AdS-CFT correspondence and a new positive energy conjecture for general relativity

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We examine the AdS-CFT correspondence when gauge theory is considered on a compactified space with supersymmetry-breaking boundary conditions. We find that the corresponding supergravity solution has a negative energy, in agreement with the expected negative Casimir energy in the field theory. The stability of the gauge theory would imply that this supergravity solution has minimum energy among all solutions with the same boundary conditions. Hence we are led to conjecture a new positive energy theorem for asymptotically locally anti–de Sitter spacetimes. We show that the candidate minimum energy solution is stable against all quadratic fluctuations of the metric. [S0556-2821(98)07824-2]

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I. INTRODUCTION

There is growing evidence for a remarkable correspondence between string theory in anti–de Sitter (AdS) spacetime and conformal field theory (CFT) [1,2,3]. In particular, type-IIB superstring theory on $AdS_5 \times S^5$ is believed to be completely equivalent to $\mathcal{N}=4$ super-Yang-Mills theory in four dimensions [1]. For many applications, it suffices to consider just the low energy limit of the superstring theory, namely, supergravity. There is a well-defined total energy for any spacetime which asymptotically approaches AdS spacetime [4], and part of the correspondence is that this energy agrees with energy in the gauge theory. For solutions which approach AdS spacetime globally, there are positive energy theorems which ensure that this energy cannot be negative [5], in agreement with the stability of the gauge theory vacuum.

Witten [6] has suggested that one can describe ordinary (i.e., nonsupersymmetric) Yang-Mills gauge theory by compactifying one direction on a circle and requiring antiperiodic boundary conditions for the fermions around the circle. In this case, the additional fermions and scalars would acquire large masses, leaving the gauge fields as the only low energy degrees of freedom. On the supergravity side, this proposal corresponds to considering spacetimes which are asymptotically AdS locally, but not globally. That is, one spatial direction is compactified on a circle asymptotically. If the spacetime topology is globally a simple product with an S^1 factor, the standard approaches [5] should still yield a positive energy theorem (see, e.g., [7,8]). However, if one considers more general topologies, e.g., for which the asymptotic circle is contractible in the interior, those techniques will not apply and hence it is uncertain if a positive energy theorem will hold. It is known that in the case of asymptotically flat spacetimes it does not: These boundary conditions allow nontrivial zero [9] and negative [10] energy solutions. In particular (for a fixed size circle at infinity), there are nonsingular solutions to Einstein's vacuum field equations with arbitrarily negative energy. Therefore, this sector of the theory is completely unstable.[1]

It is important to determine whether a similar instability arises for spacetimes which are asymptotically locally AdS. From a mathematical viewpoint, this seems rather likely [12]. Negatively curved spaces tend to be less constrained than those with positive (or zero) curvature [13]. One expects that anything that is true for asymptotically flat spacetimes should also be true for asymptotically AdS spacetimes. Of course, if the result were true for the AdS case, it would have serious consequences in the context of the AdS-CFT correspondence. A straightforward interpretation would be that the supergravity analysis is making the rather dramatic prediction that the nonsupersymmetric, strongly coupled gauge theory is unstable. However, another possibility is that this result is an indication that the correspondence fails with nonsupersymmetric boundary conditions. In the latter case, it would spoil the hope of using supergravity to learn about ordinary gauge theory.

We will show that there is a static nonsingular solution (to Einstein's equation with negative cosmological constant) with these boundary conditions which has *negative* total energy. Rather than invalidate the AdS-CFT correspondence, this particular solution has a natural interpretation in the gauge theory. Since supersymmetry is broken by antiperiodic boundary conditions on the fermions, the gauge theory on $S^1 \times R^2$ is expected to have a negative Casimir energy. Comparing the negative energy computed from supergravity and

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[1] This general result applies for any theory involving Einstein gravity in higher dimensions, including *superstring theory* [10]. The closely related positive action conjecture is also false for spacetimes which are only locally asymptotically Euclidean [11].

the Casimir energy in the weakly coupled gauge theory, we find close agreement. They have the same dependence on all parameters and disagree only by an overall factor of 3/4. This is similar to the factor of 3/4 that was noticed previously in comparisons of the entropy of the near-extremal three-branes [14]. We will show that in fact these two factors have the same origin.

The key question is whether the solution described above is the lowest energy solution with these boundary conditions. If so, there must be a new positive energy theorem which ensures that the energy of all solutions is greater than or equal to this negative value, with equality only for our particular solution. At first sight, this seems very unlikely, since the solution we discuss does not have constant curvature, supersymmetry, or any other distinguishing property which have previously characterized minimum energy solutions in general relativity. Nevertheless, we will present evidence in favor of this new positive energy theorem. We will show that the solution is a local minimum of the energy: i.e., it is stable to small fluctuations. The existence of this new theorem can be viewed as a highly nontrivial prediction of the AdS-CFT correspondence. A complete proof would provide strong evidence for the correspondence.

The outline of this paper is as follows. In the next section, we review the definition of energy for spacetimes that are asymptotically AdS. In Sec. III, we present our solutions with negative total energy and discuss their relation to the CFT. Section IV contains the statement of the new positive energy conjecture and some evidence in favor of it. In Sec. V, we consider some generalizations of the conjecture, and further discussion is given in Sec. VI.

II. ENERGY IN ANTI–de SITTER SPACETIME

The definition of total energy for spacetimes which asymptotically approach AdS spacetime was first discussed in Ref. [4]. In the following, we will adopt an equivalent definition derived in [15] (see also [16]). The total energy in general relativity is always defined relative to a background solution which has a time translation symmetry. Let the norm of the timelike Killing field be[2] $-N^2$. The energy depends only on N and on the metric of a spacelike surface which asymptotically approaches the background geometry. Starting from the action and deriving the Hamiltonian, keeping track of surface terms, one finds [15]

$$E=-\frac{1}{8\pi G}\int N(K-K_0), \tag{2.1}$$

where the integral is over a surface near infinity, K is the trace of the extrinsic curvature of this surface, and K_0 is the trace of the extrinsic curvature of a surface with the same intrinsic geometry in the background or reference spacetime.[3] This definition is very general and works for both asymptotically flat and asymptotically AdS spacetimes.

Let us illustrate this definition with a few examples. Consider the Schwarzschild-AdS solution in four dimensions:

$$ds^2=-\left(\frac{r^2}{l^2}+1-\frac{r_0}{r}\right)dt^2+\left(\frac{r^2}{l^2}+1-\frac{r_0}{r}\right)^{-1}dr^2+r^2d\Omega_2, \tag{2.2}$$

where l is related to the negative cosmological constant by $l^2=-3/\Lambda$. Consider a spatial slice of constant t in this space. At fixed r, one has a round two-sphere with area $A=4\pi r^2$. The integral of the trace of the extrinsic curvature of this sphere is easily computed as

$$\int K=n^\mu\partial_\mu A=\left(\frac{r^2}{l^2}+1-\frac{r_0}{r}\right)^{1/2}8\pi r, \tag{2.3}$$

where n^μ is the unit radial vector normal to the sphere. The background or reference spacetime is just anti–de Sitter space, i.e., Eq. (2.2) with $r_0=0$. At fixed t, the boundary surface in the background with the same intrinsic geometry as above is again a two-sphere at the same value of the radial coordinate r. Thus $\int K_0$ is simply given by Eq. (2.3) with $r_0=0$. In either case, N is constant on the sphere and asymptotically approaches $N\simeq r/l$. Substituting these expressions into Eq. (2.1) yields $E=r_0/2G_4$ as expected (where G_4 is Newton's constant in four dimensions).

This calculation is easily extended to arbitrary dimensions with the black hole metric

$$ds^2=-\left[\frac{r^2}{l^2}+1-\left(\frac{r_0}{r}\right)^{p-1}\right]dt^2+\left[\frac{r^2}{l^2}+1-\left(\frac{r_0}{r}\right)^{p-1}\right]^{-1}dr^2+r^2d\Omega_p, \tag{2.4}$$

where $d\Omega_p$ is the metric on a unit p-sphere and $l^2=-p(p+1)/2\Lambda$. Also note that $p\geqslant 2$ for the above metric. The final result for the energy is

$$E=\frac{p\Omega_p}{16\pi G_{p+2}}\,r_0^{p-1}, \tag{2.5}$$

where

$$\Omega_p=2\pi^{(p+1)/2}\Big/\Gamma\left(\frac{p+1}{2}\right)$$

is the area of a unit p-sphere and G_{p+2} is the $(p+2)$-dimensional Newton's constant.

Next consider the following asymptotically AdS metrics:

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[2]When there is more than one timelike Killing field, there are additional conserved quantities. The energy is then one component of a conserved vector (or tensor). We will focus on one timelike component and call it the energy.

[3]To leading order, N will be the same for both the given metric and the background space. Higher order differences between N and N_0 will not affect the result for the energy [15].

$$ds^2 = \frac{r^2}{l^2}\left[-\left(1-\frac{r_0^{p+1}}{r^{p+1}}\right)dt^2 + (dx^i)^2\right] + \left(1-\frac{r_0^{p+1}}{r^{p+1}}\right)^{-1}\frac{l^2}{r^2}\,dr^2, \tag{2.6}$$

where $i=1,\ldots,p$. For certain values of p, these metrics arise in the near-horizon geometry of p-branes (see, e.g., [1]). With $r_0=0$, these metrics correspond to AdS space in horospheric coordinates [17]. Once again we consider a surface of constant t. If we introduce V_p as the coordinate volume of the surfaces parametrized by x^i, then the area of a surface at fixed large r is simply $A=r^pV_p/l^p$. Computing the energy as before yields

$$E_p = \frac{pV_p}{16\pi G_{p+2}l^{p+2}}\,r_0^{p+1}. \tag{2.7}$$

E_p/V_p corresponds to the energy density of the field theory in the CFT-AdS correspondence.

There is a slight subtlety in computing the mass of the above metrics (2.6). If the directions along the brane x^i are not identified (i.e., are noncompact), then the constant r_0 can be changed by rescaling the coordinates t, r, x^i in an appropriate way. Hence the energy (2.7) is not well defined. This is not surprising, since the energy is conjugate to asymptotic time translations, and so if one rescales the time, the energy should change.[4] In the following, we will be interested in the case where at least one of the spatial directions is compactified. If we fix the periodicity of the circle (corresponding to fixing the size of the circle in the gauge theory), then r_0 cannot be rescaled. However, when some of the x^i's are compactified, the background spacetime with $r_0=0$ has a conical singularity at $r=0$. We will not worry about this singularity, since it is likely that string theory resolves it without changing the asymptotic form of the metric, which is all that is needed to compute the energy. More importantly, the lower energy solution we describe in the next section is completely nonsingular.

III. NEGATIVE ENERGY SOLUTIONS

We begin by reviewing the negative energy solutions in the asymptotically flat context [10]. It is easy to describe the initial data for these negative energy solutions. For five-dimensional solutions, the initial data consist of a four-dimensional Riemannian manifold which asymptotically approaches the flat metric on $S^1\times R^3$. Of course, within general relativity, these initial data must satisfy a number of constraint equations. However, if we set the conjugate momentum to zero, these constraints reduce to the condition that the scalar curvature vanish. As initial data, we consider the Euclidean Reissner-Nordström metric

$$ds^2 = \left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)d\tau^2 + \left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_2. \tag{3.1}$$

To avoid a conical singularity at $r=r_+\equiv m+\sqrt{m^2-q^2}$, we must periodically identify τ and so Eq. (3.1) has the desired asymptotic geometry. It also satisfies the constraint, because the Einstein tensor is proportional to the Maxwell stress tensor, which is trace free in four dimensions. We now analytically continue the parameter $q\to iq$. (Since we are interested only in the metric (3.1) and do not include a Maxwell field, we do not have to worry about the latter becoming complex.) It is now clear that we can take the mass parameter $m<0$ without the metric becoming singular. Since the size of the circle at infinity is just the period of τ which depends on both m and q, one can keep this fixed as m becomes arbitrarily negative. In fact, one finds, for a fixed period, that the curvature remains bounded as the mass becomes increasingly negative. Therefore one may conclude that such toroidal compactifications in asymptotically flat spacetimes are unstable. In theories with fermions, this instability only arises in the sector where the spin structure is asymptotically antiperiodic on one of the S^1 factors. Aside from this restriction, the analysis applies quite generally to any theory involving Einstein gravity in higher dimensions, including superstring theory [10].

We now want to know if an analogous result holds for spacetimes which are asymptotically AdS. The first thing to try is an obvious generalization of the above procedure using the Euclidean AdS Reissner-Nordström metric. When the charge in the latter is analytically continued, the metric becomes

$$ds^2 = \left(\frac{r^2}{l^2}+1-\frac{r_1}{r}-\frac{r_0^2}{r^2}\right)d\tau^2 + \left(\frac{r^2}{l^2}+1-\frac{r_1}{r}-\frac{r_0^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_2. \tag{3.2}$$

As before, this satisfies the vacuum constraints (now with negative cosmological constant) if the momenta are set equal to zero. However, there is an important difference with the asymptotically flat case. In Eq. (3.2), the proper length of the circles parametrized by τ grows with r. This means that the area of the surface at infinity grows like r^3 just like the uncompactified five-dimensional AdS spacetime. As a result, the mass is determined by the r_0^2/r^2 terms in the metric, rather than the r_1/r term. The appropriate physical boundary conditions—see the discussion in Sec. V-require that $r_1=0$. Thus this construction only yields the following one-parameter family of finite energy initial data:

$$ds^2 = \left(\frac{r^2}{l^2}+1-\frac{r_0^2}{r^2}\right)d\tau^2 + \left(\frac{r^2}{l^2}+1-\frac{r_0^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_2. \tag{3.3}$$

[4] In the full asymptotically flat p-brane solution, this is not a problem, since the scale for t is picked out by the requirement that ∂_t be a unit time translation at infinity. It is this time which corresponds to time translation in the gauge theory. In the previous metrics (2.4), the scale of r is fixed by requiring the spheres of constant radius to have an area given by $r^p\Omega_p$.

Note that one cannot change the sign of r_0^2 without introducing a naked singularity at $r=0$. With the above sign, the radial coordinate is restricted to $r \geqslant r_+$, where r_+ is the largest root of $r_+^4 + l^2(r_+^2 - r_0^2) = 0$. To avoid a conical singularity at $r = r_+$, τ is identified with period

$$\beta = \frac{2\pi l^2 r_+}{2r_+^2 + l^2}. \tag{3.4}$$

The metric (3.3) can also be obtained by analytically continuing the five-dimensional Schwarzschild AdS solution and restricting oneself to the equatorial plane of the three-spheres. One might thus expect that its mass would be positive. However, we now show that it is negative.

The area of a surface at large r is

$$A = 4\pi r^2 \beta \left(\frac{r^2}{l^2} + 1 - \frac{r_0^2}{r^2} \right)^{1/2}. \tag{3.5}$$

The integral of the extrinsic curvature becomes

$$\int K = 4\pi\beta \left[\frac{3r^3}{l^2} + 2r - \frac{r_0^2}{r} \right]. \tag{3.6}$$

The background metric is simply Eq. (3.3) with $r_0 = 0$, which is four-dimensional hyperbolic space with periodic identifications.[5] In this reference space, we need to choose a boundary surface with the same intrinsic geometry as the $S^2 \times S^1$ at fixed r above. For the S^2 geometry to agree, the radial coordinate of the surface in the background must be the same as in the original spacetime. For the proper distances along the S^1 factors to agree, the periodicity of τ in the background β_0 is related to β by

$$\left(\frac{r^2}{l^2} + 1 - \frac{r_0^2}{r^2} \right)^{1/2} \beta = \left(\frac{r^2}{l^2} + 1 \right)^{1/2} \beta_0. \tag{3.7}$$

The integral of the extrinsic curvature in the background is simply

$$\int K_0 = 4\pi\beta_0 \left[\frac{3r^3}{l^2} + 2r \right]. \tag{3.8}$$

Using Eq. (3.7), $N \simeq r/l$, and the definition of the energy, Eq. (2.1), one finds that

$$E = -\frac{\beta r_0^2}{4G_5 l}. \tag{3.9}$$

From this, one clearly sees that the energy is negative, but it is difficult to see the dependence on the size of the circle β since r_0 is implicitly related to β through Eq. (3.4) and the definition of r_+. For $r_+ \gg l$, $r_0^2 \simeq r_+^4/l^2 \simeq \pi^4 l^6/\beta^4$, and so one obtains[6]

$$E \simeq -\frac{\pi^4 l^5}{4G_5 \beta^3}. \tag{3.10}$$

Through the AdS-CFT correspondence, this analysis should be related to a gauge theory on $S^2 \times S^1$ where S^2 has radius l, S^1 has period β, and supersymmetry-breaking boundary conditions are imposed along the S^1. For small β, this is expected to have a negative Casimir energy density proportional to β^{-4} (at least at weak coupling). Hence the total energy would be negative and proportional to β^{-3}, in agreement with the above supergravity calculation.

The preceding calculations can be extended to arbitrary dimensions with the initial data metric

$$ds^2 = \left[\frac{r^2}{l^2} + 1 - \left(\frac{r_0}{r} \right)^{p-1} \right] d\tau^2 + \left[\frac{r^2}{l^2} + 1 - \left(\frac{r_0}{r} \right)^{p-1} \right]^{-1} dr^2 + r^2 d\Omega_{p-1}, \tag{3.11}$$

which satisfies the constraint equation ${}^{(p+1)}R = -p(p+1)/l^2$. Again, the geometry smoothly closes off at $r = r_+$, which is now the largest root of $r_+^{p+1} + l^2 (r_+^{p-1} - r_0^{p-1}) = 0$, provided that τ is identified with period

$$\beta = \frac{4\pi l^2 r_+}{(p+1) r_+^2 + (p-1) l^2}. \tag{3.12}$$

The final result for the energy is

$$E = -\frac{\Omega_{p-1} \beta r_0^{p-1}}{16\pi G_{p+2} l} \simeq -\frac{\Omega_{p-1}}{16\pi G_{p+2}} \left(\frac{4\pi}{p+1} \right)^{p+1} \frac{l^{2p-1}}{\beta^p}, \tag{3.13}$$

where the final formula again holds for large r_+. Where applicable, these results should be related to a quantum field theory on $S^{p-1} \times S^1$. Again, a negative energy proportional to β^{-p} can be expected to arise through the Casimir effect in the field theory.

To make a more precise comparison of the energy in AdS spacetime and the gauge theory, it would be useful to have an example of a solution with negative energy that asymptotically had topology $R^{p-1} \times S^1$. This is easily obtained by a double analytic continuation of the near-extremal p-brane solution (2.6). That is, we analytically continue this metric with both $t \to i\tau$ and $x^p \to it$. In the following, we will refer to this spacetime as the *AdS soliton*. The metric becomes

[5]Even though the spacetime resulting from these initial data is locally AdS, it is not globally static. So extra restrictions are needed to ensure energy conservation. This will not be the case for our main example discussed below.

[6]For a given β, there is another solution to Eq. (3.4) with a smaller value of r_+, but it corresponds to a configuration for which the energy which is less negative.

$$ds^2=\frac{r^2}{l^2}\left[\left(1-\frac{r_0^{p+1}}{r^{p+1}}\right)d\tau^2+(dx^i)^2-dt^2\right]$$
$$+\left(1-\frac{r_0^{p+1}}{r^{p+1}}\right)^{-1}\frac{l^2}{r^2}\,dr^2, \tag{3.14}$$

where there are now $p-1$ x^i's. Again, the coordinate r is restricted to $r\geqslant r_0$ and τ must be identified with period $\beta=4\pi l^2/(p+1)r_0$ to avoid a conical singularity at $r=r_0$. Note that this spacetime metric is globally static and completely nonsingular. For fixed t and r, the area of a surface is

$$A=\left(1-\frac{r_0^{p+1}}{r^{p+1}}\right)^{1/2} r^p\beta V_{p-1}/l^p,$$

where V_{p-1} denotes the volume of the transverse $p-1$ x^i's. The appropriate background is Eq. (3.14) with $r_0=0$, which corresponds to AdS space with periodic identifications. It is easy to see that for the background, $\int K_0=(p/l)A$. Following the above procedure, the energy is again found to be negative:

$$E=-\frac{r_0^{p+1}\beta V_{p-1}}{16\pi G_{p+2}l^{p+2}}. \tag{3.15}$$

Using the above relation between r_0 and β, this result can be rewritten as

$$E=-\frac{V_{p-1}l^p}{16\pi G_{p+2}\beta^p}\left(\frac{4\pi}{p+1}\right)^{p+1}. \tag{3.16}$$

For certain values of p, we can express this in terms of the string theory coupling g and Ramond-Ramond charge N. For example, for the three-brane $p=3$, we have $l^4=4\pi gN$ and G_5 is the ten-dimensional Newton's constant, $G_{10}=8\pi^6g^2$, divided by the volume of a five sphere of radius l, $A_5=\pi^3l^5$. We thus obtain an energy density

$$\rho_{\rm SUGRA}=\frac{E}{V_2\beta}=-\frac{\pi^2}{8}\frac{N^2}{\beta^4}. \tag{3.17}$$

We wish to compare this with the ground state energy of the gauge theory on $S^1\times R^2$, where the length of the S^1 is β. This can only be calculated directly at weak gauge coupling, where, to leading order, it reduces to the problem of determining the Casimir energy of the free field theory. The field theory is $N=4$ super-Yang Mills, which contains an SU(N) gauge field, six scalars in the adjoint representation, and their superpartner fermions. In the present case, the latter fermions are antiperiodic on the S^1. The stress-energy tensor for this theory may be found in [18]. The leading order Casimir energy may be calculated by point-splitting the fields in the energy density (i.e., T_{tt}) with the appropriate free-field Green's function and then removing the vacuum divergence before taking the limit of coincident fields [19]. The final result is

$$\rho_{\rm gauge}=-\frac{\pi^2}{6}\frac{N^2}{\beta^4}. \tag{3.18}$$

Thus we find that the negative energy density of the supergravity solution is precisely 3/4 of the Casimir energy of the weakly coupled gauge theory. This is very reminiscent of earlier results showing that the entropy of the near-extremal three-brane is precisely 3/4 the entropy of the weakly coupled gauge theory at the same temperature [14].

In retrospect, it is not surprising that the ground state energies differ by exactly the same factor as the thermal entropies, as both results can be derived as different interpretations of a common Euclidean calculation. On the field theory side, consider the Euclidean functional integral for the (weakly coupled) Yang-Mills theory on $S^1\times R^3$ where the circle has period β and antiperiodic boundary conditions for the fermions. One natural interpretation is as a thermal field theory calculation at temperature β^{-1}, and the partition function yields the free energy as $\beta F_{\rm YM}=-\log Z_{\rm YM}$. Alternatively, one may interpret one of the noncompact directions as Euclidean time t_E. In this case the same partition function, evaluated between two surfaces separated by a large difference of Euclidean time Δt_E, yields the ground state energy as $\Delta t_E E_{\rm YM}=-\log Z_{\rm YM}$. On the supergravity side, consider the Euclidean instanton obtained by analytically continuing the AdS soliton metric (3.14) with $t\rightarrow it_E$. This instanton is, of course, identical to that obtained from the black hole metric (2.6) with $t\rightarrow i\tau$ and identifying τ with the appropriate period β. (We also trivially rename $x^p=t_E$.) In the latter context, the instanton describes a thermal equilibrium of the black hole [20] at temperature β^{-1}, and the Euclidean action is interpreted as giving the black hole contribution to the free energy as $\beta F_{\rm BH}=I$. On the other hand, in the context of the AdS soliton, the same Euclidean action is simply related to the total energy (3.16) via $\Delta t_E E=I$. Now from the analysis of three-branes [14] ($p=3$), it follows that when the temperatures of the black hole and Yang-Mills theory are equated, their free energies are related by

$$F_{\rm BH}=\frac{3}{4}F_{\rm YM} \tag{3.19}$$

and hence

$$I=\frac{3}{4}(-\log Z_{\rm YM}). \tag{3.20}$$

Hence, from the preceding discussion, it also follows that we must find the same factor in relating the total energies, $E=3/4E_{\rm YM}$, and the energy densities above, as well.

The factor of 3/4 discrepancy between the two calculations does not contradict the AdS-CFT correspondence. Rather, the supergravity result (3.17) corresponds to the energy density of the gauge theory in a regime of strong coupling. To extrapolate the AdS results to weak coupling, one must include all of the higher order (in the string scale) corrections to the geometry induced by type-IIB string theory. The leading order correction to Eq. (3.14) has recently been computed in [21]. The net effect is that the Euclidean action became more negative. Thus from the preceding discussion, as expected, the energy and energy density of the corresponding AdS soliton becomes slightly more negative, improving the agreement with the weak coupling results (3.18).

In order to construct these negative energy solutions (3.14), we need $p \geqslant 1$ so that there will be one spatial direction in Eq. (2.6) to analytically continue. The case $p=1$, corresponding to three spacetime dimensions, is special. All solutions have constant curvature and hence are locally AdS. The metric (2.6) with $p=1$ is the (nonrotating) Bañados-Teitelbeim-Zanelli (BTZ) black hole [22]. We showed above that if you take the positive mass black hole and double analytically continue, then you get a solution with less mass than the zero-mass black hole. However, it is known that three-dimensional AdS spacetime itself has less mass than the $M=0$ black hole [22]. In fact, as we will now show, the double analytically continued black hole is precisely AdS globally, with no extra identifications. We start with the black hole metric (2.6) with $p=1$

$$ds^2 = -\frac{r^2-r_0^2}{l^2}\,dt^2 + \frac{l^2}{r^2-r_0^2}\,dr^2 + \frac{r^2}{l^2}\,dx^2. \quad (3.21)$$

By rescaling t,r,x, we can set $r_0 = l$. Now analytically continue in t and x as before to get

$$ds^2 = -\frac{r^2}{l^2}\,dt^2 + \left(\frac{r^2}{l^2}-1\right)^{-1} dr^2 + \left(\frac{r^2}{l^2}-1\right) d\tau^2. \quad (3.22)$$

Finally, set $\rho^2 = r^2 - l^2$ and $\phi = \tau/l$ to put this metric into the standard AdS form

$$ds^2 = -\left(\frac{\rho^2}{l^2}+1\right) dt^2 + \left(\frac{\rho^2}{l^2}+1\right)^{-1} d\rho^2 + \rho^2 d\phi^2. \quad (3.23)$$

Note that after the analytic continuation, τ should be periodically identified to avoid a conical singularity. In Eq. (3.23), this is simply the statement that ϕ is an angle with the standard periodicity of 2π.

IV. NEW POSITIVE ENERGY THEOREM?

A. Conjectures

The above qualitative and quantitative agreements between AdS energy and Casimir energy in the CFT seem to support the AdS-CFT correspondence in the nonsupersymmetric case. A crucial question though is whether the AdS soliton (3.14) is the lowest energy solution with the given boundary conditions. The aforementioned agreement would be put in peril by the existence of metrics with even lower energies.

For definiteness, let us focus on the $p=3$ case in the following. This corresponds to the near-horizon geometry of Dirichlet three-branes, for which the AdS-CFT correspondence is understood in the most detail. The AdS soliton metric (3.14) then becomes

$$ds_3^2 = \frac{r^2}{l^2}\left[\left(1-\frac{r_0^4}{r^4}\right) d\tau^2 + (dx^1)^2 + (dx^2)^2 - dt^2\right] + \left(1-\frac{r_0^4}{r^4}\right)^{-1} \frac{l^2}{r^2}\,dr^2, \quad (4.1)$$

where $r \geqslant r_0$ and τ has period $\beta = \pi l^2/r_0$. *For the remainder of this section, we will use Eq. (4.1) as our reference metric and measure energy relative to it*. We will consider metrics which asymptotically approach Eq. (4.1) in the sense that, for large r,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},$$
$$h_{\alpha\gamma} = O(r^{-2}), \quad h_{\alpha r} = O(r^{-4}),$$
$$h_{rr} = O(r^{-6}) \quad \text{with} \ \ \alpha,\gamma \neq r. \quad (4.2)$$

Derivatives of $h_{\mu\nu}$ are required to fall off one power faster. In Eq. (4.2), $\bar{g}_{\mu\nu}$ denotes the AdS soliton (4.1) (or metrics obtained from it as indicated below). Note that even though these boundary conditions allow metrics with different constants r_0 asymptotically, the periodicity of τ is fixed.

The AdS-CFT correspondence, together with the expected stability of the nonsupersymmetric gauge theory, suggests that the energy of any solution with these boundary conditions should be positive relative to Eq. (4.1). Hence we are led to formulate a new positive energy conjecture. Below, we present three different forms of this conjecture, starting with the most general and becoming more specialized. The simpler conjectures may be easier to prove, but would still be of great interest.

Conjecture 1. Consider all solutions to ten-dimensional type-IIB supergravity satisfying Eq. (4.2) [with $\bar{g}_{\mu\nu}$ denoting the product of Eq. (4.1) with a five-sphere of radius l]. Then $E \geqslant 0$, with equality if and only if $g_{\mu\nu} = \bar{g}_{\mu\nu}$.

The self-dual five-form must be nonzero to satisfy the asymptotic boundary conditions. If we make the reasonable assumptions that the other supergravity fields will only increase the energy and spacetimes which are not direct products with S^5 will also have higher energy, then the above conjecture can be reduced from ten dimensions to five as follows.

Conjecture 2. Consider all solutions to Einstein's equation in five dimensions with cosmological constant $\Lambda = -6/l^2$ satisfying Eq. (4.2) [with $\bar{g}_{\mu\nu}$ denoting the metric (4.1)]. Then $E \geqslant 0$, with equality if and only if $g_{\mu\nu} = \bar{g}_{\mu\nu}$.

In the above conjectures, the solutions are required to have at least one nonsingular spacelike surface, since otherwise one could easily construct counterexamples with naked singularities. If we assume that there is a surface with zero extrinsic curvature (i.e., a moment of time symmetry), then the constraint equations reduce to the statement that the scalar curvature is constant. We thus obtain the following.

Conjecture 3. Given a nonsingular Riemannian four-manifold with $R = -12/l^2$ satisfying Eq. (4.2) [with $\bar{g}_{\mu\nu}$ denoting the metric on a $t=$constant surface in Eq. (4.1)], then $E \geqslant 0$ with equality if and only if $g_{\mu\nu}=\bar{g}_{\mu\nu}$.

As we mentioned in the Introduction, at first sight these conjectures seem unlikely to be true. The solution (4.1) does not have constant curvature, supersymmetry, or any other special property usually associated with minimum energy solutions in general relativity. It is possible that the above conjectures fail, but there is another solution of minimum energy. However, this is unlikely, since one expects the minimum energy solution to be static and translationally invariant around the circle. One could then double analytically continue this metric to produce a new black hole solution. The ''black hole uniqueness theorems'' (which have not been proved for this case, but still are believed to be true) would then imply that this solution must be identical to Eq. (2.6) with $p=3$, which corresponds to the analytic continuation of Eq. (4.1). Furthermore, previous experience would suggest that there are time-dependent solutions of arbitrarily negative energy—see, e.g., [10].

Nevertheless, in this section we present some evidence that the above conjectures are indeed true. First, we note that under perturbations of the metric (4.1), the energy is unchanged to first order. This result in fact applies for any metric that is globally static [23] and can be seen as follows. The gravitational Hamiltonian is a function of the spatial metric and conjugate momentum, and takes the form $H(g_{ij},\pi^{ij})=\int N^\mu C_\mu + E$, where N^μ is the lapse-shift vector and C_μ are the constraints. Suppose we start with a static solution and choose N^μ to generate evolution along the time translation symmetry. Consider the variation of H with respect to g_{ij}. On the one hand, this is $\partial_t \pi^{ij}$, which vanishes since the background is static. On the other hand, the variation of the constraint will vanish whenever the perturbation solves the linearized constraints. Hence the variation of the energy must also vanish. Since E is independent of the conjugate momenta, this is sufficient to establish that the energy is an extremum.

B. Perturbative stability

While we have established that the mass of the AdS soliton is an extremum, we would like to show that it is actually a global minimum. Unfortunately, given the (nonsupersymmetric) spin structure on the asymptotic geometry, we cannot apply the spinor techniques of [24] and [5] to argue that this is the case. Instead, we must be satisfied with showing that the AdS soliton gives a local minimum of the AdS energy functional. Our perturbative approach here follows that of [4], where the stability of the AdS spacetime itself was considered. We refer the interested reader to there for a detailed discussion of the technique. Their general analysis is based on the construction of conserved charges for background solutions with Killing symmetries, in particular a time translation symmetry, and hence may be applied to the AdS soliton.

One begins by dividing the metric (globally) in a manner similar to Eq. (4.2):

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}, \tag{4.3}$$

where $\bar{g}_{\mu\nu}$ is the AdS soliton and $h_{\mu\nu}$ represents a deviation satisfying the above boundary conditions. For the moment we will work with the general d-dimensional case and later specialize to $d=5$. In order that $g_{\mu\nu}$ be still a solution, $h_{\mu\nu}$ must satisfy an equation which may be represented as a linearized Einstein equation with a nonlinear source term. Written in this form, the terms nonlinear in $h_{\mu\nu}$ may be taken to define the energy-momentum density of the gravitational field, $T^{\mu\nu}$. By virtue of the field equations, this density is covariantly conserved in the background metric, i.e., $\bar{\nabla}_\mu T^{\mu\nu}=0$. Now given a Killing vector ξ^μ of the background solution, one finds then that $T^\mu{}_\nu\xi^\nu$ is a covariantly conserved current and hence

$$E(\xi)=\frac{1}{8\pi G}\int d^{d-1}x\,\sqrt{\bar{g}}T^0{}_\nu\xi^\nu \tag{4.4}$$

is a conserved charge. If ξ^μ is a timelike vector, this quantity defines the Killing energy, i.e., the mass of the new metric (4.3) with respect to the background solution. Further, Abbott and Deser [4] show that the integrand of Eq. (4.4) is a total divergence, and so $E(\xi)$ may be written as a flux integral over a $(d-2)$-dimensional surface at infinity. The details of this calculation are not important here: however, we note that with this flux integral form one may show that this Killing energy (4.4) agrees with our previous definition of the energy (2.1) [15].

Instead, we wish to construct $E(\xi)$ (or rather the energy density $\mathcal{H}$) directly to second order in the fluctuations $h_{\mu\nu}$. Abbott and Deser [4] turn to the framework of canonical gravity for this purpose. One may view their construction as evaluating the second order term in the field equations by making a third order expansion of the action with a judicious choice of variables. The canonical variables provide a judicious choice for several reasons. First, since one wishes to focus on $T^0{}_\mu$, the third order expansion need only be carried out for terms linear in the lapse or shift, and quadratic in the spatial metric and conjugate momenta. Second, for the present background solutions of interest, we will be evaluating $\mathcal{H}$ on a time-symmetric slice, and so the background momentum variables vanish. Finally, the background-shift vector also vanishes in the solutions considered here. The net result is that one must calculate

$$\mathcal{H}=\frac{\bar{N}}{\sqrt{\bar{g}}}\left[g_{ik}g_{jl}\pi^{ij}\pi^{kl}-\frac{1}{d-2}\,\pi^2-g({}^{(d-1)}R-2\Lambda)\right], \tag{4.5}$$

with the quantity in square brackets evaluated to second order in the deviations of the spatial metric h_{ij} and the conjugate momentum deviations p^{ij}. Here ${}^{(d-1)}R$ is the intrinsic curvature scalar of the initial data surface. For convenience, the following gauge conditions are imposed on the fluctuation fields [4]:

$$p^i{}_i=0=\bar{D}^i h_{ij}, \tag{4.6}$$

where $\bar{D}^i$ is the $(d-1)$-dimensional covariant derivative on the initial data surface with the background metric. The deviations are also required to satisfied the constraint equations to linear order, which imposes

$$h^i{}_i = 0 = \bar{D}_i p^{ij}. \tag{4.7}$$

One can see that together Eqs. (4.6) and (4.7) ensure that the fluctuations are transverse and traceless with respect to the background metric. Evaluating Eq. (4.5) subject to these constraints, one arrives at the expression[7]

$$\mathcal{H} = \bar{N}\left[\frac{1}{\sqrt{\bar{g}}} p^{ij}p_{ij} + \sqrt{\bar{g}}\left(\frac{1}{4}\,(\bar{D}_k h_{ij})^2 + \frac{1}{2}\;{}^{(d-1)}\bar{R}^{ijkl}h_{il}h_{jk} - \frac{1}{2}\;{}^{(d-1)}\bar{R}^{ij}h_{ik}h_j{}^k\right)\right]. \tag{4.8}$$

Here it is immediately apparent that the momenta make a manifestly positive contribution to the energy density. Hence, if we are interested in lowering the energy of the background solution, we should set $p^{ij}=0$ and focus on the spatial metric fluctuations. For the latter, there is a gradient energy density, which is also positive, and a potential energy density, which *a priori* has no definite sign. If we consider the background to be anti–de Sitter space, for which

$${}^{(d-1)}\bar{R}_{ijkl} = -\frac{1}{l^2}\,(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}), \tag{4.9}$$

the potential becomes

$$U = +\frac{d-3}{2l^2}\,h_{ij}h^{ij}, \tag{4.10}$$

where $l^2 = -(d-2)(d-1)/2\Lambda$. Hence, in AdS space, the potential energy and hence the total energy contribution of the spatial metric fluctuations are also manifestly positive, and we may conclude that AdS space is perturbatively stable. Of course, spinor techniques [24] allow one to show that AdS space is in fact the absolute minimum energy state within that sector of the theory, i.e., for solutions which admit asymptotically constant spinors. The AdS soliton is not included in this sector, as the spin structure on the asymptotic boundary differs. Further, the latter metric does not have a Riemann tensor with the maximally symmetric form of Eq. (4.9).

As for our conjecture, the remainder of the discussion will be restricted to the AdS soliton with $p=3$, i.e., spacetime dimension $d=5$. In this case, the metric on a constant time slice in Eq. (4.1) is

$$ds^2 = \frac{r^2}{l^2}\left[\left(1-\frac{r_0^4}{r^4}\right)d\tau^2 + (dx^1)^2 + (dx^2)^2\right] + \left(1-\frac{r_0^4}{r^4}\right)^{-1}\frac{l^2}{r^2}\,dr^2. \tag{4.11}$$

In the following, we will actually refer all indices to the obvious orthonormal frame. Now the curvature of this spatial slice is given by

$${}^{(4)}\bar{R}_{\tau r\tau r} = -\frac{1}{l^2}\,(1-3y), \quad {}^{(4)}\bar{R}_{1212} = -\frac{1}{l^2}\,(1-y),$$
$${}^{(4)}\bar{R}_{\tau 1\tau 1} = -\frac{1}{l^2}\,(1+y) = {}^{(4)}\bar{R}_{\tau 2\tau 2} = {}^{(4)}\bar{R}_{r1r1} = {}^{(4)}\bar{R}_{r2r2}, \tag{4.12}$$

where $y = r_0^4/r^4$. Now the potential term in Eq. (4.8) becomes

$$\begin{aligned} U &= \frac{1}{2}\,({}^{(4)}\bar{R}^{ijkl}h_{il}h_{jk} - {}^{(4)}\bar{R}^{ij}h_{ik}h_j{}^k) \\ &= \frac{1}{l^2}\,\{(2-y)[(h_{\tau 1})^2 + (h_{\tau 2})^2 + (h_{r1})^2 + (h_{r2})^2] \\ &\quad + 2(1+y)(h_{\tau r})^2 + 2(h_{12})^2 \\ &\quad + U_{\rm diag}(h_{\tau\tau}, h_{rr}, h_{11}, h_{22})\}. \end{aligned} \tag{4.13}$$

Given that $0 \leqslant y \leqslant 1$, we see that the potential ensures the stability of all fluctuations in the off-diagonal components of the metric. Now, in evaluating the potential for the diagonal fluctuations, we first impose the traceless condition of Eq. (4.7) with $h_{rr} = -h_{\tau\tau} - h_{11} - h_{22}$. Then defining $V^a = (h_{\tau\tau}, h_{11}, h_{22})$, the remaining potential terms may be written as

$$U_{\rm diag} = \frac{1}{2l^2}\,V^a U_{ab} V^b, \tag{4.14}$$

where

$$U_{ab} = 2\begin{pmatrix} 2+2y & 1+y & 1+y \\ 1+y & 2-y & 1-2y \\ 1+y & 1-2y & 2-y \end{pmatrix}. \tag{4.15}$$

It is straightforward to determine the eigenvalues of this matrix to be

$$\lambda_0 = 2(1+y), \quad \lambda_\pm = 5 - y \pm (9 + 6y + 33y^2)^{1/2}. \tag{4.16}$$

One easily shows that λ_0 and λ_+ are positive in the range of interest, i.e., $0 \leqslant y \leqslant 1$, and hence the corresponding eigenvectors correspond to manifestly stable metric fluctuations. The most interesting case is that of λ_- for which one finds

[7] One must integrate by parts to arrive at this expression: however, the above boundary conditions (4.2) will ensure the vanishing of any boundary contributions.

$\lambda_- > 0$ for $0 \leqslant y < \frac{1}{2}$, but $\lambda_- < 0$ for $\frac{1}{2} < y \leqslant 1$. Hence the potential energy density for this eigenvector,

$$V^a_- = (v_-,1,1) \quad \text{with} \quad v_- = \frac{-1+5y-(9+6y+33y^2)^{1/2}}{2(1+y)}, \tag{4.17}$$

becomes negative in a small region near the center of the space, i.e., $r^4 < 2r_0^4$.

Thus metric fluctuations with a form where the h_{ij} are dominated by this eigenmode and only have support in this small region near $r = r_0$ seem to have the potential to be unstable, i.e., lower the energy of the background solution. However, for such fluctuations, there is a competition between the manifestly positive gradient contributions and the potential terms in the energy density (4.8). We argue below that the former terms dominate and hence these fluctuations are also stable.

Imagine that we are considering a metric fluctuation which we might characterize as $V^a = A(r) V^a_-$, where $A(r)$ is a profile, which we assume takes its maximum at $r = r_0$, monotonically decreases, in order to minimize the gradient energy, and vanishes outside $r = 2^{1/4} r_0$. The potential energy density (4.14) becomes $U = (A^2/2l^2)\lambda_-(2+v_-^2)$. With the assumption that the profile takes its maximum at $r = r_0$, the minimum of the potential energy density is

$$U_{\min} = U(r=r_0) = -8(2\sqrt{3}-3)\,\frac{A(r_0)^2}{l^2} \simeq -3.713\,\frac{A(r_0)^2}{l^2}. \tag{4.18}$$

While the complete expression for U for this fluctuation has a complicated analytic form, for the purpose of the reader's intuition we note that to within an accuracy of a few percent one may approximate this expression in the region of interest, i.e., $\frac{1}{2} < y \leqslant 1$, by the simple expression $U = U_{\min}(2y-1)A(r)^2/A(r_0)^2$. So if one imagined that the profile was constant, the potential energy density would decrease linearly as a function of y, reaching zero at $y = 1/2$.

We estimate the gradient energy as follows:

$$T = \frac{1}{4}(\bar{D}h)^2 \simeq \frac{1}{4}[2+v_-^2+(2+v_-)^2](\bar{D}A)^2, \tag{4.19}$$

where we further estimate the gradient of the profile by its maximum $A(r_0)$ divided by the proper distance between $y = 1$ and $y = 1/2$, i.e., between $r = r_0$ and $r = 2^{1/4} r_0$. The latter distance turns out to be $l\,\log(\sqrt{2}+1)/2 \simeq 0.441 l$. In the region of interest, v_- varies from -1 at $y = 1/2$ to $1-\sqrt{3} \simeq -0.73$ at $y = 1$, and so we will simply fix it to $v_- = -1$ in our estimate of T. Hence we arrive at the following estimate:

$$T_{\text{average}} \simeq 5.14\,\frac{A(r_0)^2}{l^2}. \tag{4.20}$$

Comparing Eqs. (4.18) and (4.20), we see that this average gradient energy already exceeds the minimum value of the potential energy. Hence it must be that these potentially unstable metric fluctuations in fact have a positive total energy. While our estimate of the gradient contribution may seem crude, a more detailed examination shows that in fact it greatly underestimates the energy. Properly evaluating the covariant derivatives in Eq. (4.19), accounting for the tensor properties of the fluctuations, adds more positive terms to this expression, which are roughly the same order of magnitude as those considered, i.e., A^2/l^2. Further, we have not accounted for the gauge-fixing constraint (4.6) in our calculations. This constraint fixes the form of the profile for the fluctuation considered above through $\bar{D}^i h_{ir} = 0$, which is the only nontrivial component. One finds that the profile must decay more slowly than estimated above, but that it cannot vanish at $y = 1/2$. Rather, it has infinite support, vanishing as $1/r^4$ in the asymptotic region. While this decreases the local gradient energy density, it also adds a positive potential energy density in the region $y < 1/2$. Hence the final conclusion that these fluctuations are stable remains correct in a detailed analysis.

V. GENERALIZATIONS

Although we have focused on the case $p = 3$ above, one can extend the conjecture to other dimensions, including the $p = 5$ case which is directly related to four-dimensional nonsupersymmetric gauge theories [6]. We believe the solutions (3.14) for all p are perturbatively stable, although a detailed analysis of the fluctuations has not yet been carried out.

Consider the following modification [25] of the metric (2.4):

$$ds^2 = -\left[\frac{r^2}{l^2} - 1 - \left(\frac{r_0}{r}\right)^{p-1}\right] dt^2 + \left[\frac{r^2}{l^2} - 1 - \left(\frac{r_0}{r}\right)^{p-1}\right]^{-1} dr^2 + r^2 d\sigma_p^2, \tag{5.1}$$

where

$$d\sigma_p^2 = (1+\rho^2)dz^2 + \frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega_{p-2} \tag{5.2}$$

is the metric on a p-dimensional unit hyperboloid. This metric is also a solution of Einstein's equation with negative cosmological constant and is asymptotically AdS. It is unusual since the black hole metric (5.1) (without any analytic continuation) can have negative energy [25]. Indeed, the energy is still given by Eq. (2.5) (with Ω_p denoting the area of the unit hyperbolic space[8]), but now one can let the parameter r_0^{p-1} be negative and still have a horizon. There is, however, a minimum energy possible for the black hole. For example, in the case $p = 3$, this occurs when $r_0^2 = -l^2/4$, corresponding to an energy

[8]One can either compactify this space, so that its area is finite, or work with the energy density.

$$E=-\frac{3l^2\Omega_3}{64\pi G_5}. \tag{5.3}$$

The Hawking temperature of these minimum energy black holes is zero. Are they the minimum energy configurations with these boundary conditions?

One can double analytically continue this metric, $t\to i\tau$ and $z\to it$. As usual, τ must be periodically identified with period

$$\beta=\frac{4\pi l^2 r_+}{(p+1)r_+^2-(p-1)l^2} \tag{5.4}$$

so that a constant time surface asymptotically approaches the product of hyperbolic space and a circle. The energy is again given by Eq. (3.13) and hence is negative for positive values of r_0^{p-1}. For large r_0^{p-1}, the energy is the same as the metric (3.11), since the size of the circle is much smaller than the scale of the curvature on the orthogonal space.

One might conjecture that the metrics (5.1) represent the minimum energy configurations for these boundary conditions. Since these solutions are static, they are extrema of the energy. It is likely that they are also local minima of the energy. In fact the calculations in Sec. IV show that this is the case for $p=3$. This is due to the remarkable fact that the components of the curvature of Eq. (5.1) (in an orthonormal frame) are identical[9] to Eq. (2.4). If one analytically continues in t and z and restricts oneself to a constant time surface, the curvatures are still the same. Since the potential term in the quadratic fluctuations depends only on the curvature, it will again be positive.

From the CFT viewpoint, the above metrics should describe the CFT on a product of S^1 and a hyperbolic space. Since the scalars couple to the curvature, a negative curvature space would seem to lead to an instability.[10] Thus, in this case, the apparent stability of the supergravity solution seems in contrast to the expected CFT result. We do not yet have a resolution of this puzzle.

VI. DISCUSSION

We have shown that the AdS soliton (3.14) has lower energy than AdS space with periodic identifications. Rather than producing a contradiction with the recently conjectured AdS-CFT correspondence, these results find close agreement with the negative Casimir energy of nonsupersymmetric field theory on $S^1\times R^{p-1}$.

Since the AdS soliton has extended translational symmetry, one can define not just the total energy, but a full boundary stress-energy tensor. One finds that the agreement in the energy densities, discussed here, extends to agreement of all of the components of the stress tensor [26]. In the example of the gauge theory on $S^1\times R^2$, one finds that the factor of 3/4 relating the energy densities calculated with supergravity and in the weakly coupled gauge theory is, in fact, an overall factor relating the full stress tensors calculated in these two regimes.

It is examining the full stress-energy tensor which motivated in part our choice of physical boundary conditions (4.2) in the positive energy conjecture and in our discussions in the earlier sections. For example, if one retains the r_1/r term in our first negative energy example (3.2), one might expect that the energy would be divergent. One finds though that there is a precise cancellation in the calculation so that the final result remains exactly the same as in Eq. (3.9), which was derived for $r_1=0$. This is also true for the translationally invariant solutions. However, if one considers the full boundary stress tensor, a nonvanishing r_1 produces divergences in the spatial components of the stress energy. One should expect that a less symmetric choice of the initial data surface would yield an energy density which mixes the various components of the latter tensor and, hence, diverges. Our physical boundary conditions (4.2), which rule out including $r_1\neq 0$, ensure that the energy density will remain finite for any choice of time slicing.

A precise comparison of the supergravity and gauge theory energies was only attempted for $p=3$ because this is the case in which the AdS-CFT correspondence is best understood. For this dimension, we only considered the AdS soliton solution, which corresponds to the gauge theory on $S^1\times R^2$. However, one might also consider the initial data (3.3), which would correspond to the gauge theory on $S^1\times S^2$. We calculated the supergravity energy (3.9) and need only translate it to a gauge theory expression, as in going between Eqs. (3.15) and (3.17). The final result is that Eq. (3.9) yields

$$\rho_{\rm SUGRA}=-\frac{\pi^2}{8}\frac{N^2}{\beta^4}F(\beta^2/l^2), \tag{6.1}$$

where the function satisfies $F(0)=1$. Hence, to leading order for small β, the energy density is precisely the same as for $S^1\times R^2$. This is not surprising as this result is valid in the limit where the radius of the sphere is much bigger than the period of the circle, $l\gg\beta$, and so the S^2 factor looks essentially like a flat R^2 on the latter scale. One should note that this energy is measured relative to the standard AdS background with periodic identifications, which naturally includes the curved S^2 factor in its asymptotic geometry. Thus this negative energy does not include the positive contribution which might be expected to appear as the Casimir energy of the two-sphere. It is not difficult to calculate the complete function

$$F(x)=\frac{1}{16}\left(1+\sqrt{1-\frac{2x}{\pi^2}}\right)^2\left[\left(1+\sqrt{1-\frac{2x}{\pi^2}}\right)^2+\frac{4x}{\pi^2}\right]. \tag{6.2}$$

[9] This provides a counterexample to the popular idea that if two metrics have the same curvature, they are locally isometric. The covariant derivatives of the curvature are different, showing that the metrics are indeed inequivalent. We thank S. Ross for discussions on this point.

[10] We thank J. Maldacena for pointing this out.

Taylor expanding F for small x would yield higher order corrections to the energy density. It would be interesting to see to what extent the coefficients of the higher order terms are reproduced in the weakly coupled regime of the gauge theory. This function is only defined for $x \leq \pi^2/2$ since these are the allowed values of β^2/l^2 from Eq. (3.4). [The solutions (3.3) only exist if the circle is small enough.] This suggests that there might be a jump in the ground state energy of the strongly coupled gauge theory on $S^2 \times S^1$, analogous to the Gross-Witten-like phase transition [27] discussed in [3].

The agreement of the negative energy of the AdS soliton with the expected negative Casimir energy appears to support the AdS-CFT correspondence even in the nonsupersymmetric case. But this is true only if the AdS soliton is the lowest energy solution with the given boundary conditions. This gives rise to a new type of positive energy conjecture. Although this conjecture seems unlikely to be true from a purely mathematical standpoint, we have presented evidence to support it, by showing that the AdS soliton is indeed a local minima of the energy (for the case $p=3$). It is natural to extend this conjecture to other dimensions and other asymptotic boundary conditions as we discussed in the previous section.

Our analysis of the perturbative stability of the AdS soliton with respect to metric fluctuations is closely related to the recent calculations of glueball masses in large-N QCD [28]. General arguments have been given that massless supergravity fields propagating on the AdS soliton background will have a discrete spectrum [6]. Further, by the AdS-CFT correspondence, these fluctuations should correspond to various glueball states in the large-N gauge theory [6,29]. In the context of our first form of the conjecture, the given calculations of glueball spectra [28] verify that the AdS soliton is stable against fluctuations of many of the bosonic supergravity fields, e.g., the dilaton, and Neveu-Schwarz and Ramond-Ramond antisymmetric tensors. While our perturbative calculations have not produced a precise spectrum, they do verify that a positive mass gap exists for the metric fluctuations, i.e., the spin-2 glueballs. It is interesting that among the metric excitations, our calculations indicate that the lowest energy state (i.e., the mode described as potentially unstable) must in fact contain a scalar contribution (with respect to R^2) which should actually decouple as the ultraviolet regulator is removed.

Lest the reader imagine that the AdS-CFT correspondence guarantees the local stability of all (static) supergravity solutions, we recall that this is not the case. Typically, there are many unstable stationary points in the scalar potential of the gauged supergravity theory—see, e.g., [30]. For these stationary points, there will be an AdS background, but the cosmological constant will have a value such that supersymmetry is completely broken. In the supergravity analysis, the instability arises because some of the scalars have curvature couplings which exceed the Breitenlohner-Freedman bound [31]; i.e., scalar fluctuations around the stationary point have masses which are (too) negative. It would be interesting to determine what the corresponding physics in the gauge theory is.

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