a}$$

$$3M^2_{\phi_8}+M^2_\rho=4M^2_{K^*}, \tag{15b}$$

$$3M^2_{f_8}+M^2_{a_1}=4M^2_{K_{1A}}. \tag{15c}$$

We define $M_{\eta 8}$, $M_{\phi 8}$, and M_{f8} by the Gell-Mann–Okubo formulas. From Eq. (14) follow also the equal spacing relations

$$\hat{M}^2_{\mathcal{V}}-\hat{M}^2_{\mathcal{P}}=\hat{M}^2_{\mathcal{A}}-\hat{M}^2_{\mathcal{V}}=\Delta\mathbf{1}, \tag{16}$$

which imply

$$M^2_\rho-M^2_\pi=M^2_{K^*}-M^2_K=M^2_{\phi_8}-M^2_{\eta_8}=\Delta, \tag{17a}$$

$$M^2_{a_1}-M^2_\rho=M^2_{K_{1A}}-M^2_{K^*}=M^2_{f_8}-M^2_{\phi_8}=\Delta. \tag{17b}$$

TABLE II. The ϕ_8 and f_8 masses are taken from the Gell-Mann–Okubo formulas. K_{1A} is assumed to be an equal mixture of $K_1(1270)$ and $K_1(1400)$ [1]. The quoted uncertainties are due to the a_1, $K_1(1270)$, and $K_1(1400)$ masses.

A^*-A	$\alpha'(M^2_{A*}-M^2_A)$
$a_1-\rho$	0.81 ± 0.10
$K_{1A}-K^*$	0.86 ± 0.02
$f_8-\phi_8$	0.90 ± 0.05

These equal spacing relations are the main result of this paper. Clearly not all Gell-Mann–Okubo and equal-spacing relations are independent. For instance Eqs. (15a), (15b), and (17a) together comprise four relations, three of which are independent.

The $\mathcal{V}$-$\mathcal{P}$ equal spacing rule works remarkably well (see Table I). The $\mathcal{A}$-$\mathcal{V}$ equal spacing rule is consistent within the error bars (see Table II). However, the equality of Eqs. (17a) and (17b) is not very good. The reason underlying this combination of remarkable accuracy and mediocrity is mysterious. But we emphasize that the lack of agreement between Eqs. (17a) and (17b) is nothing new; Eq. (17) is a generalization of the famous relation, Eq. (10), familiar from spectral function sum rules [11]. For instance, using Eq. (10) to pre-

[2] For a nice review of the current situation, see Ref. [12].

TABLE III. Charged and neutral kaon mass-squared splittings from the particle data group [1]. We again use $\alpha' \equiv 0.88\ \mathrm{GeV}^{-2}$.

A^*-A	$\alpha'(M_{A*}^2 - M_A^2)$
$K^{*+} - K^+$	0.4851 ± 0.0004
$K^{*0} - K^0$	0.4886 ± 0.0004

dict the a_1 mass from the ρ and π masses gives $M_{a_1} = 1080$ MeV compared with the measured value of 1260 ± 40 MeV.

F. Isospin violation and current quark masses

We now consider $m_u \neq m_d$, to leading order in chiral perturbation theory, and include an electromagnetic mass Δm_γ^2, consistent with Dashen's theorem [13] in the elements of $\hat{M}_0^2$ that carry electromagnetic charge. We can thus test the universality of the electromagnetic corrections in the $\mathcal{V}$-$\mathcal{P}$ equal spacing rule. We find

$$M_{K^{*+}}^2 - M_{K^+}^2 = M_{K^{*0}}^2 - M_{K^0}^2, \tag{18a}$$

$$M_{\rho^+}^2 - M_{\pi^+}^2 = M_{\rho^0}^2 - M_{\pi^0}^2. \tag{18b}$$

Equation (18b) is consistent within error bars. Equation (18a) is compared with experiment in Table III. The charged and neutral kaon splittings do not agree within experimental errors. Therefore this is equal-spacing rule distinguishes isospin violating contributions to the $\mathcal{P}$ and $\mathcal{V}$ octets. It is therefore of interest to calculate ratios of quark masses using $\mathcal{V}$. We have

$$\frac{m_u}{m_d} = \frac{M_{K^{*+}}^2 - M_{K^{*0}}^2 + 2M_{\pi^0}^2 - M_{\pi^+}^2}{M_{K^{*0}}^2 - M_{K^{*+}}^2 + M_{\pi^+}^2} = 0.33 \pm 0.05, \tag{19}$$

which is related by Eq. (18a) to the usual relation involving $\mathcal{P}$ alone which gives $m_u/m_d = 0.55$ [10]. The errors are due to the K^* masses [1]. We also find

$$\frac{m_s}{m_d} = \frac{M_{K^{*0}}^2 - M_{K^{*+}}^2 + 2M_{K^+}^2 - M_{\pi^+}^2}{M_{K^{*0}}^2 - M_{K^{*+}}^2 + M_{\pi^+}^2} = 16.7 \pm 0.6, \tag{20}$$

which is again related by Eq. (18a) to the usual relation involving $\mathcal{P}$ alone which gives $m_s/m_d = 20.1$. The values of the quark mass ratios implied by the chiral multiplet structure are not at odds with the usual predictions implied by chiral symmetry alone if one takes into account theoretical error due to omitted higher orders in the chiral expansion [1].

III. THE ETA TRIPLET AND THE SCALAR OCTET

Hadrons participating in representations of SU(3)×SU(3) must also participate in representations of SU(2)×SU(2). That the representations be compatible is a nontrivial consistency check and provides insight into the nature of octet-singlet mixing. We saw above that consistency of the pion representations required that ϵ_8 be the $I=0$ member of the scalar octet. Consistency of the kaon $I=\frac{1}{2}$ representation with the Goldstone boson representation will prove useful in relating matrix elements of light mesons to those of heavy mesons in the heavy quark limit, as we will see in the next section. Here we discuss the lowest lying isoscalar η. Assume η is in an irreducible representation of SU(2)×SU(2). If η is an SU(2)×SU(2) singlet then it does not communicate with other states by pion emission and absorption. This is in contradiction with the SU(3)×SU(3) assignment and is therefore ruled out. If η is in a nontrivial irreducible representation, then it must be degenerate with at least an isovector other than π. This is again in contradiction with the SU(3)×SU(3) assignments. Therefore, η must be in a reducible representation of type (b), found in Sec. II C.

Since η has $\Pi = -1$, it follows that $\Pi_{\mathrm{II}} = -1$, $\Pi_{\mathrm{III}} = 1$. All states have positive charge conjugation sign. The η representation is labeled by $\Pi G = -1$. Consultation of the particle data tables [1] makes clear that η must be joined in this representation by a scalar (isovector) $a_0(980)$ $(\Pi = +1)$, and the zero-helicity component of $f_1(1285)$ $(\Pi = -1)$. The next candidate $I=1$ state with appropriate quantum numbers to participate in the η triplet is the recently discovered $a_0(1450)$ [1]. However, this state, being heavier than $f_1(1285)$, cannot participate in the η representation. We therefore identify $|\mathrm{I}\rangle = |\eta\rangle$, $|\mathrm{II}\rangle = |f_1\rangle^{(0)}$, and $|\mathrm{III}\rangle_a = |a_0\rangle_a$. The chiral representation of η is then

$$\begin{aligned} |\eta\rangle &= \frac{1}{\sqrt{2}}\{|2,2\rangle_4 - |1,1\rangle\}, \\ |f_1\rangle^{(0)} &= \frac{1}{\sqrt{2}}\{|2,2\rangle_4 + |1,1\rangle\}, \\ |a_0\rangle_a &= |2,2\rangle_a. \end{aligned} \tag{21}$$

It follows that

$$\begin{aligned} M_\eta^2 &= \overline{M}_0^2 - \overline{M}_{\langle\bar{q}q\rangle}^2, \\ M_{f_1}^2 &= \overline{M}_0^2 + \overline{M}_{\langle\bar{q}q\rangle}^2, \\ M_{a_0}^2 &= \overline{M}_0^2, \end{aligned} \tag{22}$$

where we have defined the matrix elements

$$\langle 2,2|\hat{M}_0^2|2,2\rangle = \langle 1,1|\hat{M}_0^2|1,1\rangle \equiv \overline{M}_0^2, \tag{23a}$$

$$\langle 2,2|\hat{M}_{\langle\bar{q}q\rangle}^2|1,1\rangle \equiv \overline{M}_{\langle\bar{q}q\rangle}^2. \tag{23b}$$

From Eq. (22) follows the equal-spacing relation

$$M_{f_1}^2 - M_{a_0}^2 = M_{a_0}^2 - M_\eta^2, \tag{24}$$

which is compared with experiment in Table IV. This relation works remarkably well. Of course in two-flavor QCD there is no reason why these splittings should be related to those involving the pion quartet, Eq. (10). However, in three-flavor QCD η_8 is in the pseudoscalar octet of Goldstone

TABLE IV. Mass-squared splittings for the η triplet from the particle data group [1].

A^*-A	$\alpha'(M_{A*}^2 - M_A^2)$
$f_1 - a_0$	0.59
$a_0 - \eta$	0.59

bosons and f_8 is in the axial vector octet. Consistency requires that we treat a_0 as the $I=1$ member of the scalar octet. Noting that $\hat{M}_s^2 = \hat{M}_{\mathcal{V}}^2$ we find

$$M_{\eta_8}^2 = 2B(m+2m_s)/3,$$
$$M_{f_8}^2 = 2B(m+2m_s)/3 + 2\Delta, \qquad (25)$$
$$M_{a_0}^2 = 2Bm + \Delta,$$

which when combined with Eq. (24) gives

$$(M_{\eta_8}^2 - M_\eta^2) + (M_{f_8}^2 - M_{f_1}^2) = \frac{8}{3}(M_K^2 - M_\pi^2). \qquad (26)$$

Notice that consistency of the two- and three-flavor chiral representations requires nontrivial mixings in the presence of explicit SU(3) breaking effects. Therefore, without octet-singlet mixing the two- and three-flavor chiral representations cannot be made compatible. We can use this equation together with the Gell-Mann–Okubo formulas to predict $M_{K_{1A}} = 1315$ MeV which is consistent with the value 1340 MeV which follows from assuming K_{1A} to be an equal mixture of $K_1(1270)$ and $K_1(1400)$ [1].

Similar considerations apply to the $I=0$ member of the scalar octet. Using $M_{\epsilon_8}^2 = 2B(m+2m_s)/3 + \Delta$ and $M_\epsilon^2 = 2Bm + \Delta$ we find

$$(M_{\epsilon_8}^2 - M_\epsilon^2) = \frac{4}{3}(M_K^2 - M_\pi^2). \qquad (27)$$

Presumably ϵ, a mixture of ϵ_8 with an SU(3) singlet, is identified with $f_0(400-1200)$ [1]. Combining Eqs. (26) and (27) gives the remarkable equation

$$(M_{\eta_8}^2 - M_\eta^2) + (M_{f_8}^2 - M_{f_1}^2) = 2(M_{\epsilon_8}^2 - M_\epsilon^2). \qquad (28)$$

We emphasize that this relation is a consequence of chiral symmetry and the mass-squared matrix ansatz.

In summary, we have found the two-flavor reducible chiral representation filled out by η and its chiral partners. Consistency with the three-flavor representation is achieved only if there is octet-singlet mixing. Similar considerations apply to ϵ. We conclude on the basis of the consistency of the two- and three-flavor chiral representations of π and η that the scalar octet is $\mathcal{S} = \{a_0(980), K_0^*, \epsilon_8\}$. Consistency conditions for the kaons require discussion of $I=\frac{1}{2}$ chiral multiplets consistent with the assumed properties of the mass-squared matrix.

IV. ISOSPINOR MULTIPLETS

A. Isospinor doublets

The only representations of SU(2)×SU(2) that contain only a single $I=\frac{1}{2}$ representation of the diagonal isospin subgroup are $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$.[3] So, in general, $I=\frac{1}{2}$ meson states of definite helicity are linear combinations of any number of these irreducible representations with undetermined coefficients [6]. Mass splitting can only occur as a consequence of mixing between these representations since $\hat{M}^2$ is a sum of (0,0) ($\hat{M}_0^2$) and $(\frac{1}{2},\frac{1}{2})$ ($\hat{M}^2_{\langle \bar{q}q\rangle}$) contributions. In Ref. [6] Weinberg showed that the further assumption that these two parts of the mass-squared matrix commute imply that $I=\frac{1}{2}$ states communicate through pion transitions only in pairs and fall into chiral multiplets of the form

$$|\mathrm{I}\rangle = \frac{1}{\sqrt{2}}\{|0\ \tfrac{1}{2}\rangle - |\tfrac{1}{2}\ 0\rangle\}, \quad M_{\mathrm{I}}^2 = \mu^2 - \delta,$$
$$|\mathrm{II}\rangle = \frac{1}{\sqrt{2}}\{|0\ \tfrac{1}{2}\rangle + |\tfrac{1}{2}\ 0\rangle\}, \quad M_{\mathrm{II}}^2 = \mu^2 + \delta, \qquad (29)$$

where in the right column we exhibit the mass relations implied by the representation content. It also follows from Eq. (29) that $\langle \mathrm{I}|X_a|\mathrm{II}\rangle = T_a$ where X_a is related to the amplitude for pion transitions and $T_a = \tau_a/2$ is an SU(2) generator [6]. Note also that $\langle \mathrm{I}|X_a|\mathrm{I}\rangle = \langle \mathrm{II}|X_a|\mathrm{II}\rangle = 0$. States $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$ experience no pion transitions to states outside the multiplet. We will further discuss the axial transitions implied by this multiplet in Sec. VI.

It is possible to build more complicated representations out of the basic multiplet of Eq. (29) which are consistent with the mass-squared matrix ansatz. However, the mass-squared splittings and pion transitions will always be equivalent to what is implied by the basic multiplet of Eq. (29). We will see an example of this below.

B. K-K^* and K_0^*-K_{1A} consistency

Since the kaons carry $I=\frac{1}{2}$ they must fall into SU(2)×SU(2) representations as well as the SU(3)×SU(3) representation found in Sec. II. On the basis of our assumptions about the mass-squared matrix K and K^* must either be paired with each other in the sense of Eq. (29) or with other states. If K^* and K are not paired with each other then they do not communicate by single pion emission and absorption. But this cannot be. The SU(3)×SU(3) representation of Eq. (11) implies that $|\langle K|X_a|K^*\rangle| \neq 0$. In particular, it implies $|\langle K\|X\|K^*\rangle| = |\langle \pi\|X\|\rho\rangle|$ for reduced matrix elements in the chiral limit. Since the latter does not vanish in any limit, neither does the former. Therefore K^* and K must be paired as

[3]Note that here we label states by their isospin rather than the number of independent components. For instance, in terms or our previous notation we have $(0,\frac{1}{2}) = (1,2)$.

$$|K^*\rangle^{(0)} = \frac{1}{\sqrt{2}}\{|0\,\tfrac{1}{2}\,s\rangle + |\tfrac{1}{2}\,0\,s\rangle\},$$
$$|K\rangle = \frac{1}{\sqrt{2}}\{|0\,\tfrac{1}{2}\,s\rangle - |\tfrac{1}{2}\,0\,s\rangle\}, \tag{30}$$

where s denotes strange quark content. One can check that the SU(3)×SU(3) and SU(2)×SU(2) predictions for the pion transition matrix element $\langle K|X_a|K^*\rangle$ are consistent [9]. It is straightforward to find

$$M_{K^*}^2 - M_K^2 = 2\langle \tfrac{1}{2}\,0\,s|\hat{M}^2_{\langle \overline{q}q\rangle}|0\,\tfrac{1}{2}\,s\rangle. \tag{31}$$

Consistency with the SU(3)×SU(3) result of Eq. (17a) then implies

$$\langle \tfrac{1}{2}\,0\,s|\hat{M}^2_{\langle \overline{q}q\rangle}|0\,\tfrac{1}{2}\,s\rangle = \langle \tfrac{1}{2}\,0|\hat{M}^2_{\langle \overline{q}q\rangle}|0\,\tfrac{1}{2}\rangle = \tfrac{1}{2}\Delta, \tag{32}$$

where in the second step we have removed the s label since *this matrix element does not depend on strange quark properties*. Of course we have only shown this to be the case to leading order in chiral perturbation theory.

Similarly, K_0^* and K_{1A} must be paired as

$$|K_{1A}\rangle^{(0)} = \frac{1}{\sqrt{2}}\{|\overline{0\,\tfrac{1}{2}\,s}\rangle + |\overline{\tfrac{1}{2}\,0\,s}\rangle\},$$
$$|K_0^*\rangle = \frac{1}{\sqrt{2}}\{|\overline{0\,\tfrac{1}{2}\,s}\rangle - |\overline{\tfrac{1}{2}\,0\,s}\rangle\}, \tag{33}$$

where the bar denotes that the states are distinct from those of the K-K^* pair. It follows that

$$M_{K_{1A}}^2 - M_{K_0^*}^2 = 2\langle \overline{\tfrac{1}{2}\,0\,s}|\hat{M}^2_{\langle \overline{q}q\rangle}|\overline{0\,\tfrac{1}{2}\,s}\rangle. \tag{34}$$

Again we find the consistency condition

$$\langle \overline{\tfrac{1}{2}\,0\,s}|\hat{M}^2_{\langle \overline{q}q\rangle}|\overline{0\,\tfrac{1}{2}\,s}\rangle = \langle \overline{\tfrac{1}{2}\,0}|\hat{M}^2_{\langle \overline{q}q\rangle}|\overline{0\,\tfrac{1}{2}}\rangle = \tfrac{1}{2}\Delta. \tag{35}$$

Given Eqs. (32) and (35) we will assume that there is a *universal* matrix element in the sense that

$$\langle \tfrac{1}{2}\,0\mathscr{X}|\hat{M}^2_{\langle \overline{q}q\rangle}|0\,\tfrac{1}{2}\,\mathscr{X}\rangle = \tfrac{1}{2}\,\Delta, \tag{36}$$

where $\mathscr{X}$ represents other quantum numbers carried by the $I=\frac{1}{2}$ state. This additional assumption will prove essential in relating the light- and heavy-meson mass-squared splittings.

It might appear that the heavy meson pairs, D-D^* and B-B^*, should be paired as K-K^* is in Eq. (30). We will see that such a simplistic assignment would violate constraints due to heavy quark symmetry.

C. The heavy mesons

Heavy quark symmetry places constraints of its own on the heavy meson mass matrix. The ground state heavy mesons have light-quark spin $s_\ell = 1/2$ and $\pi_\ell = (-)$ and are denoted $P(0^-)$ and $P^*(1^-)$ [14] where P is D or B. Their masses can be expressed as

$$M_P = m_Q + \overline{\Lambda} + \{\tilde{K} + \tilde{G}\}/m_Q + O(1/m_Q^2), \tag{37}$$

$$M_{P^*} = m_Q + \overline{\Lambda} + \{\tilde{K} - \tfrac{1}{3}\tilde{G}\}/m_Q + O(1/m_Q^2), \tag{38}$$

where $\overline{\Lambda}$ is a positive contribution—independent of the heavy quark mass—and $\tilde{K}$ and $\tilde{G}$ are matrix elements of heavy quark operators which are also independent of the heavy quark mass [15]. Squaring the masses gives

$$M_{P^*}^2 - M_P^2 = -\tfrac{8}{3}\tilde{G}, \tag{39}$$

$$3M_{P^*}^2 + M_P^2 = 4(m_Q + \overline{\Lambda})^2 + 8\tilde{K}, \tag{40}$$

in the heavy quark limit. Heavy quark symmetry constrains the combination Eq. (40) to be independent of the mass-squared splitting, Eq. (39). Of course, it follows from Eq. (40) that the D-D^* and B-B^* mass-squared splittings are equal in the heavy quark limit since $\tilde{G}$ is independent of the heavy quark mass.

If P and P^* are paired in the sense of Eq. (29), the masses must be related as $\mu^2 \pm \delta$. But this would violate the heavy quark symmetry constraints on the mass-squared matrix. In order to reconcile the chiral constraint with the heavy quark symmetry constraints of Eqs. (39) and (40) we must introduce additional heavy meson states. The first excited heavy mesons (not yet observed) have $s_\ell = 1/2$ and $\pi_\ell = (+)$, and are denoted $P_0^*(0^+)$ and $P_1'(1^+)$. The unique solution to the combined chiral and heavy quark constraints is then [9]

$$M_P^2 = \mu^2 - \tfrac{3}{2}\epsilon, \quad M_{P^*}^2 = \mu^2 + \tfrac{1}{2}\,\epsilon,$$
$$M_{P_0^*}^2 = \mu^2 + \tfrac{3}{2}\epsilon, \quad M_{P'}^2 = \mu^2 - \tfrac{1}{2}\epsilon, \tag{41}$$

where $\mu^2 = (m_Q + \overline{\Lambda})^2 + 2\tilde{K} \in \hat{M}_0^2$ and $\epsilon = -4\tilde{G}/3 \in \hat{M}^2_{\langle \overline{q}q\rangle}$. Hence P and P_0^* are paired and P_1' and P^* are paired. What representation of SU(2)×SU(2) does this solution correspond to? We consider two scenarios. Scenario (a) corresponds to naive pairing of the meson states consistent with Eq. (29):

$$\text{(a)}\quad |P\rangle = |\psi_-\rangle_1, \quad |P^*\rangle^{(0)} = |\psi_+\rangle_2,$$
$$|P_0^*\rangle = |\psi_+\rangle_1, \quad |P_1'\rangle^{(0)} = |\psi_-\rangle_2, \tag{42}$$

where

$$|\psi_\pm\rangle_i = \frac{1}{\sqrt{2}}\{|0\,\tfrac{1}{2}\,Q\rangle_i \pm |\tfrac{1}{2}\,0\,Q\rangle_i\}. \tag{43}$$

The subscripts denote distinct states. The symbol Q represents quantum numbers carried by the heavy quark. With this representation content we have

$$M_P^2=\overline{\mu}^2-\overline{\epsilon},\quad M_{P*}^2=\nu^2+\kappa,$$
$$M_{P_0^*}^2=\overline{\mu}^2+\overline{\epsilon},\quad M_{P_1'}^2=\nu^2-\kappa, \tag{44}$$

where we have defined

$$_1\langle \tfrac{1}{2}\ 0\,Q|\hat{M}^2_{\langle\overline{q}q\rangle}|0\ \tfrac{1}{2}\ Q\rangle_1=\overline{\epsilon},\quad {_2}\langle \tfrac{1}{2}\ 0\,Q|\hat{M}^2_{\langle\overline{q}q\rangle}|0\ \tfrac{1}{2}\ Q\rangle_2=\kappa, \tag{45}$$

$$_1\langle \tfrac{1}{2}\ 0\,Q|\hat{M}^2_0|\tfrac{1}{2}\ 0\,Q\rangle_1=\overline{\mu}^2,\quad {_2}\langle \tfrac{1}{2}\ 0\,Q|\hat{M}^2_0|\tfrac{1}{2}\ 0\,Q\rangle_2=\overline{\nu}^2. \tag{46}$$

In the heavy quark limit the solution of Eq. (41) is recovered if $\overline{\mu}=\overline{\nu}=\mu$, $\overline{\epsilon}=3\epsilon/2$ and $\kappa=\epsilon/2$. We then have

$$_1\langle \tfrac{1}{2}\ 0|\hat{M}^2_{\langle\overline{q}q\rangle}|0\ \tfrac{1}{2}\rangle_1=3\ {_2}\langle \tfrac{1}{2}\ 0|\hat{M}^2_{\langle\overline{q}q\rangle}|0\ \tfrac{1}{2}\rangle_2, \tag{47}$$

where we have removed the Q label since these matrix elements are independent of heavy quark properties. This equation is clearly incompatible with the universality conjecture. We therefore rule out scenario (a) as a viable chiral representation of the heavy mesons.

As pointed out above, we can instead make combinations of Eq. (29) which give *identical* predictions for the observable mass-squared splittings and pion transitions and yet which order matrix elements differently:

$$\text{(b)}\quad |P\rangle=\frac{1}{\sqrt{2}}\{|\psi_-\rangle_1+|\psi_-\rangle_2\},$$
$$|P^*\rangle^{(0)}=\frac{1}{\sqrt{2}}\{|\psi_+\rangle_1-|\psi_+\rangle_2\},$$
$$|P_0^*\rangle=-\frac{1}{\sqrt{2}}\{|\psi_+\rangle_1+|\psi_+\rangle_2\}, \tag{48}$$
$$|P_1'\rangle^{(0)}=-\frac{1}{\sqrt{2}}\{|\psi_-\rangle_1-|\psi_-\rangle_2\}.$$

We then have

$$M_P^2=(\overline{\mu}^2+\nu^2)-(\overline{\epsilon}+\kappa),\quad M_{P*}^2=(\overline{\mu}^2-\nu^2)+(\overline{\epsilon}-\kappa),$$
$$M_{P_0^*}^2=(\overline{\mu}^2+\nu^2)+(\overline{\epsilon}+\kappa),\quad M_{P_1'}^2=(\overline{\mu}^2-\nu^2)-(\overline{\epsilon}-\kappa), \tag{49}$$

where we have defined

$$_i\langle \tfrac{1}{2}\ 0\,Q|\hat{M}^2_{\langle\overline{q}q\rangle}|0\ \tfrac{1}{2}\ Q\rangle_j=\delta_{ij}\overline{\epsilon}+(1-\delta_{ij})\kappa, \tag{50}$$

$$_i\langle \tfrac{1}{2}\ 0\,Q|\hat{M}^2_0|\tfrac{1}{2}\ 0\,Q\rangle_j=\delta_{ij}\overline{\mu}^2+(1-\delta_{ij})\nu^2. \tag{51}$$

In the heavy quark limit the solution of Eq. (41) is recovered if $\overline{\mu}=\mu$, $\nu=0$, $\overline{\epsilon}=\epsilon$ and $\kappa=\epsilon/2$. We then have

$$_i\langle \tfrac{1}{2}\ 0\,Q|\hat{M}^2_{\langle\overline{q}q\rangle}|0\ \tfrac{1}{2}\ Q\rangle_j=i\langle \tfrac{1}{2}\ 0|\hat{M}^2_{\langle\overline{q}q\rangle}|0\ \tfrac{1}{2}\rangle_j=\tfrac{1}{2}(\delta_{ij}+1)\epsilon, \tag{52}$$

$$_i\langle \tfrac{1}{2}\ 0\,Q|\hat{M}^2_0|\tfrac{1}{2}\ 0\,Q\rangle_j=\delta_{ij}\mu^2, \tag{53}$$

where in Eq. (52) we have made use of the fact that ϵ is independent of heavy quark properties. This assignment of states is consistent with universality. Moreover, one can easily check that universality is not inconsistent with $1/m$ corrections. Therefore scenario (b) is a viable chiral representation. We have

$$M_{P_0^*}^2-M_{P_1'}^2=M_{P*}^2-M_P^2$$
$$=2\ {_1}\langle \tfrac{1}{2}\ 0|\hat{M}^2_{\langle\overline{q}q\rangle}|0\ \tfrac{1}{2}\rangle_1=2\ {_2}\langle \tfrac{1}{2}\ 0|\hat{M}^2_{\langle\overline{q}q\rangle}|0\ \tfrac{1}{2}\rangle_2. \tag{54}$$

On the basis of the universality assumption of Eq. (36) we then find

$$M_{P*}^2-M_P^2=\Delta=M_{K*}^2-M_K^2, \tag{55}$$

where the last equality follows from Eq. (32). Equation (55) is compared with experiment in Table I. This derivation is rendered less persuasive by the universality conjecture. Nevertheless, it is the best that we can do. We will give an independent test of universality in the next section.

V. A NOTE ON THE BARYONS

The excited $I=\frac{1}{2}$ cascade $\Xi(1530)$ decays to Ξ and a pion with a branching ratio of 1 [1]. We therefore expect these states to be paired in the sense of Eq. (29). Hence we have another test of universality. Note that since baryons are fermions, here we are focusing on $|\lambda|=1/2$ transitions. We have

$$|\Xi(1530)\rangle=\frac{1}{\sqrt{2}}\{|0\ \tfrac{1}{2}\ 2s\rangle+|\tfrac{1}{2}\ 0\ 2s\rangle\},$$
$$|\Xi\rangle=\frac{1}{\sqrt{2}}\{|0\ \tfrac{1}{2}\ 2s\rangle-|\tfrac{1}{2}\ 0\ 2s\rangle\}, \tag{56}$$

where $2s$ denotes the strange quark content and it is implied that these are $|\lambda|=\frac{1}{2}$ states. It follows that

$$M^2_{\Xi(1530)}-M^2_{\Xi}=2\langle \tfrac{1}{2}\ 0\ 2s|\hat{M}^2_{\langle\overline{q}q\rangle}|0\ \tfrac{1}{2}\ 2s\rangle=\Delta, \tag{57}$$

TABLE V. Lowest lying baryons of a given character. Masses are central values from the particle data group, and $\alpha' = 0.88\ \mathrm{GeV}^{-2}$.

A^*-A	$\alpha'(M_{A^*}^2 - M_A^2)$
$\Delta - N$	0.56
$\Sigma(1385) - \Lambda$	0.59
$\Sigma(1385) - \Sigma$	0.44
$\Xi(1530) - \Xi$	0.54
$\Sigma_c - \Lambda_c$	0.70

where the last equality follows from the universality assumption of Eq. (36). This predicts a $\Xi(1530) - \Xi$ mass-squared splitting of 0.50 in units of $1/\alpha'$, as compared to the observed splitting of 0.54. Of the baryon pairs listed in Table V, this splitting is in fact closest to the ρ-π splitting, thus providing a gratifying test of the universality conjecture.

Other baryons require separate discussion. For instance, the $I=\frac{1}{2}$ baryons made out of three light quarks clearly have single-pion transitions to states with $I=\frac{3}{2}$. So states such as N and Δ will in general fill our reducible combinations of any number of $(0,\frac{1}{2})$, $(\frac{1}{2},0)$, $(0,\frac{3}{2})$, $(\frac{3}{2},0)$, $(1,\frac{1}{2})$, and $(\frac{1}{2},1)$ irreducible representations of SU(2)×SU(2) [6]. Many multiplets can be constructed consistent with the allowed mass-squared splittings. One might further expect that the $I=0$ and $I=1$ baryons have a quartet structure similar to that of the light mesons.

VI. REGGE BEHAVIOR AND QCD

It might seem mysterious that using the full chiral algebra and special assumptions about the chiral transformation properties of the mass-squared matrix we recovered results that are predicted by hadronic string models. Consider, however, that *the basic assumptions that have been made in this paper are in one-to-one correspondence with statements of Regge asymptotic behavior*. This correspondence is exhibited in Table VI and was demonstrated long ago by Weinberg in Ref. [5] and further developed in Ref. [6]. The constraints on the mass-squared matrix are known as superconvergent sum rules. Hadronic string models exhibit very soft asymptotic behavior; in fact, they satisfy an infinite number of superconvergence relations [16]. We required only two superconvergence relations to derive equal spacing relations for the low-lying mesons.

TABLE VI. The equivalence of the first and second columns is exact in the tree-graph approximation (large-N_c for pion-meson scattering). The first row implies that, for each helicity, mass eigenstates fill out generally reducible representations of $SU(2)_L \times SU(2)_R$. That the two parts of $\hat{M}^2$ commute is a statement of a Z_2 symmetry in the $I=\frac{1}{2}$ sector.

Hadrons	Regge in $\pi\alpha \to \pi\beta$
$SU(2)_L \times SU(2)_R$	$\alpha_1(0) < 1$
$\hat{M}^2 = \hat{M}_0^2 + \hat{M}_{\langle\overline{q}q\rangle}^2$	$\alpha_2(0) < 0,\ \alpha_0(0) = 1$
$Z_2 \leftrightarrow [\hat{M}_0^2, \hat{M}_{\langle\overline{q}q\rangle}^2] = 0$	$\alpha_0(0) < 0\ \ \alpha \neq \beta$

We have seen that the algebraic relation $[\hat{M}_0^2, \hat{M}_{\langle\overline{q}q\rangle}^2] = 0$, which is a statement about diffraction in pion-hadron scattering (see Table VI), fixes the reducible chiral representations filled out by the mesons. Here we will give a symmetry interpretation of this relation. Recall the chiral multiplet structure of the $I=\frac{1}{2}$ states implied by the mass-squared matrix constraints and given by Eq. (29). In order to better demonstrate our point consider this multiplet with an arbitrary mixing angle θ:

$$|\mathrm{I}\rangle = \sin\,\theta|0\ \tfrac{1}{2}\rangle - \cos\,\theta|\tfrac{1}{2}\ 0\rangle,$$
$$|\mathrm{II}\rangle = \cos\,\theta|0\ \tfrac{1}{2}\rangle + \sin\,\theta|\tfrac{1}{2}\ 0\rangle. \tag{58}$$

We define the axial couplings $\langle \mathrm{I}|X_a|\mathrm{II}\rangle \equiv g_{\mathrm{I,II}\pi} T_a$, $\langle \mathrm{I}|X_a|\mathrm{I}\rangle \equiv g_{\mathrm{I,I}\pi} T_a$, and $\langle \mathrm{II}|X_a|\mathrm{II}\rangle \equiv g_{\mathrm{II,II}\pi} T_a$. Since $X_a|0,\frac{1}{2}\rangle = T_a|0,\frac{1}{2}\rangle$ and $X_a|\frac{1}{2},0\rangle = -T_a|\frac{1}{2},0\rangle$ it follows from Eq. (58) that $g_{\mathrm{I,II}\pi} = \sin 2\theta$, $g_{\mathrm{I,I}\pi} = -\cos 2\theta$, and $g_{\mathrm{II,II}\pi} = \cos 2\theta$.

It is possible to understand why the mixing angles are fixed so that each representation has equal weight, by assuming that there is a Z_2 symmetry which permutes chiral representations as $(0,\frac{1}{2}) \leftrightarrow (\frac{1}{2},0)$. If $\sin\theta = \cos\theta = 1/\sqrt{2}$ then $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$ form a Z_2 doublet and we can assign Z_2 charges as

hadron	Z_2
$\|\mathrm{I}\rangle$	-1
$\|\mathrm{II}\rangle$	1
π	-1

Since $\theta = \pi/4$, $g_{\mathrm{I,I}\pi} = g_{\mathrm{II,II}\pi} = 0$ as they must since they are not Z_2 respecting transitions. Hence the assumption of a Z_2 symmetry is in this case equivalent to the mass-squared matrix ansatz [6]. The Z_2 interpretation of the constraint $[\hat{M}_0^2, \hat{M}_{\langle\overline{q}q\rangle}^2] = 0$ is more subtle for the $I=0,1$ states. We have seen that the mass-squared constraint fixes the representations and the mixing angles as in the $I=\frac{1}{2}$ case. However there is clearly no new conserved charge reflected in the spectrum and so the Z_2 symmetry interpretation is not so straightforward. This subtlety has been considered in Ref. [7].

Here we suggest a QCD-based interpretation of the Z_2 symmetry in the $I=\frac{1}{2}$ sector. Consider an underlying quark description with an SU(2)×SU(2) flavor symmetry and the desired Z_2 symmetry. There can be any number of heavy quarks which transform as chiral singlets. Assume that the light quarks transform as $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ with respect to SU(2)×SU(2). The Z_2 transformation permutes quarks with opposite chiral charges: i.e., $(0,\frac{1}{2}) \leftrightarrow (\frac{1}{2},0)$. It is important to realize that this symmetry is *not* ordinary parity; the Z_2 symmetry is an internal symmetry, not a spacetime symmetry. Therefore, the Z_2 symmetry must act independently on left- and right-handed quarks. For instance, a left-handed

Weyl fermion of charge $(0,\frac{1}{2})$ must have a left-handed partner of charge $(\frac{1}{2},0)$. If parity is conserved, then the quark description must have equal numbers of left- and right-handed quarks assigned to each chiral charge, $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$. This implies that the SU(2)×SU(2) flavor symmetry in the underlying quark description with the Z_2 symmetry must be *vectorlike* [17]. This is easy to see by noticing that SU(2)×SU(2) invariant mass operators can always be formed when Z_2 is a symmetry [9].

Clearly the Z_2 symmetry is not a symmetry of the QCD Lagrangian in which the SU(2)×SU(2) flavor symmetry is *chiral*. But what then do we learn from the relation between the Z_2 symmetry and sum rules derived from Regge asymptotic behavior? Consider the following argument. An underlying quark description with the Z_2 symmetry is vectorlike and is therefore automatically consistent with the Nielsen-Ninomiya theorem [17]. This implies that the quark description can be defined on the lattice with an *exact* SU(2)×SU(2) symmetry at nonzero lattice spacing. In practice such a description will have four flavors of quarks and so will look similar to QCD with doublers [9]. The origin of the doublers on the lattice has a physical interpretation in terms of chiral anomalies [18]. Since a gauge theory with an intrinsic cutoff cannot feel the effects of anomalies, each left- and right-handed Weyl fermion in the theory will have a doubler of opposite chiral charge. However, only for special values of the vacuum parameters of the theory will there be a Z_2 symmetry corresponding to permutations of these charges. For instance, the Z_2 symmetry can be spontaneously broken by quark condensates. This would correspond to a low-energy theory with $[\hat{M}_0^2,\hat{M}^2_{\langle\bar{q}q\rangle}]\neq 0$. It is intriguing that the Z_2 symmetry is also broken *unless* $\bar{\theta}$ takes *CP* conserving values [9,19].

It is surprising that this Z_2 symmetry has physical consequences relevant to low-energy QCD. A possible explanation is that the Z_2 symmetry will play a role in any nonperturbative definition of QCD where the flavor symmetries are unbroken by the regulator.[4] Since the lattice is the only known means of defining QCD in the nonperturbative region, and no lattice definition exists in which the flavor symmetries are chiral and unbroken, this hypothesis is safe. In essence, this hypothesis raises the Nielsen-Ninomiya theorem from a statement specific to the lattice to the level of a general physical principle.

There is some evidence backing our hypothesis. The equal-spacing relations have been tested in lattice QCD using improved lattice actions [21]. Improved actions do substantially better at generating universal (constant) mass-squared splittings than do Wilson or staggered fermion actions. In the Wilson action the chiral symmetry breaking Wilson term is dimension 5 whereas in the improved action it is dimension seven [21]. It is conceivable that only an action improved to all orders will explicitly realize the Z_2 symmetry and give precisely constant mass-squared splittings.

[4]This viewpoint is elaborated in Ref. [20].

VII. SUMMARY AND CONCLUSION

There has been little progress in understanding empirically successful predictions of Regge theory and string models of hadrons from QCD. One piece of phenomenology that has received little attention is the remarkable equality of various hadron mass-squared splittings predicted long ago by hadronic string models.

In this paper we have used symmetry arguments to derive a cornucopia of equal spacing relations. Our arguments rest on consequences of the full chiral algebra with two and three flavors of quarks, together with an ansatz for the chiral symmetry transformation properties of the hadronic mass-squared matrix. We showed that $M_\rho^2-M_\pi^2=M_{K*}^2-M_K^2$ to leading order in m_q. This and other equal spacing relations for the masses of the low-lying pseudoscalar, vector, scalar and axialvector octets are our most robust results. We also considered isospin violation and gave new determinations of ratios of current quark masses. The requirement that all states fill out representations of SU(2)×SU(2) *and* SU(3)×SU(3) gave several interesting results. It enabled us to determine members of the scalar octet that are ambiguous from the point of view of SU(3) alone and it required that there be nontrivial octet-singlet mixings. It further allowed us to express the kaon mass-squared splittings in terms of an $I=\frac{1}{2}$ matrix element which is independent of strange quark properties. We conjectured that this matrix element is universal. We then found the chiral representations of the heavy mesons that are consistent with the mass-squared matrix ansatz and heavy quark symmetry and showed that the relevant mass-squared splittings are determined by the universal matrix element. This result gave $M_{K*}^2-M_K^2=M_{P*}^2-M_P^2$ where P represents D or B. Unfortunately, the necessity of conjecturing the existence of a universal matrix element renders this derivation less persuasive than the derivation of the light-meson mass-squared splittings. We then discussed the baryons and gave an independent successful test of the universality conjecture using the cascades.

In Sec. VI we attempted an interpretation of the mass-squared matrix constraints in the context of QCD. First we compared our derivation to that of hadronic string models and showed that the mass-squared matrix ansatz maps precisely to superconvergent sum rules in pion-hadron scattering. We showed that these sum rules for $I=\frac{1}{2}$ mesons are equivalent to assigning a novel Z_2 symmetry to physical states. Finally, we gave an interpretation of this Z_2 symmetry in the context of lattice QCD. We suggested that a Z_2 permutation of chiral charges is a fundamental property of a gauge theory with an intrinsic cutoff.

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