Improved lattice gauge field Hamiltonian

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(Received 12 February 1998; revised manuscript received 21 August 1998; published 12 January 1999)

The improvement problem of lattice gauge field theories is discussed within the Hamiltonian formulation. For a pure gauge theory, we derive an improved quantum Hamiltonian from a lattice Lagrangian free of $O(a^2)$ errors in the classical limit. We do this by the transfer matrix method, but we also show that the alternative via Legendre transformation gives identical results. The resulting color-electric energy is an infinite series, which is expected to be rapidly convergent. For the purpose of practical calculations, we construct a simpler improved Hamiltonian, which includes only nearest-neighbor interactions. We also consider tadpole improvement and the structure of Lüscher-Weisz improvement. As a check of the improved Hamiltonian we compute the gluon dispersion relation and find that the $O(a^2)$ errors disappear. [S0556-2821(99)04301-5]

PACS number(s): 11.15.Ha

I. INTRODUCTION

A systematic pursuit of Symanzik's improvement program [1] has recently led to major progress in lattice QCD, opening the possibility to approach continuum physics on coarse lattices. After the early work by Lüscher and Weisz [2], recently, much progress has been made by the ALPHA Collaboration concerning nonperturbative improvement of the fermion Lagrangian, by implementing PCAC (partial conservation of axial vector current). See, e.g., the review by Lüscher [3]. For the pure gauge actions, however, nobody knows how to perform nonperturbative improvement. The main impetus in this field came from Lepage's tadpole improvement [4].

In this paper we want to address the question of how this improvement scheme can be formulated for a lattice gauge theory in the Hamiltonian approach. Although standard lattice gauge theory has been very successful over the last two decades, there are areas where progress has been quite slow. Examples are the dynamical computation of the S matrix and cross sections, QCD at finite baryon density, or the computation of QCD structure functions in the region of small x_B and Q^2. This situation calls for the development of new methods, and in our opinion the lattice Hamiltonian approach is a viable alternative [5] which should be explored. The Hamiltonian approach corresponds to considering a continuous time, i.e., $a_t=0$. Similar ideas have been pursued recently by workers in standard lattice gauge theory by considering anisotropic lattices with lattice spacings $a_t \ll a_s$. This has the purpose of improving the computation of the mass spectrum [4,6].

As a first step, we restrict ourselves to the problem of improvement of pure gauge theory. Let us recall that the standard improvement program consists of several steps, by starting from the Wilson action (see the review article by Lepage [4]): First defining a classically improved action, second performing tadpole improvement and third introducing additional quantum corrections (Lüscher-Weisz improvement). Hereby the tadpole improvement is considered as a convenient step in order to improve the rate of convergence of the third step. We now discuss how to carry over these ideas to the Hamiltonian formulation. Different strategies are possible, let us explain those for the case of classical improvement.

Strategy 1. Construct the classical Hamiltonian corresponding to the classical Wilson action. Improve this classical Hamiltonian and quantize this Hamiltonian according to the rules of canonical quantization. This yields a *classically improved quantum Hamiltonian.* We should mention that a classically improved Lagrangian to be used for classical calculations relevant for heavy ion collisions has been recently given by Moore [7].

Strategy 2. Starting from the classically improved Wilson action, construct via the transfer matrix a classically improved quantum Hamiltonian.

Strategy 3. Starting from the classical Wilson action, construct first the corresponding quantum Hamiltonian via the transfer matrix. This yields the Kogut-Susskind Hamiltonian $H(E,U)$ where U and E are the link variables and their canonical conjugate momenta. The usual expansion in powers

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of the field variables A and their conjugate $\partial/\partial A$ yields then the standard expression $\int d^3x((\partial/\partial A)^2+B^2(A))$ up to corrections of $O(a^2)$. This Kogut-Susskind Hamiltonian may be improved by adding corrections such that an better agreement with this *formal* continuum limit operator is obtained.

Different strategies exist also for the further improvements (tadpole and Lüscher-Weisz) with respect to the quantum Hamiltonian. In principle, the construction of these quantum corrections should start from new perturbative calculations in the Hamiltonian framework. The coefficients of a fully improved Wilson action (as given, e.g., by Lepage) can only be used as starting point for the transfer matrix to obtain a fully improved Hamiltonian if the action is expressed on a lattice with the time spacing being much smaller than the spatial spacing ($a_t \ll a_s$).

In this paper, we mainly discuss the problem of classical improvement. As a first result we show explicitly that the first strategy—the canonical quantization of a classical lattice gauge theory—is a viable alternative to the second strategy—using the transfer matrix—leading to the same quantum Hamiltonian, but in a more direct way.

The classically improved quantum Hamiltonian obtained in this way has the mathematical structure of a kinetic part with an infinite number of terms. The reason for this structure is given by the fact that the inversion of a nearly local matrix leads to a nonlocal matrix. This being an undesirable feature from the point of view of practical calculations, we show that it is possible to use the nonuniqueness of the improved action to obtain an improved quantum Hamiltonian containing only nearest neighbor interaction terms.

Finally we discuss the structure of the quantum Hamiltonian related to the tadpole and Lüscher-Weisz corrections. The determination of the corresponding coefficients, however, will be deferred to a future investigation. Although we do not discuss the third strategy in detail, a fully consistent computation of the quantum corrections should eventually lead to the same result as the first two strategies. Note that we discuss here only the improved Hamiltonian for the purpose to compute the spectrum. The general existence of such improved Hamiltonians is discussed in standard many-body theory in the context of model space calculations ([8,9]). Like in the action formulation general observables require particular improved operators.

Calculations of the glueball spectrum using the coupled cluster method based on the standard Kogut-Susskind Hamiltonian have been done by Luo *et al.* [10] and Schütte *et al.* [11]. By incorporation of an improved Hamiltonian one would expect reliable results already in lower order of the coupled cluster truncation compared to the standard Kogut-Susskind Hamiltonian. Calculations are under way [12]. In order to check if the proposed improved Hamiltonian actually gives improvement, we have computed the gluon dispersion relation. We find that the $O(a^2)$ errors disappear. This is presented in the Appendix.

II. FROM WILSON ACTION TO KOGUT-SUSSKIND HAMILTONIAN

A. Canonical method via Legendre transformation

Before deriving the improved Hamiltonian, we describe in a pedagogical manner how to obtain the standard Kogut-Susskind lattice Hamiltonian [13] from the classical lattice Lagrangian using the Legendre transformation [14–16], and canonical quantization. Wilson's Euclidean lattice action is given by ($a \equiv a_s$)

$$S_E = \frac{a}{a_t}\frac{2N_c}{g^2}\sum_{t-\Box}(1-P_\Box) + \frac{a_t}{a}\frac{2N_c}{g^2}\sum_{s-\Box}(1-P_\Box). \quad (1)$$

Here the notation of Ref. [17] has been used, i.e., $t-\Box$ stands for timelike and $s-\Box$ for spacelike plaquettes, respectively, and

$$P_\Box = \frac{1}{N_c}\mathrm{Re\ Tr}(U_\Box). \quad (2)$$

For later use we need to distinguish between the Euclidean and Minkowski action as well as Lagrangian. Its relation is defined, when going from Minkowski to Euclidean time by the transformation $it \to t$, $\exp[iS_M] \to \exp[-S_E]$, $L_M \to -L_E$ and $S_M = \int dt L_M$ as well as $S_E = \int dt L_E$. Thus the Euclidean lattice Lagrangian is the following:

$$\begin{aligned} L_E &= -\frac{2N_c a}{g^2 a_t^2}\sum_{x,i}(P_{io}-1) - \frac{2N_c}{g^2 a}\sum_{x,i<j}(P_{ij}-1) \\ &= -\frac{a}{g^2 a_t^2}\sum_{x,i}\mathrm{Tr}(U_{i0}(x)+U_{i0}^\dagger(x)-2) \\ &\quad -\frac{1}{g^2 a}\sum_{x,i<j}\mathrm{Tr}(U_{ij}(x)+U_{ij}^\dagger(x)-2). \end{aligned} \quad (3)$$

Here $U_{ij}(x,t)$ denotes a spacelike plaquette where the first link goes in direction i and the second link goes in direction j, and $U_{i0}(x,t)$ denotes the corresponding timelike plaquette. One should note that $P_{i0}=P_{0i}$ and $P_{ij}=P_{ji}$ for plaquettes. In the temporal gauge $U_0(x,t)=1$, the timelike plaquette becomes a function only of the link variables $U_i(x,t)$ ($i=1,2,3$) and we have

$$P_{i0} = \frac{1}{N_c}\mathrm{Re\ Tr}(U_{i0}) = \frac{1}{N_c}\mathrm{Re\ Tr}(U_i(x,t)U_i^\dagger(x,t+a_t)). \quad (4)$$

We want to construct a classical Lagrangian defining trajectories of generalized coordinates $U_i(x,t)$ and generalized velocities, where the variable t is now continuous. We assume that the action corresponding to this Lagrangian is given by the continuum limit $a_t \to 0$ of the lattice action which also yields the dependence on the generalized velocities.

In order to construct this Lagrangian, we introduce a Taylor expansion in time and write, for a fixed (i,x) up to errors of $O(a_t^3)$,

$$P_{i0}-1=\frac{1}{N_c}\mathrm{Re\ Tr}[U(t)U^\dagger(t+a_t)-1]$$
$$=\frac{1}{N_c}\mathrm{Re\ Tr}\left(U(t)\left[U^\dagger(t)+a_t\dot{U}^\dagger(t)+\frac{a_t^2}{2}\ddot{U}^\dagger(t)\right]-1\right)$$
$$=\frac{a_t^2}{2N_c}\mathrm{Re\ Tr}[U(t)\ddot{U}^\dagger(t)]$$
$$=-\frac{a_t^2}{2N_c}\mathrm{Tr}[\dot{U}^\dagger(t)\dot{U}(t)]$$
$$=-\frac{a_t^2}{2N_c}\mathrm{Tr}[\dot{q}(t)\dot{q}(t)]. \quad (5)$$

Here, we denote $U(t)\equiv U_i(x)\equiv U_i(x,t)$, and we have introduced a generalized velocity $\dot{q}$ (corresponding to the angular velocities of the classical top theory) which is an element of the $SU(N_c)$ Lie algebra,

$$\dot{q}_i(x)=\dot{q}_i^\alpha(x)\lambda^\alpha=-i\dot{U}_i(x)U_i^\dagger(x)=iU_i(x)\dot{U}_i^\dagger(x). \quad (6)$$

The $SU(N_c)$ generators λ^a are normalized to $tr(\lambda^a\lambda^b)=\delta^{ab}/2$. Going to the limit $a_t\to 0$ and performing a transition from Euclidean to Minkowski space ($t\to it$) yields the classical lattice Lagrangian

$$L_M=\frac{a}{2g^2}\sum_{x,i}\dot{q}_i^\alpha(x)\dot{q}_i^\alpha(x)+\frac{1}{g^2a}\sum_{x,i<j}\mathrm{Tr}(U_{ij}+U_{ij}^\dagger-2). \quad (7)$$

Here we denote $U_{ij}\equiv U_{ij}(x)\equiv U_{ij}(x,t)$. For a classical canonical formulation we introduce the conjugate momenta

$$E_j^\alpha(x)=\frac{\partial L_M}{\partial\dot{q}_j^\alpha(x)}=\frac{a}{g^2}\dot{q}_j^\alpha(x)=\frac{2ai}{g^2}\mathrm{Tr}[\lambda^\alpha U_j(x)\dot{U}_j^\dagger(x)],$$
$$E_j(x)=E_j^\alpha(x)\lambda^\alpha=\frac{a}{g^2}\dot{q}_j(x). \quad (8)$$

The standard Legendre transformation leads then to the following classical lattice Hamiltonion:

$$H=\sum_{x,i}\frac{\partial L_M}{\partial\dot{q}_i^\alpha(x)}\dot{q}_i^\alpha(x)-L_M$$
$$=\frac{g^2}{2a}\sum_{x,i}E_i^\alpha(x)E_i^\alpha(x)-\frac{1}{g^2a}\sum_{x,i<j}\mathrm{Tr}(U_{ij}+U_{ij}^\dagger-2). \quad (9)$$

Recalling $\dot{q}_i\to ga\dot{A}$, we convince ourselves that $E_i\approx a^2\dot{A}_i/g$. Therefore $E_i(x)$ is the approximated color-electric field on the lattice.

To quantize this classical theory, we proceed according to the rules of quantization of the classical top theory [18]. This results in the prescription that the quantum mechanical states are functions of the link variables $U_i(x)$ and that the canonical conjugates $E_i(x)$ become the operators of infinitesimal left multiplication. A generalization of the standard quantum mechanical formula $x+a=e^{-i\hat{p}a}xe^{i\hat{p}a}$ yields, for the link variables,

$$e^{i\epsilon^\alpha\lambda^\alpha}U_j(x)=e^{-i\epsilon^\alpha E_j^\alpha(x)}U_j(x)e^{i\epsilon^\alpha E_j^\alpha(x)}. \quad (10)$$

Variables corresponding to different lattice links are considered to be independent. This yields the commutation relations

$$[U_i(x),E_j^\alpha(y)]=\lambda^\alpha U_i(x)\,\delta_{x,y}\delta_{i,j},$$
$$[U_i^\dagger(x),E_j^\alpha(y)]=-U_i^\dagger(x)\lambda^\alpha\delta_{x,y}\delta_{i,j}. \quad (11)$$

Since the operators $e^{i\epsilon^\alpha E_j^\alpha(x)}$ yield a representation of the gauge group $SU(N_c)$, we obtain for the $E_j(x)$ the commutation relations of the Lie algebra

$$[E_j^\alpha(x),E_j^\beta(x)]=if^{\alpha\beta\gamma}E_j^\gamma(x). \quad (12)$$

The quantization of the classical Hamiltonian, Eq. (9), by use of the commutation relations Eq. (11) gives the standard quantum Hamiltonian of Kogut and Susskind.

B. The transfer matrix method

The construction of the Kogut-Susskind Hamiltonian from the Wilson action via the transfer matrix method has been first established by Creutz [17] (see also Ref. [19]). Here we recall the basic steps which may be used also for the construction of the improved Hamiltonian as discussed below:

$$S_E=\int dtL_E=\sum_t a_t[L_0(q(t),q(t+a_t))$$
$$+L_1(q(t))]+O(a_t^2). \quad (13)$$

Hereby, L_0 is kinetic part of the Lagrangian which couples the system at time t to that at time $t+a_t$. Invoking the Baker-Cambell-Hausdorf formula and going to the limit $a_t\to 0$ [20], the Hamiltonian is eventually given by

$$H=H_0+L_1, \quad (14)$$

where the nontrivial part H_0 is related to L_0 via the functional integral kernel of the corresponding time evolution operator (transfer matrix formalism).

Let us recall that relation for the simple example of standard one-body quantum mechanics of free motion [20] where one has (we put the mass $m=1$)

$$L_0(q',q)=\frac{1}{2a_t^2}(q'-q)^2, \quad (15)$$

with $q'=q(t+a_t)$, $q=q(t)$. The discrete time-evolution, which relates the generator H_0 to the kernel L_0, is given by

$$(e^{-a_t H_0}\psi)(q) = N\int dq' e^{-a_t L_0(q,q')}\psi(q'), \tag{16}$$

where N is some unimportant normalization factor. Using

$$\psi(q') = e^{(q'-q)\nabla}\psi(q), \tag{17}$$

yields

$$e^{-a_t H_0} = N\int dq' e^{-a_t L_0(q,q')} e^{(q'-q)\nabla}. \tag{18}$$

In this case, and also for the case of scalar field theory [20], this integral is analytically computable for finite a_t. It yields the usual result $H_0 = -\Delta/2$.

In order to clarify the notations and the particularities for non-Abelian gauge field theories, we first recall how to obtain from the transfer matrix method the standard Kogut-Susskind Hamiltonian. We start by the decomposition of the action as given by Eq. (1) and introduce the temporal gauge. The corresponding Lagrangian is given by

$$L_0(U(t+a_t), U(t)) = \frac{a}{g^2 a_t^2}\sum_{x,i} \mathrm{Tr}(2 - U_i(x,t+a_t) \times U_i^\dagger(x,t) - \mathrm{H.c.}),$$

$$L_1(U(t)) = \frac{1}{g^2 a}\sum_{x,i<j} \mathrm{Tr}(2 - U_{ij}(x,t) - \mathrm{H.c.}). \tag{19}$$

Here L_0 corresponds to the kinetic part and L_1 to the potential part, respectively. The kinetic part of the Lagrangian is given by the plaquettes involving different times. Using the notation analogous to Eq. (15),

$$L_0(U'_{x,j}, U_{x,j}) = \frac{a}{g^2 a_t^2}\sum_{x,j} \mathrm{Tr}[2 - (V_{x,j} + V_{x,j}^\dagger)],$$

$$V_{x,j} = U'_{x,j} U_{x,j}^\dagger, \tag{20}$$

where U corresponds to the time slice t and U' to the time slice $t+a_t$, respectively. It is well known that the quantum mechanics of $SU(N_c)$ gauge theory and that of the quantum mechanical top are closely related [13,18]. Thus the relation between the wave function at time slice t and time slice $t+a_t$, in analogy to Eq. (17), involves the standard color electric operators $E^\alpha, \alpha = 1, \ldots, N_c^2 - 1$,

$$\Psi(U') = e^{i\omega^\alpha E^\alpha}\Psi(U), \tag{21}$$

where the numbers ω^α are the parameters of a group element $g_V \in SU(N_c)$ such that

$$g_V = e^{i\omega^\alpha \lambda^\alpha} = U' U^{-1} = V. \tag{22}$$

The analogy to Eq. (18) the Hamiltonian H_0 in the case of one link variable $V = U' U^\dagger$ is given by

$$e^{-a_t H_0} = N\int dU' e^{-a_t L_0(V)} e^{i\omega^\alpha E^\alpha}, \tag{23}$$

where $L_0(V)$ is given by Eq. (20). We use the invariance of the Haar measure yielding $dU' = dV$ and note that like in Eqs. (17),(21), the operators E^a have to be treated as commuting with U and U'. The integral in Eq. (23) can not be evaluated analytically for finite time translations a_t. However, for the determination of H_0 one may consider the limit $a_t \to 0$. In this case, the variables V approach the identity and it is legitimate to use the approximation for the L_0 term

$$\mathrm{Tr}(V + V^\dagger) = 2\,\mathrm{Tr}(\cos\lambda^\alpha \omega^\alpha) = 2\left[N_c - \frac{1}{4}\omega^2 + O(\omega^4)\right]. \tag{24}$$

Writing the group integral $\int dU$ as Haar measure $\int \Pi_a d\omega^a \det_{Jac}$ yields for Eq. (23) a Gaussian integral in analogy to Eq. (18), with the exponent

$$\frac{a}{2a_t g^2}\omega^\alpha\omega^\alpha + iE^\alpha\omega^\alpha = \frac{a}{2a_t g^2}\left(\omega^\alpha + iE^\alpha\frac{a_t g^2}{a}\right)^2 + \frac{a_t g^2}{2a}E^\alpha E^\alpha. \tag{25}$$

Taking the sum over the spacelike links this reproduces the standard kinetic term of the Kogut-Susskind Hamiltonian

$$H_0 = \frac{g^2}{2a}\sum_{y,j} E_j^\alpha(x) E_j^\alpha(x). \tag{26}$$

III. CLASSICAL IMPROVEMENT OF KOGUT-SUSSKIND HAMILTONIAN

A. Continuum behavior of classical improved action

The Wilson action reproduces the classical continuum action only up to errors of $O(a^2)$. It is possible to add to the Wilson action new terms such that these $O(a^2)$ errors are canceled [1,21,4]. In order to construct the corresponding improved Hamiltonian, one needs a generalization to lattices with $a_t \neq a_s \equiv a$. We first discuss the classical continuum behavior of the Wilson action. For spacelike plaquettes one has (see Ref. [4])

$$P_{ij} = \frac{1}{N_c}\mathrm{Re}\,\mathrm{Tr}(U_{ij}) \to 1 - \frac{g^2 a^4}{2N_c}\left(\mathrm{Tr}[F_{ij}F_{ij}] + \frac{a^2}{12}\mathrm{Tr}[F_{ij}(\mathcal{D}_i^2 + \mathcal{D}_j^2)F_{ij}]\right). \tag{27}$$

For convenience, the continuum limit of a loop is expressed in terms of the field strength tensor and its covariant derivative the center of the loop x_0. According to Ref. [4] for timelike plaquettes one has to consider the path ordered integral

$$\oint A\cdot dx \to \int_{-a/2}^{a/2} dx_i \int_{-a_t/2}^{a_t/2} dt \Bigg[F_{i0}(x_0) + \frac{1}{2} x_{i'} x_{j'} (\mathcal{D}_{i'} \mathcal{D}_{j'} F_{i0}) \Big|_{x=x_0} \Bigg]$$

$$\to a a_t F_{i0}(x_0) + \frac{a_t a^3}{24} (\mathcal{D}_i^2) F_{i0}(x_0) + \frac{a a_t^3}{24} (\mathcal{D}_0^2) F_{i0}(x_0). \tag{28}$$

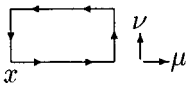
The last term can be neglected since $a_t \ll a$. Therefore

$$P_{i0} = \frac{1}{N_c} \mathrm{Re\ Tr}(U_{i0}) \to \frac{1}{N_c} \mathrm{Re\ Tr} \Bigg[1 - \frac{1}{2} \Big(g \oint A\cdot dx \Big)^2 \Bigg]$$

$$\to \frac{1}{N_c} \mathrm{Re\ Tr} \Bigg[1 - \frac{g^2 a^2 a_t^2}{2} \Big(F_{i0} + \frac{1}{24} a^2 \mathcal{D}_i^2 F_{i0} \Big)^2 \Bigg]$$

$$\to 1 - \frac{g^2 a^2 a_t^2}{2N_c} \Big(\mathrm{Tr}[F_{i0} F_{i0}] + \frac{a^2}{12} \mathrm{Tr}[F_{i0} \mathcal{D}_i^2 F_{i0}] \Big) + O(a^2 a_t^4). \tag{29}$$

In order to compensate these $O(a^2)$ errors, one may add new terms to the Wilson action [22,4]. One of these terms is given by a rectangular loop,

$$R_{\mu\nu} = \frac{1}{N_c} \mathrm{Re\ Tr}$$ 

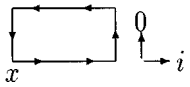
(30)

For a spacelike loop one has in particular

$$R_{ij} = \frac{1}{N_c} \mathrm{Re\ Tr} [U_i(x) U_i(x + a\hat{i}) U_j(x + 2a\hat{i}) \times U_i^\dagger(x + a\hat{j} + a\hat{i}) U_i^\dagger(x + a\hat{j}) U_j^\dagger(x)]$$

$$\to 1 - \frac{g^2 a^4}{2N_c} \Big(4\,\mathrm{Tr}[F_{ij} F_{ij}] + \frac{a^2}{3} \mathrm{Tr}[F_{ij} (4\mathcal{D}_i^2 + \mathcal{D}_j^2) F_{ij}] \Big). \tag{31}$$

Considering timelike loops, there are two possibilities. First, one has a $2a \times a_t$ rectangular loop

$$R_{i0} = \frac{1}{N_c} \mathrm{Re\ Tr}$$ 

(32)

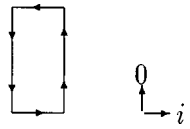
yielding

$$R_{i0} = \frac{1}{N_c} \mathrm{Re\ Tr} [U_i(x,t) U_i(x + a\hat{i}, t) \times U_0(x + 2a\hat{i}, t) U_i^\dagger(x + a\hat{i}, t + a_t) U_i^\dagger(x, t + a_t) U_0^\dagger(x,t)]$$

$$\to 1 - \frac{g^2 a^2 a_t^2}{2N_c} \mathrm{Tr\ Re} \Big(2F_{i0} + \frac{a^2}{3} \mathcal{D}_i^2 F_{i0} \Big)^2$$

$$\to 1 - \frac{g^2 a^2 a_t^2}{2N_c} \Big(4\,\mathrm{Tr}[F_{i0} F_{i0}] + \frac{4a^2}{3} \mathrm{Tr}[F_{i0} \mathcal{D}_i^2 F_{i0}] \Big). \tag{33}$$

Secondly, one has a $2a_t \times a$ rectangular loop

$$R_{0i} = \frac{1}{N_c} \mathrm{Re\ Tr}$$ 

(34)

This term corresponds to advancing two steps in time direction. The conventional transfer matrix corresponds to an advance of a single step in time direction. Thus it is not compatible with the definition of the transfer matrix. We may disregard this term because the improvement terms in the Lagrangian are not uniquely determined [4]. Taking into account only the first term is sufficient.

Therefore, we make the following ansatz for the classically improved Euclidean lattice Lagrangian [4]:

$$L_t = -\frac{2N_c a}{g^2 a_t^2} \sum_{x,i} \Big[C_1' \frac{P_{i0} + P_{0i}}{2} + C_2' R_{i0} \Big] + \mathrm{const.}$$

$$C_1' = 4/3 \ \text{ and } \ C_2' = -1/12,$$

$$L_s = -\frac{2N_c}{g^2 a} \sum_{x, i<j} \Big[C_1 \frac{P_{ij} + P_{ji}}{2} + C_2 (R_{ij} + R_{ji}) \Big] + \mathrm{const},$$

$$C_1 = 5/3 \ \text{ and } \ C_2 = -1/12,$$

$$L_E = L_t + L_s. \tag{35}$$

We remark that an equivalent expression for the classically improved Lagrangian was obtained by Moore [7].

B. Improved Hamiltonian via Legendre transformation

Now we proceed as in Sec. (II A) to construct a classical Lagrange function in Minkowski space in terms of the generalized coordinates $U_i(x)$ and the generalized velocities $\dot{q}_i(x)$ as defined in Eq. (6). Working in the temporal gauge and denoting $\bar{U}(t) = U_i(x + a\hat{i}, t)$ yields

$$
\begin{aligned}
R_{i0} &= \frac{1}{N_c}\mathrm{Re\ Tr}[\overline{U}(t)\overline{U}^{\dagger}(t+a_t)U^{\dagger}(t+a_t)U(t)] \\
&\to \frac{1}{N_c}\mathrm{Re\ Tr}\left\{\overline{U}(t)\left[\overline{U}^{\dagger}(t)+a_t\dot{\overline{U}}^{\dagger}(t)+\frac{a_t^2}{2}\ddot{\overline{U}}^{\dagger}(t)\right]\left[U^{\dagger}(t)+a_t\dot{U}^{\dagger}(t)+\frac{a_t^2}{2}\ddot{U}^{\dagger}(t)\right]U(t)\right\} \\
&= \frac{1}{N_c}\mathrm{Re\ Tr}\left[\frac{a_t^2}{2}\overline{U}(t)\ddot{\overline{U}}^{\dagger}(t)+\frac{a_t^2}{2}\ddot{U}^{\dagger}(t)U(t)+a_t^2\overline{U}(t)\dot{\overline{U}}^{\dagger}(t)\dot{U}^{\dagger}(t)U(t)\right]+\mathrm{const} \\
&\to -\frac{1}{N_c}a_t^2\mathrm{Tr}\left[\frac{1}{2}\dot{q}_i(x+a\hat{i})\dot{q}_i(x+a\hat{i})+\frac{1}{2}\dot{q}_i(x)\dot{q}_i(x)+\dot{Q}_i(x)\dot{q}_i(x+a\hat{i})\right]+\mathrm{const},
\end{aligned}
\tag{36}
$$

where we have introduced the variable

$$
\dot{Q}_i(x)=U_i(x)^{\dagger}\dot{q}_i(x)U_i(x). \tag{37}
$$

This gives the following classical improved lattice Lagrangian in Minkowski space

$$
\begin{aligned}
L_M=\frac{a}{g^2}\sum_{x,i}\ \mathrm{Tr}[&(C_1'+2C_2')\dot{q}_i(x)\dot{q}_i(x) \\
&+2C_2'\dot{Q}_i(x)\dot{q}_i(x+a\hat{i})]-L_s.
\end{aligned}
\tag{38}
$$

This Lagrangian can be written in the form

$$
L_M=\frac{1}{2}\frac{a}{g^2}\sum_{\sigma,\rho}\ \dot{q}_{\sigma}(t)M_{\sigma\rho}(U(t))\dot{q}_{\rho}(t)-L_s,
$$

where $\sigma=(x,i,\alpha)$, $\rho=(y,j,\beta)$,

$$
\begin{aligned}
M_{\sigma,\rho}(U(t))=&(C_1'+2C_2')\delta_{\sigma,\rho}+4C_2'\delta(x,y-a\hat{i})\delta_{i,j} \\
&\times\mathrm{Tr}[U_i^{\dagger}(x)\lambda^{\alpha}U_i(x)\lambda^{\beta}].
\end{aligned}
\tag{39}
$$

The matrix M is not symmetric. However, it can be shown that only the symmetric part of M will contribute to the Hamiltonian. Thus we introduce

$$
\begin{aligned}
L^{sym}_{\sigma,\rho}=&\delta(x,y-a\hat{i})\delta_{i,j}\mathrm{Tr}[U^{\dagger}_{x\to y}\lambda^{\alpha}U_{x\to y}\lambda^{\beta}] \\
&+\delta(x,y+a\hat{i})\delta_{i,j}\mathrm{Tr}[U^{\dagger}_{y\to x}\lambda^{\beta}U_{y\to x}\lambda^{\alpha}],
\end{aligned}
\tag{40}
$$

which allows to write

$$
M^{sym}_{\sigma,\rho}=\frac{1}{2}(M+M^t)_{\sigma,\rho}=(C_1'+2C_2')\delta_{\sigma,\rho}+2C_2'L^{sym}_{\sigma,\rho}. \tag{41}
$$

Inspection shows that L_{sym} and hence M_{sym} are real, symmetric matrices. Then the Lagrangian reads

$$
L_M=\frac{1}{2}\frac{a}{g^2}\sum_{\sigma,\rho}\ \dot{q}_{\sigma}(t)\ \ M^{sym}_{\sigma\rho}(U(t))\ \ \dot{q}_{\rho}(t)-L_s. \tag{42}
$$

Via Legendre transformation, the classical improved Hamiltonian is obtained

$$
\begin{aligned}
H&=\sum_{x,i}\frac{\partial L_M}{\partial\dot{q}_i^{\alpha}(x)}\dot{q}_i^{\alpha}(x)-L_M=H_0+V, \\
H_0&=\frac{1}{2}\frac{g^2}{a}\sum_{\sigma,\rho}E_{\sigma}(M^{sym})^{-1}_{\sigma,\rho}E_{\rho}, \\
V&=-\frac{2N_c}{g^2a}\sum_{x,i<j}\left[C_1\frac{P_{ij}+P_{ji}}{2}+C_2(R_{ij}+R_{ji})\right].
\end{aligned}
\tag{43}
$$

The color-electric field E_{σ} is given by the conjugate momentum, being related to the generalized velocity $\dot{q}_{\sigma}$ via,

$$
E_{\sigma}=\frac{\partial L}{\partial\dot{q}_{\sigma}}=\frac{a}{g^2}\sum_{\rho}\ M^{sym}_{\sigma,\rho}\dot{q}_{\rho}. \tag{44}
$$

The color-electric field, obeys commutation relations with the link variables given by Eq. (11).

1. Hopping expansion and algebraic properties of M_{sym}

Taking a closer look to the kinetic part of the improved Hamiltonian reveals that via M^{-1}_{sym} an infinite number of terms enters into the Hamiltonian. In analogy to the hopping parameter expansion [19], which expresses the propagator in terms of powers of a hopping matrix, we introduce K^{sym}

$$
\begin{aligned}
M_{sym}&=m_0[1+K_{sym}]=m_0[1+k_0L_{sym}] \\
m_0&=C_1'+2C_2'=\frac{7}{6}, \\
k_0&=\frac{2C_2'}{C_1'+2C_2'}=-\frac{1}{7},
\end{aligned}
\tag{45}
$$

to obtain

$$
M^{-1}_{sym}=\frac{1}{m_0}[1-K_{sym}+K^2_{sym}-K^3_{sym}+\cdots]. \tag{46}
$$

While K_{sym} involves only link variables between next neighbor lattice sites, higher powers of K_{sym} involve links extending over several lattice sites. Using the notation

$$U_{x\to x+(N+1)a\hat{i}}=U_i(x)U_i(x+a\hat{i})\cdots U_i(x+Na\hat{i}), \quad (47)$$

we generalize the definition of L_{sym} to

$$L^{(N)sym}_{\sigma,\rho}=\delta(x,y-Na\hat{i})\,\delta_{i,j}\mathrm{Tr}[U^{\dagger}_{x\to y}\lambda^{\alpha}U_{x\to y}\lambda^{\beta}] + \delta(x,y+Na\hat{i})\,\delta_{i,j}\mathrm{Tr}[U^{\dagger}_{y\to x}\lambda^{\beta}U_{y\to x}\lambda^{\alpha}], \quad N=0,1,2,\ldots, \quad (48)$$

where $L^{(0)}_{sym}=1$, and $L^{(1)}_{sym}=L_{sym}$. A little algebra shows that the matrix $L^{(N)}_{sym}$ obeys the following product rule:

$$L^{(p)}_{sym}L^{(q)}_{sym}=\frac{1}{2}L^{(p+q)}_{sym}+\frac{1}{2}L^{(|p-q|)}_{sym}. \quad (49)$$

Thus we obtain, for the lowest powers of K_{sym},

$$\begin{aligned}
K_{sym}&=k_0L^{(1)}_{sym},\\
K^2_{sym}&=k_0^2\left[\frac{1}{2}L^{(0)}_{sym}+\frac{1}{2}L^{(2)}_{sym}\right],\\
K^3_{sym}&=k_0^3\left[\left(\frac{1}{2}+\frac{1}{4}\right)L^{(1)}_{sym}+\frac{1}{4}L^{(3)}_{sym}\right],\\
K^4_{sym}&=k_0^4\left[\left(\frac{1}{4}+\frac{1}{8}\right)L^{(0)}_{sym}+\frac{1}{2}L^{(2)}_{sym}+\frac{1}{8}L^{(4)}_{sym}\right],\\
K^5_{sym}&=k_0^5\left[\left(\frac{1}{2}+\frac{1}{8}\right)L^{(1)}_{sym}+\left(\frac{1}{4}+\frac{1}{16}\right)L^{(3)}_{sym}+\frac{1}{16}L^{(5)}_{sym}\right],\\
K^6_{sym}&=k_0^6\left[\left(\frac{1}{4}+\frac{1}{16}\right)L^{(0)}_{sym}+\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\right)L^{(2)}_{sym}\right.\\
&\quad\left.+\left(\frac{1}{8}+\frac{1}{16}\right)L^{(4)}_{sym}+\frac{1}{32}L^{(6)}_{sym}\right],\\
&\vdots
\end{aligned} \quad (50)$$

It has the general structure

$$K^n_{sym}=k_0^n\sum_{p=0}^{n}\kappa^{(n)}_pL^{(p)}_{sym}. \quad (51)$$

The coefficients of lowest order are

$$\begin{aligned}
&\kappa^{(0)}_0=1,\\
&\kappa^{(1)}_0=0,\ \ \kappa^{(1)}_1=1,\\
&\kappa^{(2)}_0=\frac{1}{2},\ \ \kappa^{(2)}_1=0,\ \ \kappa^{(2)}_2=\frac{1}{2},\\
&\vdots
\end{aligned} \quad (52)$$

The coefficients $\kappa^{(n)}_p$ vanish except when n and p are both even or both odd. Using Eq. (51), we express M^{-1}_{sym} by

$$M^{-1}_{sym}=\frac{1}{m_0}\sum_{p=0}^{\infty}\mu_pL^{(p)}_{sym}, \qquad \mu_p=\sum_{n=p}^{\infty}(-k_0)^n\kappa^{(n)}_p. \quad (53)$$

As result, starting from an improved Lagrangian with a finite number of terms, one obtains for the improved Hamiltonian an expression given by an infinite number of terms.

In the following we will explore more of the algebraic structure of M_{sym} and obtain analytic expressions for the hopping expansion coefficients $\kappa^{(n)}_p$. This will be useful in what follows. We introduce

$$J_{\sigma,\rho}=2\,\delta(x,y-a\hat{i})\,\delta_{i,j}\mathrm{Tr}[U^{\dagger}_{x\to y}\lambda^{\alpha}U_{x\to y}\lambda^{\beta}]. \quad (54)$$

A little algebra shows that

$$J^n_{\sigma,\rho}=2\,\delta(x,y-na\hat{i})\,\delta_{i,j}\mathrm{Tr}[U^{\dagger}_{x\to y}\lambda^{\alpha}U_{x\to y}\lambda^{\beta}], \quad (55)$$

and

$$JJ^t=J^tJ=1, \quad (56)$$

i.e., J is a real, orthogonal matrix. Comparison with Eqs. (40),(48) shows

$$\begin{aligned}
L_{sym}&=\frac{1}{2}(J+J^t),\\
L^{(p)}_{sym}&=\frac{1}{2}(J^p+(J^t)^p),\\
K_{sym}&=\frac{k_0}{2}(J+J^t),\\
M_{sym}&=m_0\left[1+\frac{k_0}{2}(J+J^t)\right].
\end{aligned} \quad (57)$$

Using Eq. (56), M_{sym} can be factorized,

$$M_{sym}=\frac{m_0}{1+C^2}(1+CJ)(1+CJ^t), \quad (58)$$

if C is chosen as solution of

$$k_0=\frac{2C}{1+C^2}. \quad (59)$$

Solutions are $C=-7\pm4\sqrt{3}$. Note that J being a real, orthogonal matrix, which has eigenvalues of modulus one, and $|C|\neq1$, thus the matrix M_{sym} can be inverted and M^{-1}_{sym} is well defined. Moreover, we note that M_{sym} is a positive definite matrix. This can be seen directly from Eq. (58), which factorizes M_{sym} into a matrix times its Hermitian conjugate. Also, a lower bound can be estimated using Eq. (57). J being orthogonal implies $||J||=1$. Thus R_J defined by $R_J=\frac{1}{2}(J+J^t)$, being a real, symmetric matrix like M_{sym}, obeys

$||R_J|| \leqslant 1$. Then an arbitrary state vector ϕ of unit norm yields $|\langle\phi|R_J|\phi\rangle| \leqslant 1$. Then Eq. (57) implies

$$\langle\phi|M_{sym}|\phi\rangle = m_0 + m_0 k_0 \langle\phi|R_J|\phi\rangle = \frac{7}{6} - \frac{1}{6}\langle\phi|R_J|\phi\rangle \geqslant 1, \tag{60}$$

showing also that M_{sym} is positive. To summarize the properties of M_{sym}, this is a real, symmetric, positive definite and nonsingular matrix. This property is needed for the construction of the Hamiltonian via the transfer-matrix, in particular for doing the Gaussian integral.

Factorization of M_{sym}, via Eq. (58), allows to express the kinetic energy term H_0 of the Hamiltonian, Eq. (43), as follows:

$$\begin{aligned} H_0 &= \frac{g^2}{2a}\frac{1+C^2}{m_0}\sum_\nu \left[\sum_\rho (1+CJ)^{-1}_{\nu\rho} E_\rho\right]^2 \\ &= \frac{g^2}{a}\frac{1+C^2}{m_0}\mathrm{Tr}\sum_{x,i}\,[E_i(x) - C U_i(x) E_i(x+a\hat{i}) U_i^\dagger(x) \\ &\quad + C^2 U_i(x) U_i(x+a\hat{i}) E_i(x+2a\hat{i}) U_i^\dagger(x+a\hat{i}) \\ &\quad \times U_i^\dagger(x) - \cdots]^2. \end{aligned} \tag{61}$$

Note that this is an expansion in terms of C and J.

Analytic expressions for the coefficients of the hopping expansion can be obtained in the following way:

$$K^n_{sym} = \left(\frac{k_0}{2}\right)^n (J+J^t)^n = \left(\frac{k_0}{2}\right)^n \sum_{p=0}^n \binom{n}{p} J^p (J^t)^{n-p}. \tag{62}$$

Because this expression is a symmetric matrix and making use of Eqs. (56),(57), one may write

$$K^n_{sym} = \left(\frac{k_0}{2}\right)^n \sum_{p=0}^n \alpha_p^{(n)} (J^p + (J^t)^p) = \left(\frac{k_0}{2}\right)^n \sum_{p=0}^n \alpha_p^{(n)} 2 L^{(p)}_{sym}. \tag{63}$$

Comparison of coefficients yields

$$\begin{aligned} p=0:\ \alpha_0^{(n)} &= \frac{1}{2}\binom{n}{n/2} \quad \text{if } n \text{ is even, zero else,} \\ p \geqslant 1:\ \alpha_p^{(n)} &= \binom{n}{(n+p)/2} \quad \text{if } n,p \text{ are both even or both odd, zero else.} \end{aligned} \tag{64}$$

Comparison with Eq. (51) eventually yields for the hopping expansion coefficients $\kappa_p^{(n)}$ the following expression:

$$\kappa_p^{(n)} = 2^{-n+1}\alpha_p^{(n)}. \tag{65}$$

C. Improved Hamiltonian via transfer matrix

We start from the classically improved Euclidean Lagrangian, given by Eq. (35). It is built from spacelike plaquettes P_{ij}, timelike plaquettes P_{i0} and corresponding rectangular loops R_{ij} and R_{i0}. We now want to show that the transfer matrix method yields the same Hamiltonian as has been obtained in the previous section via Legendre transformation. Let us consider the timelike part of the Lagrangian, which yields the kinetic part of the Hamiltonian. The spacelike part yields the potential part in a trivial way. Using the temporal gauge, one has

$$P_{i0} = \frac{1}{2N_c}\mathrm{Tr}[U_i(x,t)U_i^\dagger(x,t+a_t) + U_i(x,t+a_t)U_i^\dagger(x,t)] = \frac{1}{2N_c}\mathrm{Tr}[V_i(x,t) + V_i^\dagger(x,t)], \tag{66}$$

using the notation $V_i(x,t) = U_i(x,t+a_t)U_i^\dagger(x,t)$. Similarly one obtains for the rectangular loop

$$R_{i0} = \frac{1}{2N_c}\mathrm{Tr}[V_i(x,t)U_i(x,t)V_i(x+a\hat{i},t)U_i^\dagger(x,t) + U_i(x,t)V_i^\dagger(x+a\hat{i},t)U_i^\dagger(x,t)V_i^\dagger(x,t)]. \tag{67}$$

The Hamiltonian is defined via the transfer matrix like in Eq. (23). Because we consider $a_t \to 0$, the group integral will be dominated by group elements of $SU(N_c)$ in the neighborhood of the unit element. Thus one can expand the group elements $V_i(x,t)$ in a Taylor series of the Lie group parameters $\omega^\alpha_{x,i}(t)$,

$$V_i(x,t)=\exp[i\omega_{x,i}(t)]=1+i\omega_{x,i}(t)-\frac{1}{2}\omega_{x,i}(t)^2+O(\omega^3), \tag{68}$$

where we denote $\omega_{x,i}(t)=\Sigma_\alpha \omega^\alpha_{x,i}(t)\lambda^\alpha$. Thus we arrive at

$$\mathrm{Tr}[V_i(x,t)+V_i^\dagger(x,t)]=2N_c-\frac{1}{2}\sum_\alpha \omega^\alpha_{x,i}(t)^2+O(\omega^3), \tag{69}$$

and hence in the notation of Eq. (39),

$$\sum_{x,i}\mathrm{Tr}[V_i(x,t)+V_i^\dagger(x,t)]=-\frac{1}{2}\sum_{\sigma,\rho}\omega_\sigma(t)\delta_{\sigma\rho}\omega_\rho(t)+O(\omega^3)+\text{const.} \tag{70}$$

Carrying out the corresponding steps for the rectangular term, one obtains

$$\begin{aligned}\sum_{x,i}&\mathrm{Tr}[V_i(x,t)U_i(x,t)V_i(x+a\hat{i},t)U_i^\dagger(x,t)\\&+U_i(x,t)V_i^\dagger(x+a\hat{i},t)U_i^\dagger(x,t)V_i^\dagger(x,t)]\\&=-\sum_{\sigma,\rho}\omega_\sigma(t)\left[\delta_{\sigma\rho}+\frac{1}{2}J_{\sigma\rho}\right]\omega_\rho(t)+O(\omega^3)+\text{const},\end{aligned} \tag{71}$$

where J is given by Eq. (54). Inserting this into the timelike Lagrangian L_t yields eventually

$$L_t=\frac{1}{2}\frac{a}{a_t^2g^2}\sum_{\sigma,\rho}\omega_\sigma(t)[(C_1'+2C_2')\delta_{\sigma\rho}+4C_2'J^{sym}_{\sigma\rho}]\omega_\rho(t). \tag{72}$$

One should note that the matrix J is not symmetric. However, to the Lagrangian only the symmetric part $J_{sym}=(J+J^t)/2$ contributes. Note further that $J_{sym}=L_{sym}$ and $M_{sym}=(C_1'+2C_2')1+4C_2'J_{sym}$, being real, symmetric matrices. Thus we arrive at

$$L_t=\frac{1}{2}\frac{a}{a_t^2g^2}\sum_{\sigma,\rho}\omega_\sigma(t)M^{sym}_{\sigma\rho}\omega_\rho(t). \tag{73}$$

The transfer matrix is then given by

$$\begin{aligned}&\exp[-a_tH_0+O(a_t^2)]\\&=\int\left[\prod_{x,i}dV_i(x)\right]\exp[-a_tL_t(V_i(x))]\\&\quad\times\exp\left[i\sum_{x,i,\alpha}\omega^\alpha_{x,i}E^\alpha_i(x)\right]\\&=\int\left[\prod_\sigma d\omega_\sigma\det{}_{Jac}\right]\\&\quad\times\exp\left[-\frac{1}{2}\frac{a}{a_tg^2}\sum_{\sigma,\rho}\omega_\sigma M^{sym}_{\sigma\rho}\omega_\rho+i\sum_\sigma\omega_\sigma E_\sigma\right]\\&=N\exp\left[-\frac{1}{2}\sum_{\sigma,\rho}E_\sigma\left(\frac{a}{a_tg^2}M^{sym}\right)^{-1}_{\sigma\rho}E_\rho\right].\end{aligned} \tag{74}$$

Thus we obtain

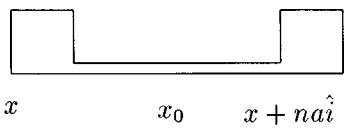
$$H_0=\frac{g^2}{2a}\sum_{\sigma,\rho}E_\sigma(M^{-1}_{sym})_{\sigma\rho}E_\rho \tag{75}$$

in agreement with the result, Eq. (43), obtained via Legendre transformation.

IV. IMPROVED HAMILTONIAN GIVEN BY FINITE NUMBER OF TERMS

As was shown in the previous section, the kinetic energy of the classical improved Hamiltonian obtained directly from the corresponding action is given by an infinitive series of terms. Even though the series is rapidly convergent, such an Hamiltonian is too complicated for a practical calculation. Recalling that the purpose of classical improvement is to push the $O(a^2)$ error to order $O(a^4)$, we show here how to construct a simpler improved Hamiltonian corresponding to a finite number of terms to achieve such a goal. In the previous section we have seen that the infinite number of terms in the Hamiltonian arises due to the inversion of the matrix M_{sym}, which itself has only a finite number of terms. Thus it is plausible that in order to obtain a Hamiltonian given by a finite number of terms, one needs to start from a Lagrangian corresponding to a matrix M_{sym} with an infinite number of terms. Such a construction is possible, because the Lagrangian leading to improvement is not unique. We start by considering the following type of Wilson loop, which emerges as a generalization of the $2a\times a_t$ loop R_{i0} to a $(n+1)a\times a_t$ loop parallel transporter $R_{ni,0}$ given by

$$R_{ni,0}=\frac{1}{N_c}\,\mathrm{Re}\,\mathrm{Tr}$$



(76)

In the temporal gauge it corresponds to the expression

$$R_{ni,0}=\frac{1}{N_c}\mathrm{Re\ Tr}[U_i(x)U_i(x+a\hat{i})\cdots \times U_i(x+(n-1)a\hat{i})U_i(x+na\hat{i}) \times U_i^\dagger(x+na\hat{i},t+a_t)U_i^\dagger(x+(n-1)a\hat{i})\cdots \times U_i^\dagger(x+a\hat{i})U_i^\dagger(x,t+a_t)]. \quad (77)$$

Note that for $n=1, R_{ni,0}$ coincides with R_{i0}. The path-ordered integral of such a Wilson loop is given by

$$\oint A\cdot dx\to\int dx_i dt\left[F_{i0}(x_0)+\frac{1}{2}x_{i'}x_{j'}(\mathcal{D}_{i'}\mathcal{D}_{j'}F_{i0})\Big|_{x=x_0}\right] =2aa_tF_{i0}(x_0)+\frac{1}{2}a_t\mathcal{D}_i^2F_{i0}(x_0)\left[\int_{(n-1)a/2}^{(n+1)a/2}x^2dx+\int_{-(n+1)a/2}^{-(n-1)a/2}x^2dx\right] =2aa_tF_{i0}(x_0)+\frac{3n^2+1}{12}a_ta^3\mathcal{D}_i^2F_{i0}(x_0). \quad (78)$$

Therefore, we obtain the following continuum behavior for the above Wilson loop

$$R_{ni,0}\to\frac{1}{N_c}\mathrm{Re\ Tr}\left[1-\frac{1}{2}\left(\oint A\cdot dx\right)^2\right] \to 1-\frac{g^2a^2a_t^2}{2N_c}(4\mathrm{Tr}[F_{i0}F_{i0}] +(n^2+1/3)a^2\mathrm{Tr}[F_{i0}\mathcal{D}_i^2F_{i0}]). \quad (79)$$

One verifies for $n=1$ that Eq. (79) coincides with Eq. (33), as should be.

We make the following ansatz for the Euclidean lattice Lagrangian

$$L_t=-\frac{2N_ca}{g^2a_t^2}A'\left[B'\sum_{x,i}P_{io}(x)+\sum_{n=1}^{\infty}C'^n\sum_{x,i}R_{ni,0}(x)\right]. \quad (80)$$

In order that the usual continuum limit of the Lagrangian is obtained and the $O(a^2)$ error is canceled, we imply from the continuum behavior of P_{i0}, Eq. (29), and of $R_{ni,0}$, Eq. (79), that the following conditions hold,

$$A'\left[B'+4\sum_{n=1}^{\infty}C'^n\right]=1, \quad (81)$$

$$A'\left[\frac{B'}{12}+\sum_{n=1}^{\infty}\left(n^2+\frac{1}{3}\right)C'^n\right]=0. \quad (82)$$

We have deliberately introduced the coefficient B'. Choosing

$$B'=1-2\sum_{n=1}^{\infty}C'^n, \quad (83)$$

results in a simple expression of the Lagrangian expressed in terms of generalized coordinates and velocities. Using

$$\sum_{n=1}^{\infty}n^2C'^n=C'\frac{\partial}{\partial C'}\left(C'\frac{\partial}{\partial C'}\right)\sum_{n=1}^{\infty}C'^n=\frac{C'(1+C')}{(1-C')^3}, \quad (84)$$

we obtain

$$A'=\frac{1-C'}{1+C'}, \qquad B'=\frac{1-3C'}{1-C'}, \quad (85)$$

and C' is a root of

$$C'^3+11C'^2+11C'+1=0. \quad (86)$$

This equation has three real roots, given by

$$C'=-1, \qquad C'=-5\pm2\sqrt{6}. \quad (87)$$

The root closest to zero is $C_0'=-5+2\sqrt{6}=-0.101021\ldots$. In order to obtain the kinetic energy, we express $R_{ni,0}$ in terms of generalized coordinates and velocities,

$$R_{ni,0}\to\frac{1}{N_c}\mathrm{Re\ Tr}\left[\left[U_i^\dagger(x)+a_t\dot{U}_i^\dagger(x)+\frac{1}{2}a_t^2\ddot{U}_i^\dagger(x)\right]U_i(x)U_i(x+a\hat{i})\cdots U_i(x+(n-1)a\hat{i})U_i(x+na\hat{i}) \times\left[U_i^\dagger(x+na\hat{i})+a_t\dot{U}_i^\dagger(x+na\hat{i})+\frac{1}{2}a_t^2\ddot{U}_i^\dagger(x+na\hat{i})\right]U_i^\dagger(x+(n-1)a\hat{i})\cdots U_i^\dagger(x+a\hat{i})\right] \to\frac{1}{N_c}\mathrm{Tr}\left[1-\frac{1}{2}a_t^2\dot{q}_i(x)\dot{q}_i(x)-\frac{1}{2}a_t^2\dot{q}_i(x+na\hat{i})\dot{q}_i(x+na\hat{i})-a_t^2\dot{Q}_i(x+(n-1)a\hat{i})\dot{q}_i(x+na\hat{i})\right], \quad (88)$$

where we have introduced

$$\dot{Q}_i(x+(n-1)a\hat{i})=U_i^\dagger(x+(n-1)a\hat{i})\cdots U_i^\dagger(x+a\hat{i})U_i^\dagger(x)\dot{q}_i(x)U_i(x)U_i(x+a\hat{i})\cdots U_i(x+(n-1)a\hat{i}). \tag{89}$$

Note again, for $n=1$, $\dot{Q}_i(x+(n-1)a\hat{i})$ coincides with $\dot{Q}_i(x)$ defined in Eq. (37). Thus we can write the timelike part of the Minkowski lattice Lagrangian,

$$L_t^M=\frac{a}{g^2}A'\sum_{x,i}\ \mathrm{Tr}\left[\dot{q}_i(x)\dot{q}_i(x)+2\sum_{n=1}^{\infty}\ C'^n\dot{Q}_i(x+(n-1)a\hat{i})\dot{q}_i(x+na\hat{i})\right]. \tag{90}$$

To find the kinetic energy of the improved Hamiltonian, it is convenient to express this in terms of the matrix J, defined by Eq. (54),

$$L_t^M=\frac{a}{2g^2}\sum_{\sigma,\rho}\ \dot{q}_\sigma M'_{\sigma\rho}\dot{q}_\rho, \tag{91}$$

where

$$M'=A'\left[1+2\sum_{n=1}^{\infty}\ C'^nJ^n\right]=A'\frac{1+C'J}{1-C'J}. \tag{92}$$

The color-electric field is expressed as

$$E_\sigma=\frac{\partial L}{\partial\dot{q}_\sigma}=\frac{a}{g^2}\sum_{\rho}\ M'^{sym}_{\sigma,\rho}\dot{q}_\rho. \tag{93}$$

where

$$M'^{sym}=\frac{1}{2}(M'+M'^t)=A'\frac{1-C'^2}{(1-C'J)(1-C'J^t)}=\frac{(1-C')^2}{(1+C'^2)-C'(J+J^t)}. \tag{94}$$

If we choose C' such that $|C'|\neq 1$, e.g. $C'_0=-0.101021\ldots$, then M'_{sym} is a real, symmetric, positive and nonsingular matrix. Finally, we obtain the corresponding kinetic energy of the improved Hamiltonian, given by

$$\begin{aligned}H_0&=\frac{1}{2}\frac{g^2}{a}\sum_{\sigma,\rho}\ E_\sigma(M'^{sym})^{-1}_{\sigma,\rho}E_\rho\\&=\frac{g^2}{2a}\sum_{\sigma,\rho}\left[\frac{1+C'^2}{(1-C')^2}E_\sigma\delta_{\sigma,\rho}E_\rho-\frac{C'}{(1-C')^2}E_\sigma(J+J^t)_{\sigma,\rho}E_\rho\right]\\&=\frac{g^2}{2a}\sum_{\sigma,\rho}\left[\frac{1+C'^2}{(1-C')^2}E_\sigma\delta_{\sigma,\rho}E_\rho-\frac{2C'}{(1-C')^2}E_\sigma J_{\sigma,\rho}E_\rho\right]\\&=\frac{g^2}{a}\mathrm{Tr}\sum_{x,i}\left[\frac{1+C'^2}{(1-C')^2}E_i(x)E_i(x)-\frac{2C'}{(1-C')^2}\ U_i(x)^\dagger E_i(x)U_i(x)E_i(x+a\hat{i})\right].\end{aligned} \tag{95}$$

It consists of only two terms, which makes it convenient for practical calculations.

V. TADPOLE IMPROVEMENT

In the preceding section, we derived an improved Hamiltonian for gluons with no classical $O(a^2)$ corrections. An important further step of the improvement program is to take into account quantum corrections by adding suitable additional terms. Without such improvement, only part of the $O(a^2)$ errors are canceled. According to Lepage and Mackenzie, tadpole improvement is a simple, but important second step. In fact most of the tadpole contributions can be removed just by dividing each link operator U_μ by the mean u_μ of the link. For asymmetric lattices, $a_t\ll a_s$, and small enough a_t we have $u_t=1$ for time-like directions. In the Hamiltonian formulation, the mean u_s of a spacelike link is defined by

$$u_s = \langle \Omega | P_{ij} | \Omega \rangle^{1/4}, \tag{96}$$

where $|\Omega\rangle$ is the vacuum of the improved Hamiltonian. Thus tadpole improvement of the lattice Lagrangian $L = L_t + L_s$, where L_t is given by Eq. (80) and L_s by Eq. (35), corresponds to the replacements

$$\begin{aligned} P_{ij} &\to P_{ij}/u_s^4, \\ R_{ij} &\to R_{ij}/u_s^6, \\ P_{i0} &\to P_{i0}/u_s^2, \\ R_{ni,0} &\to R_{ni,0}/u_s^{2n+2}. \end{aligned} \tag{97}$$

This is equivalent to the following replacement of constants

$$\begin{aligned} C_1 &\to C_1/u_s^4, \\ C_2 &\to C_2/u_s^6, \\ g_t &\to g_t u_s, \\ C' &\to C'/u_s^2, \end{aligned} \tag{98}$$

where we put $g = g_t$ in Eq. (80). For the transition to the Hamiltonian these redefinitions of the coefficients can be taken over yielding for the ''two-term'' tadpole improved Hamiltonian ($C' = -0.101021$)

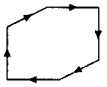
$$\begin{aligned} H &= H_0 + V, \\ H_0 &= \frac{g_t^2 u_s^2}{a} \mathrm{Tr} \sum_{x,i} \left[\frac{1 + C'^2/u_s^4}{(1 - C'/u_s^2)^2} E_i(x) E_i(x) \right. \\ &\quad \left. - \frac{2C'/u_s^2}{(1 - C'/u_s^2)^2} U_i(x)^\dagger E_i(x) U_i(x) E_i(x + a\hat{i}) \right], \\ V &= -\frac{2N_c}{g_s^2 a} \sum_{x,i<j} \left[\frac{C_1}{u_s^4} \frac{P_{ij} + P_{ji}}{2} + \frac{C_2}{u_s^6} (R_{ij} + R_{ji}) \right]. \end{aligned} \tag{99}$$

Here, we have introduced different couplings in the kinetic and potential terms in order to allow for a ''speed of light'' correction as discussed in Ref. [23] (see below).

VI. FURTHER PERTURBATIVE IMPROVEMENT

Tadpoles have been identified as an essential part of the problem when approaching the continuum limit of quantum field theory on the lattice. A systematic perturbative calculation on the lattice has been performed by Lüscher and Weisz [2]. This leads to the determination of additional terms in the Lagrangian needed to compensate errors. It turns out that such a further improved Lagrangian (for details see Refs. [24,25]) contains the same plaquettes and planar rectangle loop terms which occurred before, but with suitably redefined coefficients, plus a new term, being a nonplanar ''parallelogram'' loop, given by

$$C_{\mu\nu\sigma} = \frac{1}{N_c} \mathrm{Re\,Tr} \tag{100}$$



It corresponds to

$$\begin{aligned} C_{x,\mu\nu\sigma} = \frac{1}{N_c} \mathrm{Re\,Tr} (& U_{x,\mu} U_{x+a\hat{\mu},\nu} U_{x+a\hat{\mu}+a\hat{\nu},\sigma} \\ & \times U^{-1}_{x+a\hat{\nu}+a\hat{\sigma},\mu} U^{-1}_{x+a\hat{\sigma},\nu} U^{-1}_{x,\sigma}). \end{aligned} \tag{101}$$

The corresponding term occurring in the Lagrangian is proportional to

$$\sum_{x,\mu<\nu<\sigma} C_{x,\mu\nu\sigma}. \tag{102}$$

The structure of the corresponding improved Hamiltonian can be inferred from the improved Lagrangian as before: One introduces different lattice spacings $a_s = a$ and a_t and constructs the Hamiltonian by Legendre transformation and canonical quantization. Here, we refrain from discussing details and only give the general structure of emerging Hamiltonian.

(1) The *plaquette and planar rectangle loop* terms will give a part of the improved Hamiltonian which has the same form as before, only the weights of the individual terms will be different.

(2) The *spacelike parallelogram loop* terms (i.e. $\mu\nu\sigma$ spacelike) will yield a corresponding additional term in the potential part of the Hamiltonian.

(3) The *timelike parallelogram loop* terms (where either μ or ν or σ is time-like, the other two indices being spacelike) produce a large number of different contributions to the Hamiltonian (with well defined weights). The final result for the improved Lagrangian has the structure

$$L = L_t(\dot{q}, U) + L_s(U), \tag{103}$$

with

$$L_t = \frac{a}{2g_t^2} \left[\sum_{\sigma,\rho} \dot{q}_\sigma M_{\sigma\rho}(U) \dot{q}_\rho + \sum_\sigma A_\sigma(U) \dot{q}_\sigma + \mathrm{H.c.} \right]. \tag{104}$$

A new feature is the occurrence of a term linear in $\dot{q}$. As before, $M(U)$ is a symmetric matrix of the form

$$M = 1 + \tilde{M}, \tag{105}$$

allowing the definition of M^{-1} by a geometric series expansion. Legendre transformation and quantization yields a Hamiltonian of the structure

$$H = H_0 + V, \tag{106}$$

with

$$H_0=\frac{g_t^2}{2a}\sum_{\sigma,\rho}\ [E_\sigma M^{-1}_{\sigma\rho}E_\rho-(A_\sigma M^{-1}_{\sigma\rho}E_\rho+\text{H.c.})$$
$$-2A_\sigma M^{-1}_{\sigma\rho}A_\rho]. \tag{107}$$

VII. DISCUSSION

For the purpose of a numerical calculation, in particular for a comparison with lattice Monte Carlo results, the following points are important.

(1) As discussed in Refs. [26,23], the scales related to the regularization of the gauge field theory in the Hamiltonian formulation as opposed to the Euclidean path integral formulation are different. This difference can be accounted for by introducing spacelike (g_s) and timelike couplings (g_t) which have a well defined relation to the ''Lagrangian coupling'' g. In one-loop approximation this relation is of the type

$$\frac{1}{g_\mu^2}=\frac{1}{g^2}+c_\mu, \tag{108}$$

where c_μ depends on the space-time dimension and on the type of the gauge group and is given in detail in Refs. [26,27,23].

(2) Because of this difference in the nature of the lattice regularization, all perturbative calculations which determine some non-classical improvement in the sense of Lüscher-Weisz have to be redone. Such a calculation can be done on an asymmetric Euclidean lattice with $a_t \ll a_s$ (see Ref. [23]).

(3) Tadpole improvement which has been considered by Lepage [25] in the Lagrangian framework corresponds in the Hamiltonian framework to an expression given by Eq. (99).

(4) A systematic determination of the Lüscher-Weisz improvement terms on asymmetric lattices in the Hamiltonian framework has still to be done. Since these additional corrections turn out to be small in the standard Euclidean framework (see Ref. [25])—the most important correction coming from the inclusion of the tadpole terms—in should be worthwhile to work with the improved Hamiltonian given by Eq. (99), e.g., for the numerical simulation of glueballs.

To summarize, we have investigated in this paper two schemes of improvement of the Kogut-Susskind Hamiltonian: If one starts from Lepage's Lagrangian, which is preferable for Monte Carlo simulations in the Lagrangian formulation, the corresponding Hamiltonian is given by an infinite series of terms which contain terms with arbitrary long range. In contrast, we have shown that by starting from a suitable Lagrangian with an infinite number of terms , one can get an improved Hamiltonian consisting of a finite small number of terms. This should be preferable for numerical computations in the Hamiltonian framework. Finally we should stress that we have restricted ourselves within this paper to elucidate the structure of the Hamiltonian improvement problem for a pure lattice Yang-Mills theory, i.e., for the Kogut-Susskind Hamiltonian. Including fermions, the improvement problem is even more vital since the standard action with Wilson fermions contains already $O(a)$ errors due to the removement of the mirror fermions by a Wilson term. It should be interesting and important to work out the analogue of the recent exciting development of the $O(a^2)$ improvement [3] in the Hamiltonian framework.

ACKNOWLEDGMENTS

H.K. would like to acknowledge support by NSERC Canada. X.Q.L. is supported by the National Science Fund for Distinguished Young Scholars of China. X.Q.L. and S.H.G. would also like to acknowledge additional support by the National Natural Science Foundation of China, Ministry of Education, and Hong Kong Foundation of Zhongshan University Advanced Research Center. X.Q.L. is grateful to the hospitality offered by the colleagues of the Physics Department, Université Laval, where part of the work was done. We also thank K. Chao, X. Fang, H. Jirari, T. Huang, J. Liu, Z. Mei, N. Scheu, and J. Wu for useful discussions.

APPENDIX

As a simple example for the validity of our improvement scheme, we derive in this appendix the dispersion relation for the (classical) gauge fields.

1. Dispersion relation in the continuum

From the Lagrangian equations of motion and using a weak field approximation, $A_i^\alpha(x)\ll 1$, we obtain

$$\frac{d}{dt}F^\alpha_{0i}-\sum_j\ \partial_j F^\alpha_{ji}=0. \tag{A1}$$

In the temporal gauge, this is equivalent to

$$\ddot{A}_i^\alpha(x)=\sum_j\ (\delta_{ij}\nabla^2-\partial_i\partial_j)A_j^\alpha(x). \tag{A2}$$

Its Fourier transformation leads to the expected dispersion relation

$$\omega_i^2=\sum_j\ (\delta_{ij}k^2-k_ik_j), \tag{A3}$$

which means that only the transverse momentum components contribute to the dispersion relation.

2. Dispersion relation for the Kogut-Susskind Hamiltonian

Once we obtain a generalized momentum according to the standard definition $E_\sigma=\partial L_M/\dot{q}_\sigma$ the conjugate variables q_σ and E_σ should satisfy the Poisson bracket relation $\{q_\rho,E_\sigma\}=\delta_{\sigma,\rho}$. The corresponding commutation relation is

$$[q_\rho,E_\sigma]=i\,\delta_{\sigma,\rho}. \tag{A4}$$

Where we have as before $\sigma=(x,i,\alpha)$. From the Kogut-Susskind Hamiltonian

$$H=\frac{g^2}{2a}\sum_\sigma\ E_\sigma E_\sigma-\frac{2N_c}{g^2a}\sum_{x,i<j}\ P_{ij}(x), \tag{A5}$$

and the commutation relation Eq. (A4), one shows that the conjugate variables q and E satisfy automatically the Hamilton equations of motion

$$\dot{q}_\sigma = i[H, q_\sigma] = \frac{g^2}{a} E_\sigma,$$

$$\dot{E}_\sigma = i[H, E_\sigma]. \tag{A6}$$

Combining these two equations, we obtain the equation of motion in a second order differential form

$$\ddot{q}_\sigma = \frac{g^2}{a}\dot{E}_\sigma = i\frac{g^2}{a}[H, E_\sigma]. \tag{A7}$$

According to Eq. (29), the definition of the generalized coordinate and the continuum behavior of the timelike plaquette P_{i0} tells us

$$\mathrm{Tr}\dot{q}_i(x)^2 \approx g^2 a^2 \mathrm{Tr}\left[F_{i0}(x) + \frac{a^2}{24}\mathcal{D}_i^2 F_{i0}(x)\right]^2, \tag{A8}$$

from which we have

$$\dot{q}_i^\alpha(x) \approx ga\left[\dot{A}_i^\alpha(x) + \frac{a^2}{24}\mathcal{D}_i^2 \dot{A}_i^\alpha(x)\right], \tag{A9}$$

or

$$q_i^\alpha(x) \approx ga\left(1 + \frac{a^2}{24}\mathcal{D}_i^2\right) A_i^\alpha(x) = gaZ_i^{-1} A_i^\alpha(x), \tag{A10}$$

Z is an operator defined below. This equation tells us that due to the $O(a^2)$ error in the kinetic energy, the generalized coordinate $q_i^\alpha(x)$ differs from the continuum one $gaA_i^\alpha(x)$ by order of $O(a^2)$. The continuum gauge field is expressed in terms of the generalized coordinate by inverting Eq. (A10):

$$A_i^\alpha(x) \approx \frac{1}{ga}\left(1 - \frac{a^2}{24}\partial_i^2\right) q_i^\alpha(x) = \frac{1}{ga} Z_i q_i^\alpha(x). \tag{A11}$$

Neglecting the higher order terms of A we have that

$$Z_i = 1 - \frac{a^2}{24}\partial_i^2, \tag{A12}$$

which deviates from the identity because of the $O(a^2)$ error in the kinetic energy. In the weak field approximation, the gauge field tensor and its square can be expressed in terms of the generalized coordinates by

$$\begin{aligned} F_{ij}^2 &\approx \frac{1}{g^2 a^2}[Z_j^2(\partial_i q_j)^2 - Z_i^2(\partial_j q_i)^2 \\ &\quad - 2Z_i Z_j \partial_i q_j \partial_j q_i] \\ &\to \frac{1}{g^2 a^2}(-q_j Z_j^2 \partial_i^2 q_j - q_i Z_i^2 \partial_j^2 q_i \\ &\quad + 2q_j Z_i Z_j \partial_j \partial_i q_i). \end{aligned} \tag{A13}$$

The continuum behavior of the spacelike plaquette is

$$\begin{aligned} P_{ij} - 1 &\to -\frac{g^2 a^4}{2N_c}\left(\mathrm{Tr}[F_{ij}F_{ij}] + \frac{a^2}{12}\mathrm{Tr}[F_{ij}(\mathcal{D}_i^2 + \mathcal{D}_j^2)F_{ij}]\right) \\ &\approx \frac{a^2}{2N_c}\mathrm{Tr}\left\{ q_j Z_j^2\left[1 + \frac{a^2}{12}(\partial_i^2 + \partial_j^2)\right]\partial_i^2 q_j \right. \\ &\quad \left. + q_i Z_i^2\left[1 + \frac{a^2}{12}(\partial_i^2 + \partial_j^2)\right]\partial_j^2 q_i + \cdots \right\}, \end{aligned} \tag{A14}$$

where $\cdots$ denotes cross terms plus higher order ones. Using Eqs. (A5) and Eq. (A14), Eq. (A7) becomes

$$\ddot{q}_i^\alpha(x) = Z_i^2 \sum_j [(\delta_{ij}\nabla^2 - \partial_i\partial_j) + O(a^2\partial^4)] q_j^\alpha(x). \tag{A15}$$

By Fourier transformation, we obtain the dispersion relation

$$\omega_i^2 = \left(1 + \frac{a^2}{12}k_i^2\right)\left[\sum_j (\delta_{ij}k^2 - k_i k_j) + O(a^2 k^4)\right]. \tag{A16}$$

In comparison to the continuum dispersion relation Eq. (A3), there are $O(a^2 k_i^2)$ and $O(a^2 k^4)$ errors. The first one comes from the error in the kinetic energy, and the second one comes from the error in the potential energy.

3. Dispersion relation for the improved Hamiltonians

Improvement of the spacelike part leads to the absence of the $a^2 F_{ij}(\mathcal{D}_i^2 + \mathcal{D}_j^2)F_{ij}/12$ term in Eq. (A14). Consequently, the $O(a^2\partial^4)$ term in Eq. (A15) disappears so that [up to $O(a^4)$]

$$i[H, E_i^\alpha(x)] = \frac{a}{g^2} Z_i^2 \sum_j (\delta_{ij}k^2 - k_i k_j) q_j^\alpha(x). \tag{A17}$$

The timelike part of any improved Lagrangian has the general form

$$L_t^{Imp} = \frac{a}{2g^2}\sum_{\sigma,\rho} \dot{q}_\sigma M_{\sigma\rho}\dot{q}_\rho = \frac{a^3}{2}\sum_{\sigma,\rho} \dot{A}_\sigma Z_\sigma^{-1} M_{\sigma\rho} Z_\rho^{-1}\dot{A}_\rho, \tag{A18}$$

where Eq. (A10) has be used. Up to $O(a^4)$, this part is required to satisfy

$$L_t^{Imp}=\frac{a^3}{2}\sum_\sigma \dot{A}_\sigma \dot{A}_\sigma . \tag{A19}$$

Therefore, the matrix M should be

$$M_{ij}^{\alpha\beta}\approx Z_i^{-2}\delta_{ij}\delta^{\alpha\beta}\approx \delta_{ij}\delta^{\alpha\beta}\left(1-\frac{a^2}{12}\partial_i^2\right), \tag{A20}$$

which fits exactly to the two improved Hamiltonian proposed before. The kinetic energy of the improved Hamiltonians has the following form:

$$H_0^{Imp}=\frac{g^2}{2a}\sum_\sigma E_\sigma M_{\sigma\rho}^{-1}E_\rho . \tag{A21}$$

According to the equation of motion,

$$\dot{q}_\sigma = i[H,q_\sigma]=\frac{g^2}{a}M_{\sigma\rho}^{-1}E_\rho ,$$

$$\ddot{q}_\sigma=\ddot{q}_i^\alpha(x)=\frac{g^2}{a}M_{\sigma\rho}^{-1}\dot{E}_\rho$$

$$=i\frac{g^2}{a}M_{\sigma\rho}^{-1}[H,E_\rho]=M_{\sigma\rho}^{-1}Z_{i'}^2\sum_j (\delta_{i'j}\nabla^2-\partial_{i'}\partial_j)q_j^\beta(x)$$

$$=\sum_j (\delta_{ij}\nabla^2-\partial_i\partial_j)q_j^\alpha(x), \tag{A22}$$

where Eq. (A20) has been used. Its Fourier transformation leads to the correct dispersion relation up to $O(a^4)$. This tells us that once the kinetic and potential energies in the Hamiltonian are properly improved, the dispersion relation is certainly improved.

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