Another look at $\pi\pi$ scattering in the scalar channel

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We set up a general framework to describe $\pi\pi$ scattering below 1 GeV based on chiral low-energy expansion with possible spin-0 and -1 resonances. Partial wave amplitudes are obtained with the N/D method, which satisfy unitarity, analyticity, and approximate crossing symmetry. A comparison with the phase shift data in the $J=0$ channel favors a scalar resonance near the ρ mass. [S0556-2821(99)00901-7]

I. INTRODUCTION

Although quantum chromodynamics has long been accepted as the fundamental theory of the strong interaction, the spectrum of hadrons composed of light quarks still poses many unanswered questions. Even below 1 GeV, the old controversy on the existence of the σ meson, an isospin-0 scalar boson strongly coupled to the $\pi\pi$ system, remains unanswered. Recently, there have been some renewed interests in this problem both theoretically [1–11] and experimentally [12–14]. Some analyses favor the existence of σ. The particle reappeared in the Particle Data Group compilation [15] as ''$f_0(400-1200)$ or σ'' after an absence for more than two decades, though it is cautiously stated that ''the interpretation of this entry as a particle is controversial.'' There is no good agreement on its mass among the recent studies. For example, Törnqvist and Roos [7] have used a ''unitarized quark model'' to fit the meson-meson S wave amplitudes and claimed the existence of a very broad σ with a mass of ~ 860 MeV. Ishida *et al.* [11] fit the $\pi\pi$ S wave amplitudes to an S matrix model and find the σ mass of 585 ± 20 MeV. Although these results seem conclusive within their frameworks, the disagreement of the derived σ mass may imply that quantitative conclusions are quite model dependent, casting some doubt in the very existence of σ. In any case, it is not easy to assess how model-independent are their conclusions.

In this paper, we look at this problem from a somewhat different point of view. We try to minimize the necessary model assumptions by a simple approach which only assumes the chiral flavor symmetry and general constraints on the amplitudes such as analyticity, unitarity, and crossing symmetry. We notice that crossing symmetry is not taken into account in the recent works discussed above.

As we put more emphasis on theoretical transparency than aiming at a perfect fit to the data, we concentrate on $\pi\pi$ scattering below 1 GeV and work in the chiral limit with massless pions. Since the pions are the Goldstone bosons of the spontaneously broken $SU(2)\times SU(2)$ symmetry, the form of the pion interactions is tightly constrained at low energies by the symmetry. If we expand the $\pi\pi$ elastic scattering amplitude around $s=t=0$, chiral symmetry demands the amplitude vanishes at $s=t=0$ and the linear terms in s, t are determined in terms of the pion decay constant f_π. These

and terms in higher order in s, t can be described systematically if one uses the machinery of the chiral Lagrangian.

Although this expansion around the origin gives a good description of the amplitude at low energies, it breaks down when one approaches the mass of the lowest-lying hadrons (resonances) other than pions. These resonances manifest themselves as a pole on the second sheet in the scattering amplitude. We are thus led to start with a simple form of the amplitude which has relevant poles (corresponding to possible spin-0 isospin-0 σ and spin-1 isospin-1 ρ resonances) and has the behavior consistent with chiral symmetry (low-energy theorem).

If there is a resonance in the s channel, the same resonance is also exchanged in the t and u channels because of crossing symmetry. In a study of strongly interacting Higgs sector [16] we found that the crossed-channel exchange of a vector resonance has a large reflection in the $J=0$ partial wave. In most of the recent model studies of the σ meson, this effect is not explicitly taken into consideration.[1] It is one of the motivations of this work to assess the importance of the crossed channel ρ exchange in the scalar channel.

As we are concerned with the strong interaction, the amplitude constructed in this way tend to violate unitarity near the pole. To obtain a unitary amplitude, we first project to partial waves (we will be concerned with $J=0$ and $J=1$ channels), and use the N/D method to unitarize the partial wave amplitudes. This method gives amplitudes which has the correct analytic properties with cuts on the real axis. In this respect, it is superior to the K matrix or Padé unitarization scheme. Although the procedure is not exactly crossing symmetric, the deviation from the symmetry is controlled and mostly limited to the region near the pole.

In Sec. II, we summarize the general characteristics of the $\pi\pi$ elastic scattering amplitude and set up our chirally symmetric ''model-independent'' amplitude with possible poles. Two simple cases, ''no σ'' and ''degenerate $\rho-\sigma$'' are discussed in detail. In Sec. III, we calculate the partial wave

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[1]The importance of the crossed channel exchanges has also been emphasized in Ref. [17] based on the original works [6,18]. Our result with ρ exchange agrees qualitatively with the corresponding result in Ref. [6].

amplitudes from the invariant amplitudes in Sec. II. Unitarization of the amplitudes is performed using the N/D method. Relation of our method to the low-energy chiral expansion is clarified in Sec. IV. In Sec. V, we determine the parameters in the amplitudes and compare them with the phase shift data. We summarize and conclude with some remarks in Sec. VI.

II. CHARACTERISTICS OF $\pi\pi$ AMPLITUDE

The $\pi\pi$ system has three independent isospin channels. In terms of Mandelstam variables, the invariant amplitude for the process $\pi^i + \pi^j \rightarrow \pi^k + \pi^\ell$ has the form

$$\mathcal{M}_{ijk\ell}(s,t) = A(s,t)\delta_{ij}\delta_{k\ell} + A(t,s)\delta_{ik}\delta_{j\ell} + A(u,t)\delta_{i\ell}\delta_{jk},$$
(2.1)

where $i,j,\ldots,=1,2,3$ are isospin indices [with $\pi^\pm = (\pi^1 \pm i\pi^2)/\sqrt{2}$, $\pi^0 = \pi^3$]. The variable s is the c.m. energy squared, $t = -s(1-\cos\theta)/2$ and $u = -s(1+\cos\theta)/2$ with $\cos\theta$ denoting the c.m. scattering angle. Note that $s+t+u = 0$. There is only one analytic function $A(s,t)$ because of crossing symmetry. It satisfies $A(s,t) = A(s,u)$ due to Bose symmetry. The last term in Eq. (2.1) thus may be rewritten as $A(u,s)\delta_{i\ell}\delta_{jk}$.

Chiral symmetry low energy theorem demands that A behaves near $s=t=0$ as

$$A(s,t) = \frac{s}{f_\pi^2} + \mathcal{O}(s^2, st, t^2),$$
(2.2)

where $f_\pi \approx 93$ MeV is the pion decay constant. The structure of the second term will be discussed later.

The expansion breaks down by the existence of a resonance. We expect that possible lowest-lying resonances are in $I=J=0$ and $I=J=1$ channels. In the narrow width approximation, the contribution of these resonances may be written as

$$A(s,t) = \frac{g_\sigma^2 s}{m_\sigma^2 - s}$$
(2.3)

for the scalar exchange (we write s in the numerator instead of a constant, to be consistent with the low-energy theorem; this corresponds to adding a contact interaction such as that in the σ model) and

$$A(s,t) = g_\rho^2 \left(\frac{s-u}{m_\rho^2 - t} + \frac{s-t}{m_\rho^2 - u} \right)$$
(2.4)

for the vector exchange (the numerator is the minimal dependence to assure spin 1 and has the same form as the gauge boson exchange).

The tail of these exchange amplitude contributes to the slope of the amplitude at the origin. If we assume that these two resonances saturate the low-energy theorem, we find

$$\frac{g_\sigma^2}{m_\sigma^2} + \frac{3g_\rho^2}{m_\rho^2} = \frac{1}{f_\pi^2}.$$
(2.5)

This condition, applied to the electroweak symmetry breaking, has been used in our previous study of the strong WW scattering [16], in which we have obtained partial wave amplitudes consistent with unitarity and analyticity. For hadron physics, it turns out that the condition is too strong to explain the observed width of the ρ meson. Even if one maximizes the vector coupling and includes the enhancing effect of unitarization, the resulting width is too small by $\sim 20\%$. Thus we are led to relax the condition (2.5) to increase the $\rho\pi\pi$ coupling g_ρ. This may be done by subtracting the $\mathcal{O}(s)$ part from the exchange amplitudes and add a suitable $\mathcal{O}(s)$ term to $A(s,t)$ instead. This procedure gives

$$A(s,t) = \frac{s}{f_\pi^2} + \frac{g_\sigma^2 s^2}{m_\sigma^2(m_\sigma^2 - s)} + \frac{g_\rho^2}{m_\rho^2}\left(\frac{t(s-u)}{m_\rho^2 - t} + \frac{u(s-t)}{m_\rho^2 - u} \right).$$
(2.6)

At the lowest order, the ρ width may be reproduced if one takes g_ρ around the KSRF value [19] $g_\rho^2 = m_\rho^2/2f_\pi^2$. The price to pay is the worse high-energy behavior.

Expanding Eq. (2.6) to second order, we have

$$A(s,t) \simeq \frac{s}{f_\pi^2} + \frac{g_\sigma^2 s^2}{m_\sigma^4} + \frac{g_\rho^2}{m_\rho^4}(-2s^2 + t^2 + u^2).$$
(2.7)

This will be used later in matching with chiral Lagrangian.

To assess the possible existence of σ, we will compare the two cases (1) no σ (ρ only) and (2) degenerate $\rho-\sigma$. We now discuss motivations for these choices.

(1) *No σ meson.* In the nonrelativistic quark model, the lowest-lying S wave mesons are pseudoscalar (π, η) and vector (ρ, ω). Scalar mesons are P wave states and are expected to have similar masses as the other P wave states, the axial vector and tensor mesons which lie in the 1200–1300 MeV range. As we will be concerned with the scattering amplitude below 1 GeV, such mesons in this mass range have small effect and we can simply take $g_\sigma = 0$ to illustrate this case. We may recall that the pion electromagnetic form factor is rather well described [20] by the hypothesis of ρ dominance. The coupling of ρ to pions given by the KSRF relation [19]

$$g_\rho^2 = \frac{m_\rho^2}{2f_\pi^2}$$
(2.8)

reproduces the ρ width quite well.

(2) *Degenerate $\rho-\sigma$.* Since the light quarks are essentially massless compared to the QCD scale, there is no reason that nonrelativistic quark model reliably describe the spectrum. In the string-type picture of hadrons, the spectrum of the states has a tower structure and the vector meson is accompanied by a scalar daughter. This situation in the narrow width approximation is realized in the Veneziano amplitude [21].

The degeneracy of ρ and σ is also suggested in the framework of nonlinear realization of the SU(2)×SU(2) chiral flavor symmetry developed by Weinberg [22]. Algebraization of the Adler-Weisberger sum rule results in the mass matrix structure with this degeneracy, again in the narrow

width (large N_{color}) limit. The couplings g_ρ and g_σ are found to be equal and has the same strength as the KSRF coupling

$$g_\sigma^2 = g_\rho^2 = \frac{m_\rho^2}{2f_\pi^2}. \qquad (2.9)$$

The Veneziano amplitude also gives $g_\rho = g_\sigma$ but the size of the coupling is different, as we will now discuss. The Veneziano $\pi\pi$ scattering amplitude takes a simpler form for the charge eigenstates $\pi^+\pi^- \to \pi^+\pi^-$. With the constraints of chiral symmetry, it reads [23]

$$B_4(s,t) = -\frac{2m_\rho^2}{\pi f_\pi^2} \frac{\Gamma[(1-s/m_\rho^2)/2]\Gamma[(1-t/m_\rho^2)/2]}{\Gamma(u/2m_\rho^2)}. \qquad (2.10)$$

Vanishing of the amplitude at $s = t = 0$ demands that the intercept of the Regge trajectory is 1/2, and the overall coefficient is determined by the scale of chiral symmetry breaking f_π. The invariant amplitude A is related to Eq. (2.10) by the relation $A(s,t) = [B_4(s,t) + B_4(s,u) - B_4(t,u)]/2$.

As is well known, the amplitude (2.10) has an infinite number of poles both in the s and t channels. The lowest-lying poles are at $s = m_\rho^2$ and $t = m_\rho^2$, at which the amplitude behaves as

$$B_4(s,t) \sim \begin{cases} \dfrac{2m_\rho^2(m_\rho^2+t)}{\pi f_\pi^2(m_\rho^2-s)} & (s \sim m_\rho^2), \\[3mm] \dfrac{2m_\rho^2(m_\rho^2+s)}{\pi f_\pi^2(m_\rho^2-t)} & (t \sim m_\rho^2). \end{cases} \qquad (2.11)$$

Expanding Eq. (2.11) in partial waves, one finds that a scalar and a vector state are degenerate at m_ρ. The corresponding couplings are

$$g_\sigma^2 = g_\rho^2 = \frac{m_\rho^2}{\pi f_\pi^2}. \qquad (2.12)$$

The chiral Veneziano amplitude may be approximated by the form (2.6) with appropriate couplings and masses in the energy region of our interest, where the higher poles have small effect.

III. PARTIAL WAVE AMPLITUDES

The invariant amplitude can be expanded in terms of partial waves for states having definite isospin I:

$$a_{IJ}(s) = \frac{1}{64\pi} \int_{-1}^{1} d\cos\theta \, P_J(\cos\theta) \, \mathcal{M}^{(I)}(s,t), \qquad (3.1)$$

with $t = -s(1-\cos\theta)/2$ and

$$\mathcal{M}^{(I=0)} = 3A(s,t) + A(t,s) + A(u,s), \qquad (3.2a)$$

$$\mathcal{M}^{(I=1)} = A(t,s) - A(u,s), \qquad (3.2b)$$

$$\mathcal{M}^{(I=2)} = A(t,s) + A(u,s). \qquad (3.2c)$$

Elastic unitarity requires $\operatorname{Im} a_{IJ}^{-1}(s) = -1$ and the amplitude can be written in terms of the phase shift δ_{IJ} as

$$a_{IJ} = e^{i\delta_{IJ}} \sin\delta_{IJ}. \qquad (3.3)$$

Inelastic channels $(4\pi, \ldots,)$ are known to be negligible below the $K\bar{K}$ threshold [24,25], in accordance with the expectation based on chiral symmetry [the $\pi\pi \to 4\pi$ cross section starts at the order $\sim s^4/(4\pi f_\pi)^8$].

We project the subtracted pole amplitude (2.6) into partial waves, which we denote by $\overset{\circ}{a}_{IJ}$. We find

$$\overset{\circ}{a}_{00} = \frac{1}{16\pi}\left\{ \frac{s}{f_\pi^2} + g_\sigma^2[\tfrac{3}{2}f_r(s/m_\sigma^2) + f_{0\sigma}(s/m_\sigma^2)] \right.$$
$$\left. + 2g_\rho^2 f_{0\rho}(s/m_\rho^2) \right\}, \qquad (3.4a)$$

$$\overset{\circ}{a}_{20} = \frac{1}{16\pi}\left[-\frac{s}{2f_\pi^2} + g_\sigma^2 f_{0\sigma}(s/m_\sigma^2) - g_\rho^2 f_{0\rho}(s/m_\rho^2) \right], \qquad (3.4b)$$

$$\overset{\circ}{a}_{11} = \frac{1}{16\pi}\left\{ \frac{s}{6f_\pi^2} + g_\sigma^2 f_{1\sigma}(s/m_\sigma^2) \right.$$
$$\left. + g_\rho^2[\tfrac{1}{3}f_r(s/m_\rho^2) + f_{1\rho}(s/m_\rho^2)] \right\}, \qquad (3.4c)$$

where

$$f_r(x) = \frac{x^2}{1-x}, \qquad (3.5a)$$

$$f_{0\sigma}(x) = \frac{1}{x}\log(1+x) - 1 + \frac{x}{2}, \qquad (3.5b)$$

$$f_{0\rho}(x) = \left(\frac{1}{x} + 2\right)\log(1+x) - 1 - \frac{3}{2}x, \qquad (3.5c)$$

$$f_{1\sigma}(x) = \frac{1}{x}\left(\frac{2}{x} + 1\right)\log(1+x) - \frac{2}{x} - \frac{x}{6}, \qquad (3.5d)$$

$$f_{1\rho}(x) = \left(\frac{1}{x} + 2\right)\left(\frac{2}{x} + 1\right)\log(1+x) - \frac{2}{x} - 4 - \frac{x}{6}. \qquad (3.5e)$$

These functions may alternatively be obtained from the n_α functions defined in [16], Eq. (13) by subtracting the $\mathcal{O}(x)$ term. Near $x=0$, these functions behave as

$$f_r(x) \simeq x^2, \tag{3.6a}$$

$$f_{0\sigma}(x) \simeq \frac{1}{3} x^2, \tag{3.6b}$$

$$f_{0\rho}(x) \simeq -\frac{2}{3} x^2, \tag{3.6c}$$

$$f_{1\sigma}(x) \simeq -\frac{1}{6} x^2, \tag{3.6d}$$

$$f_{1\rho}(x) \simeq \frac{1}{6} x^2. \tag{3.6e}$$

For large values of the couplings, these amplitudes badly violates unitarity near the resonances. We use the N/D method to obtain amplitudes satisfying elastic unitarity and analyticity. This method is superior to K matrix or Padé unitarization scheme in that it automatically provides an amplitude having correct analytic behavior. We thus write

$$a_{IJ} = \frac{N_{IJ}}{D_{IJ}}, \tag{3.7}$$

and use single N/D iteration by setting $N_{IJ} = \overset{\circ}{a}_{IJ}$ [given by Eq. (3.4)]. The denominator function is determined by analyticity

$$\mathrm{Im}\, D_{IJ}(s) = -N_{IJ}(s)\theta(s) \tag{3.8}$$

(we assume the contribution of inelastic channels is not important), which symbolically gives

$$D_{IJ}(s) = -\frac{1}{\pi} \int_0^\infty \frac{ds'}{s'-s} N_{IJ}(s'). \tag{3.9}$$

Since $N_{IJ}(s) \sim s$ at $s \to \infty$, the dispersion integral has to be subtracted twice. One of the subtraction constants is fixed by the normalization condition $D_{IJ}(0)=1$ (remember that our amplitude $\overset{\circ}{a}_{IJ}$ is constructed to be exact near $s=0$, which requires this condition), and the second constant determines the $\mathcal{O}(s)$ behavior of $D_{IJ}(s)$ as will be discussed later.

To write down the explicit functional form of D, we define the function $d_\alpha(x)$ with the property

$$\mathrm{disc}\, d_\alpha(x) \equiv d_\alpha(x+i\epsilon) - d_\alpha(x-i\epsilon) = 2\pi i f_\alpha(x)\theta(x) \tag{3.10}$$

by

$$d_\alpha(x) + (c_\alpha \log R + c'_\alpha)x = x \int_0^R \frac{dy}{y(y-x)} f_\alpha(y) \quad (R \to \infty), \tag{3.11}$$

where we demand $d_\alpha(x) \sim x^2$ near $x=0$. This is accomplished by separating the $\mathcal{O}(x)$ term of the integral as the second term in the left-hand side (LHS) of Eq. (3.11). Though the integral diverges logarithmically for $R \to \infty$, $d_\alpha(x)$ thus defined is finite in this limit. We find

$$d_r(x) = -\frac{x^2}{1-x} \log(-x), \tag{3.12a}$$

$$d_{0\sigma}(x) = \frac{1}{x} L(x) + \left(1 - \frac{x}{2}\right) \log(-x) - 1 + \frac{x}{4}, \tag{3.12b}$$

$$d_{0\rho}(x) = \left(\frac{1}{x} + 2\right) L(x) + \left(1 + \frac{3}{2} x\right) \\ \times \log(-x) - 1 - \frac{7}{4} x, \tag{3.12c}$$

$$d_{1\sigma}(x) = \frac{1}{x}\left(\frac{2}{x} + 1\right) L(x) + \left(\frac{2}{x} + \frac{x}{6}\right) \\ \times \log(-x) - \frac{2}{x} - \frac{1}{2} + \frac{x}{36}, \tag{3.12d}$$

$$d_{1\rho}(x) = \left(\frac{1}{x} + 2\right)\left(\frac{2}{x} + 1\right) L(x) + \left(\frac{2}{x} + 4 + \frac{x}{6}\right) \\ \times \log(-x) - \frac{2}{x} - \frac{9}{2} - \frac{35}{36} x, \tag{3.12e}$$

with

$$L(x) = -\mathrm{Li}_2(-x) - \log(-x)\log(1+x), \tag{3.13}$$

and

$$c_r = -1, \quad c_{0\sigma} = \frac{1}{2}, \quad c_{0\rho} = -\frac{3}{2}, \quad c_{1\sigma} = c_{1\rho} = -\frac{1}{6}, \tag{3.14}$$

$$c'_r = 0, \quad c'_{0\sigma} = -\frac{1}{4}, \quad c'_{0\rho} = \frac{7}{4}, \quad c'_{1\sigma} = -\frac{1}{36}, \quad c'_{1\rho} = \frac{35}{36}. \tag{3.15}$$

Then we can write

$$N_{00} = \frac{1}{16\pi}\left\{\frac{s}{f_\pi^2} + g_\sigma^2[\tfrac{3}{2}f_r(s/m_\sigma^2) + f_{0\sigma}(s/m_\sigma^2)]\right.$$

$$\left. + 2g_\rho^2 f_{0\rho}(s/m_\rho^2)\right\}, \tag{3.16a}$$

$$N_{20} = \frac{1}{16\pi}\left[-\frac{s}{2f_\pi^2} + g_\sigma^2 f_{0\sigma}(s/m_\sigma^2) - g_\rho^2 f_{0\rho}(s/m_\rho^2)\right], \tag{3.16b}$$

$$N_{11} = \frac{1}{16\pi}\left\{\frac{s}{6f_\pi^2} + g_\sigma^2 f_{1\sigma}(s/m_\sigma^2) + g_\rho^2[\tfrac{1}{3}f_r(s/m_\rho^2)\right.$$

$$\left. + f_{1\rho}(s/m_\rho^2)]\right\}, \tag{3.16c}$$

$$D_{00} = 1 - d_{00}'s - \frac{1}{16\pi^2}\left\{-\frac{s}{f_\pi^2}\log\left(\frac{-s}{\mu^2}\right) + g_\sigma^2[\tfrac{3}{2}d_r(s/m_\sigma^2)\right.$$

$$\left. + d_{0\sigma}(s/m_\sigma^2)] + 2g_\rho^2 d_{0\rho}(s/m_\rho^2)\right\}, \tag{3.17a}$$

$$D_{20} = 1 - d_{20}'s + \frac{1}{16\pi^2}\left[-\frac{s}{2f_\pi^2}\log\left(\frac{-s}{\mu^2}\right)\right.$$

$$\left. + g_\sigma^2 d_{0\sigma}(s/m_\sigma^2) - g_\rho^2 d_{0\rho}(s/m_\rho^2)\right], \tag{3.17b}$$

$$D_{11} = 1 - d_{11}'s - \frac{1}{16\pi^2}\left\{-\frac{s}{6f_\pi^2}\log\left(\frac{-s}{\mu^2}\right) + g_\sigma^2 d_{1\sigma}(s/m_\sigma^2)\right.$$

$$\left. + g_\rho^2[\tfrac{1}{3}d_r(s/m_\rho^2) + d_{1\rho}(s/m_\rho^2)]\right\}. \tag{3.17c}$$

The coefficient d_{IJ}' corresponds to the second subtraction constant and depends implicitly on μ, which cancels the explicit μ dependence of the amplitudes.

The N/D unitarization breaks crossing symmetry because it treats the s channel distinctly from the other channels. The deviation from symmetry is proportional to $D-1$, because the N function is crossing symmetric by construction. Thus our unitarized amplitude is approximately crossing symmetric away from the resonance.

Our procedure apparently gives three independent subtraction constants. However, there can be at most two independent ones. To see this, we now turn to the discussion of $\pi\pi$ scattering in the chiral Lagrangian language.

IV. CHIRAL LAGRANGIAN UP TO ∂^4 ORDER

Interactions of pions at low energies can be described by the chiral Lagrangian, an effective Lagrangian with nonlinearly realized chiral symmetry, which is an expansion in the number of derivatives. The Lagrangian with terms up to the order ∂^4 takes the form [in the exact SU(2)$\times$SU(2) limit we are working]

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4, \tag{4.1}$$

$$\mathcal{L}_2 = \frac{f_\pi^2}{4}\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U), \tag{4.2a}$$

$$\mathcal{L}_4 = L_1[\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U)]^2 + L_2\mathrm{Tr}(\partial_\mu U^\dagger \partial_\nu U)\mathrm{Tr}(\partial^\mu U^\dagger \partial^\nu U), \tag{4.2b}$$

with

$$U = \exp(i\pi^i \tau^i / f_\pi), \tag{4.3}$$

where π^i ($i=1, 2, 3$) denotes the pion field and τ^i is the Pauli matrix. The parameters in Eq. (4.1) are in principle calculable from QCD, but in practice can be regarded as parameters to be determined from experiments.

The tree level $\pi\pi$ scattering amplitude derived from the Lagrangian (4.1) is

$$A(s,t) = \frac{s}{f_\pi^2} + \frac{8s^2}{f_\pi^4}L_1 + \frac{4(t^2+u^2)}{f_\pi^4}L_2. \tag{4.4}$$

Comparing the tree chiral amplitude (4.4) with the subtracted pole amplitude (2.6), we identify

$$L_1 = \frac{g_\sigma^2 f_\pi^4}{8m_\sigma^4} - \frac{g_\rho^2 f_\pi^4}{4m_\rho^4}, \quad L_2 = \frac{g_\rho^2 f_\pi^4}{4m_\rho^4}. \tag{4.5}$$

It may be seen that the two coefficients reflect the underlying dynamics. The scalar exchange gives $L_2=0$, and the vector exchange is characterized by the relation $L_1+L_2=0$. In the ρ-σ degenerate case with equal couplings, we have $2L_1 + L_2 = 0$.

Independent determination of these parameters has been done using the D wave $\pi\pi$ phase shift [26] or $K \to \pi\pi\ell\nu$ decays [27]. These data exclude the case $L_2=0$. The other two cases of ρ only and degenerate ρ-σ are compatible with the data.

At $\mathcal{O}(s^2)$, the contribution of one-loop graphs with the $\mathcal{L}_2$ vertices has to be included:

$$A(s,t) = \frac{1}{16\pi^2 f_\pi^4}\left\{\frac{1}{2}s^2\left[\frac{1}{\epsilon} - \log\left(\frac{-s}{\mu^2}\right)\right]\right.$$

$$+ \frac{1}{6}t(t-u)\left[\frac{1}{\epsilon} - \log\left(\frac{-t}{\mu^2}\right)\right]$$

$$+ \frac{1}{6}u(u-t)\left[\frac{1}{\epsilon} - \log\left(\frac{-u}{\mu^2}\right)\right]$$

$$\left. + \frac{5}{9}s^2 + \frac{13}{18}(t^2+u^2)\right\}. \tag{4.6}$$

We have used dimensional regularization with $D=4-2\epsilon$ spacetime dimensions. (In the usual convention, $1/\epsilon$ should be interpreted as $1/\epsilon - \gamma_E + \ln 4\pi$.) Notice that the one-loop amplitude contains terms required by unitarity and analyticity at $\mathcal{O}(s^2)$. The logarithmic divergences can be absorbed into the parameter L_i as

$$L_1^r(\mu) = L_1 + \frac{1}{16\pi^2} \frac{1}{24} \left(\frac{1}{\epsilon} + 1 \right), \qquad (4.7a)$$

$$L_2^r(\mu) = L_2 + \frac{1}{16\pi^2} \frac{1}{12} \left(\frac{1}{\epsilon} + 1 \right), \qquad (4.7b)$$

where we have followed the renormalization prescription of Gasser and Leutwyler [26]. The amplitude in terms of the renormalized parameters is

$$A(s,t) = \frac{s}{f_\pi^2} + \frac{8s^2}{f_\pi^4} L_1^r + \frac{4(t^2+u^2)}{f_\pi^4} L_2^r + \frac{1}{16\pi^2 f_\pi^4}$$
$$\times \left[-\frac{1}{2} s^2 \log\left(\frac{-s}{\mu^2} \right) - \frac{1}{6} t(t-u) \log\left(\frac{-t}{\mu^2} \right) \right.$$
$$\left. -\frac{1}{6} u(u-t) \log\left(\frac{-u}{\mu^2} \right) + \frac{2}{9} s^2 + \frac{7}{18}(t^2+u^2) \right].$$
$$(4.8)$$

This gives the general form of the amplitude up to order $\mathcal{O}(s^2)$ compatible with chiral symmetry. Expanding the $\mathcal{O}(E^4)$ chiral amplitude (4.8) into partial waves, we find for $J \le 1$

$$a_{00} = \frac{1}{16\pi} \left\{ \frac{s}{f_\pi^2} + \frac{s^2}{f_\pi^4} \left[\frac{44}{3} L_1^r + \frac{28}{3} L_2^r \right. \right.$$
$$\left. \left. + \frac{1}{16\pi^2} \left(-\log\frac{-s}{\mu^2} - \frac{7}{18} \log\frac{s}{\mu^2} + \frac{17}{12} \right) \right] \right\}, \quad (4.9a)$$

$$a_{20} = \frac{1}{32\pi} \left\{ -\frac{s}{f_\pi^2} + \frac{s^2}{f_\pi^4} \left[\frac{16}{3} L_1^r + \frac{32}{3} L_2^r \right. \right.$$
$$\left. \left. + \frac{1}{16\pi^2} \left(-\frac{1}{2} \log\frac{-s}{\mu^2} - \frac{11}{18} \log\frac{s}{\mu^2} + \frac{17}{12} \right) \right] \right\},$$
$$(4.9b)$$

$$a_{11} = \frac{1}{96\pi} \left\{ \frac{s}{f_\pi^2} + \frac{s^2}{f_\pi^4} \left[-8L_1^r + 4L_2^r \right. \right.$$
$$\left. \left. + \frac{1}{16\pi^2} \left(-\frac{1}{6} \log\frac{-s}{\mu^2} + \frac{1}{6} \log\frac{s}{\mu^2} + \frac{1}{9} \right) \right] \right\}.$$
$$(4.9c)$$

Now we are ready to discuss the connection with the partial waves obtained in Sec. III. Since chiral symmetry allows only two independent $\mathcal{O}(s^2)$ parameters, the three coefficients d_{00}', d_{20}', and d_{11}' cannot be arbitrary. Expanding $a = N/D$ obtained in the previous section up to $\mathcal{O}(s^2)$, we have

$$a_{00} \simeq \frac{1}{16\pi} \left\{ \frac{s}{f_\pi^2} + s^2 \left[\frac{d_{00}'}{f_\pi^2} + \frac{11g_\sigma^2}{6m_\sigma^4} - \frac{4g_\rho^2}{3m_\rho^4} \right. \right.$$
$$\left. \left. - \frac{1}{16\pi^2 f_\pi^4} \log\left(\frac{-s}{\mu^2} \right) \right] \right\}, \qquad (4.10a)$$

$$a_{20} \simeq \frac{1}{32\pi} \left\{ -\frac{s}{f_\pi^2} + s^2 \left[\frac{d_{20}'}{f_\pi^2} + \frac{2g_\sigma^2}{3m_\sigma^4} + \frac{4g_\rho^2}{3m_\rho^4} \right. \right.$$
$$\left. \left. - \frac{1}{32\pi^2 f_\pi^4} \log\left(\frac{-s}{\mu^2} \right) \right] \right\}, \qquad (4.10b)$$

$$a_{11} \simeq \frac{1}{96\pi} \left\{ \frac{s}{f_\pi^2} + s^2 \left[\frac{d_{11}'}{f_\pi^2} - \frac{g_\sigma^2}{m_\sigma^4} + \frac{3g_\rho^2}{m_\rho^4} \right. \right.$$
$$\left. \left. - \frac{1}{96\pi^2 f_\pi^4} \log\left(\frac{-s}{\mu^2} \right) \right] \right\}. \qquad (4.10c)$$

Comparing Eq. (4.10) with Eq. (4.9), we immediately find that the $\log(-s)$ terms obtained here are just as given by the general chiral Lagrangian, although $\log s$ terms are absent in Eq. (4.10). The appearance of the former terms is the result of s channel unitarity and analyticity of the N/D amplitudes. The latter terms, which reflects the crossed channel singularity, are not incorporated in our procedure which is not exactly crossing symmetric. The effect of these logarithmic terms is unimportant if we choose μ to be around m_ρ, since the coefficient is small. Neglecting the logarithmic and related constant terms, we may identify

$$\frac{3}{4} d_{00}' f_\pi^2 + \frac{11g_\sigma^2 f_\pi^4}{8m_\sigma^4} - \frac{g_\rho^2 f_\pi^4}{m_\rho^4} = 11L_1^r + 7L_2^r, \qquad (4.11a)$$

$$\frac{3}{4} d_{20}' f_\pi^2 + \frac{g_\sigma^2 f_\pi^4}{2m_\sigma^4} + \frac{g_\rho^2 f_\pi^4}{m_\rho^4} = 4L_1^r + 8L_2^r, \qquad (4.11b)$$

$$d_{11}' f_\pi^2 - \frac{g_\sigma^2 f_\pi^4}{m_\sigma^4} + \frac{3g_\rho^2 f_\pi^4}{m_\rho^4} = -8L_1^r + 4L_2^r. \qquad (4.11c)$$

This gives one consistency condition for the three subtraction coefficients

$$5d_{20}' = 4(d_{00}' + d_{11}'), \qquad (4.12)$$

which has to hold regardless of the dynamics. In addition, we can impose the dynamics-dependent relation between the chiral Lagrangian parameters discussed below Eq. (4.5) on L_1^r and L_2^r in Eq. (4.11). We find

$$\frac{1}{11} d_{00}' = -\frac{1}{6} d_{11}' = \frac{1}{4} d_{20}' \qquad (4.13)$$
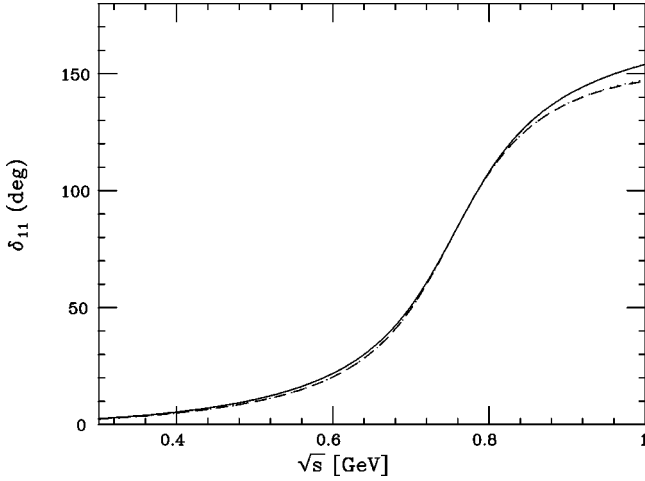
for scalar only ($g_\rho = 0$),

FIG. 1. The $I=J=1$ $\pi\pi$ phase shift. The solid curve is for only ρ exchange with the KSRF coupling, and the dashed (dotted) curve for degenerate ρ and σ with the Veneziano (KSRF/Weinberg) coupling. The latter two curves are almost indistinguishable.

$$\frac{1}{4}d'_{00} = -\frac{1}{9}d'_{11} = -\frac{1}{4}d'_{20} \tag{4.14}$$

for vector only ($g_\sigma = 0$), and

$$4d'_{00} = d'_{11} = d'_{20} \tag{4.15}$$

for the equal contribution of both ($g_\sigma/m_\sigma^2 = g_\rho/m_\rho^2$). These conditions reduce the number of independent subtraction constants to 1.

V. COMPARISON WITH DATA

Let us first discuss the P wave amplitude a_{11}. Experimentally, this amplitude is dominated by the ρ resonance. Since the existence of ρ is well established and the parameters are well measured, we use the mass $m_\rho = 769$ MeV and width 151 MeV as inputs (as well as $f_\pi = 93$ MeV). Since we work in the chiral limit, we correct the measured ρ width for the P wave phase space factor β^3 to obtain the ideal width $\Gamma_\rho = 187$ MeV. We find that the result for the S wave is not sensitive to the inclusion of this correction.

The subtraction constant d'_{11} may be fixed for a given set of model parameters (g_ρ, g_σ) by the condition that the unitarized amplitude gives the correct width Γ_ρ. For a unitarized amplitude a, we define the width by

$$\left.\frac{d}{ds}a^{-1}(s)\right|_{s=m^2} = -\frac{1}{m\Gamma}, \tag{5.1}$$

where the mass m is defined by $a(m^2) = i$. This gives for the N/D amplitude

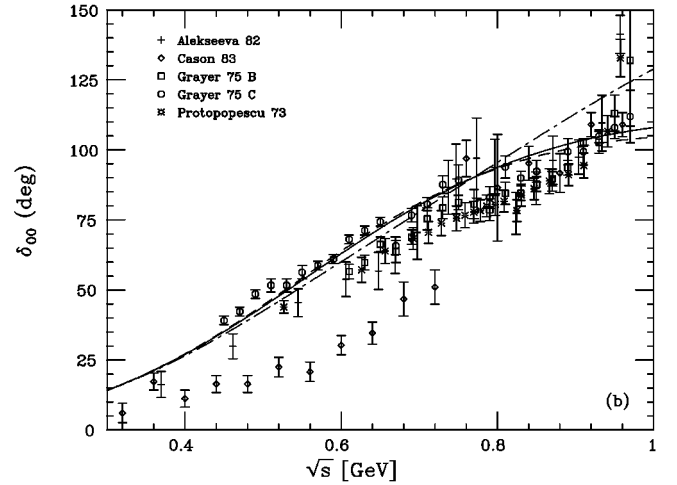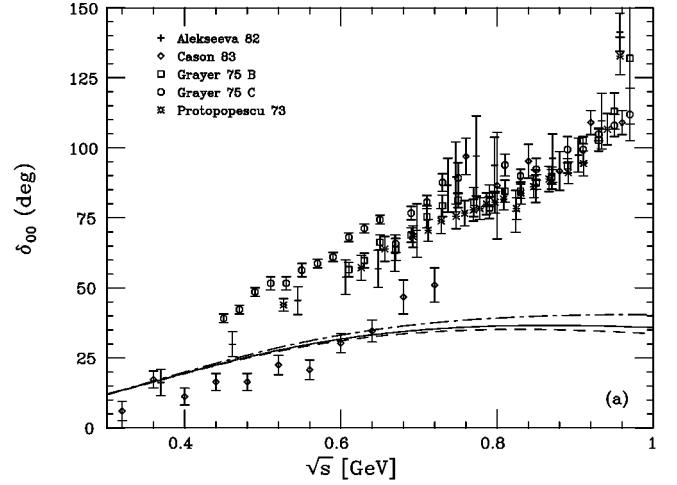$$\Gamma_\rho = \frac{\Gamma_\rho^0}{\mathrm{Re}\,D_{11}(m_\rho^2)}, \tag{5.2}$$

where

FIG. 2. The $I=J=0$ $\pi\pi$ phase shift with (a) ρ exchange only; (b) degenerate σ and ρ exchanges for the KSRF/Weinberg (dashed), Veneziano (dot-dash), and intermediate $g_\rho^2 = 0.45 m_\rho^2/2f_\pi^2$ (solid) couplings. Some experimental data are also shown.

$$\Gamma_\rho^0 = \frac{g_\rho^2 m_\rho}{48\pi}. \tag{5.3}$$

We thus obtain

$$d'_{11}m_\rho^2 = 1 - \frac{g_\rho^2 m_\rho}{48\pi\Gamma_\rho} + \frac{m_\rho^2}{96\pi^2 f_\pi^2}\log\frac{m_\rho^2}{\mu^2}$$
$$- \frac{1}{16\pi^2}\left[g_\sigma^2\mathrm{Re}\,d_{1\sigma}(m_\rho^2/m_\sigma^2) + g_\rho^2\left(\frac{3\pi^2}{4} - \frac{257}{36}\right)\right]. \tag{5.4}$$

In the "ρ only" case, we can drop the term with $d_{1\sigma}$ in Eq. (5.4). In the degenerate case $m_\sigma = m_\rho$ with $g_\sigma = g_\rho$, Eq. (5.4) simplifies to

$$d'_{11}m_\rho^2 = 1 - \frac{g_\rho^2 m_\rho}{48\pi\Gamma_\rho} + \frac{m_\rho^2}{96\pi^2 f_\pi^2}$$
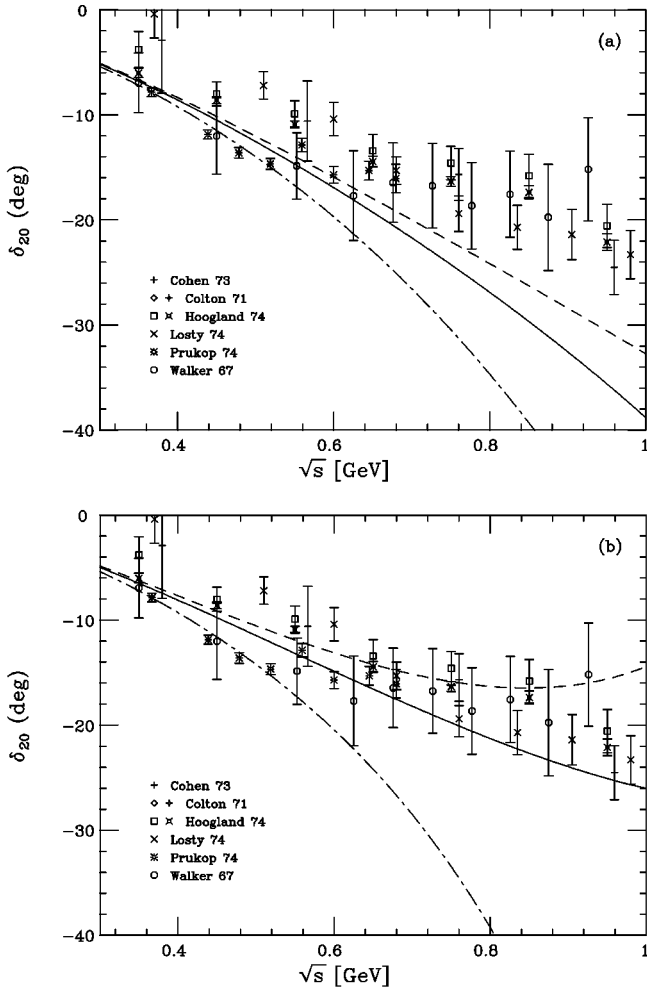$$\times \log\frac{m_\rho^2}{\mu^2} - \frac{g_\rho^2}{16\pi^2}\left(\pi^2 - \frac{173}{18}\right). \tag{5.5}$$

FIG. 3. The $I=2$, $J=0$ $\pi\pi$ phase shift with (a) ρ exchange only; (b) degenerate σ and ρ exchanges for the KSRF/Weinberg (dashed), Veneziano (dot-dash), and intermediate $g_\rho^2=0.45m_\rho^2/2f_\pi^2$ (solid) couplings. Some experimental data are also shown.

This procedure gives a P wave phase shift with the ρ mass and width reproducing the experimental value. It may be thought as renormalizing the coupling with the "on-shell" ρ width, though not in the sense of conventional perturbative expansion. Shown in Fig. 1 is the P wave phase shift for three cases, ρ only for Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) coupling (2.8) and degenerate $\rho-\sigma$ for Veneziano (2.12) and KSRF-Weinberg (2.9) couplings. The difference in the "bare" coupling g_ρ gives very slight change in the phase shift. There is a small difference in the region away from the resonance depending whether the σ exists or not.

We can now determine the two other subtraction constants d'_{00} and d'_{20} from the relations discussed at the end of the previous section: Eq. (4.14) for the ρ-only case, Eq. (4.15) for the degenerate case. It is then possible to calculate the two $J=0$ phase shifts using these parameters.

In Fig. 2(a), we show the calculated $I=J=0$ phase shift in the ρ only scenario for three choices of g_ρ (KSRF/Weinberg, Veneziano, and an intermediate coupling $g_\rho^2=0.45m_\rho^2/f_\pi^2$). The experimental data [25,28–30] are also

plotted. Although the reflection of the crossed channel ρ exchange gives a substantial effect, it can account only about half of the observed phase shift. The phase shift in the degenerate ρ-σ scenario for the same couplings is shown in Fig. 2(b). The agreement with the data is reasonable. It is rather difficult to determine the best value of the coupling from this data.

The phase shift for the exotic channel $I=2$, $J=0$ is shown in Fig. 3 with the experimental data [31–36]. The ρ exchange [Fig. 3(a)] gives slightly larger phase shifts than the data. Unlike the $I=0$ phase shift, the result is very sensitive to the magnitude of the coupling, especially for degenerate ρ-σ exchanges [Fig. 3(b)]. The intermediate coupling of $g_\rho^2\simeq0.45m_\rho^2/f_\pi^2$ reproduces the data quite well.

VI. CONCLUSIONS

We have proposed a general "model-independent" framework of the $\pi\pi$ scattering based on chiral low-energy expansion and possible resonances in the $I=J=0$ and $I=J=1$ channels. To cope with the strong interaction of pions, we use the N/D formalism to obtain partial wave amplitudes which satisfy unitarity, analyticity, and approximate crossing symmetry. The result is compared to the experimental phase shift data and we find preference for a σ resonance with a mass similar to the ρ meson. Without σ, the ρ exchange in the crossed channel can give substantial reflection in the scalar channel, but the effect is not large enough to explain the measured phase shift.

In this work, we have examined two clearcut cases with ρ only, and degenerate ρ-σ with the same coupling strengths. There is certainly some room to improve the fit if we regard the σ mass and coupling as free parameters. It is also desirable to include the effect of the pion mass, which we have neglected in the present study. These questions will be addressed in a future study.

Theoretically, $\pi\pi$ scattering is the simplest laboratory of the low-energy strong interaction. Unfortunately, no new experiment has been done since early 1980's and the most recent result is in some disagreement with older data. (We note that more recent experiments on the σ meson utilize "Pomeron-Pomeron" scattering or $p\bar{p}$ annihilation.) New experiments with more precision are clearly desirable. Systematic uncertainties may also be reduced. In fact, the existing data involve some extrapolation because they are extracted from the reaction $\pi N\to\pi\pi N$. It would be much more welcome if direct beam-beam $\pi\pi$ experiment can be done.

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