Einstein manifolds and conformal field theories

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In light of the anti–de Sitter space conformal field theory correspondence, it is natural to try to define a conformal field theory in a large N, strong coupling limit via a supergravity compactification on the product of an Einstein manifold and anti–de Sitter space. We consider the five-dimensional manifolds T^{pq} which are coset spaces $[SU(2)\times SU(2)]/U(1)$. The central charge and a part of the chiral spectrum are calculated, respectively, from the volume of T^{pq} and the spectrum of the scalar Laplacian. Of the manifolds considered, only T^{11} admits any supersymmetry: it is this manifold which characterizes the supergravity solution corresponding to a large number of $D3$-branes at a conifold singularity, discussed recently by Klebanov and Witten. Through a field theory analysis of anomalous three point functions we are able to reproduce the central charge predicted for the T^{11} theory by supergravity: it is $\frac{27}{32}$ of the central charge of the $\mathcal{N}=2\,\mathbf{Z}_2$ orbifold theory from which it descends via a renormalization group flow. [S0556-2821(98)03524-3]

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I. INTRODUCTION

A large number of coincident $D3$-branes in a smooth spacetime induce a supergravity metric which is locally five-dimensional anti–de Sitter space $(\mathrm{AdS}_5)\times S^5$. It is in the context of this geometry that the AdS conformal field theory (CFT) correspondence was first developed [1–3]. Considerable thought has already been given to $D3$-branes on orbifold singularities [4,5], as components of F-theory vacua [6], in the presence of $D7$-branes and orientifolds [7–9] and more recently on conifold singularities [10]. In each case the geometry can be worked out, and properties of the $D3$-brane world-volume theory can be deduced from the holographic prescriptions of Refs. [2, 3].

An alternative approach, which we take in this paper, is to start with a geometry and see how its properties relate via the holographic correspondence to conformal field theory. The conjecture is implicit in Ref. [3] that *any* anti–de Sitter vacuum of string theory or M theory defines a conformal field theory.[1] The simplest examples apart from S^5 involve the coset manifolds T^{pq} considered in Ref. [11]. Of these, only T^{11} preserves some supersymmetry ($\mathcal{N}=1$, which is $\frac{1}{4}$ of maximal considering conformal invariance). One is entitled to wonder what status the other T^{pq} compactifications have as string vacua. In any case, whatever exotic quantum field theoretic behavior reflects the usual pathologies of nonsupersymmetric string vacua, it is at least a fascinating consequence of holography that a conformal fixed point at large N and strong coupling exists and is characterized by compactified classical type-II B supergravity.

In Sec. II we use the volume of the Einstein manifold to compute the central charge of the conformal field theory. The results tally with field theory in the case of orbifolds, but for the coset manifolds the central charge is typically irrational, which rules out any weakly interacting Lagrangian description. For the case of T^{11} supported by N units of $D3$-brane charge, the central charge is $\frac{27}{16}$ times the central charge for $\mathcal{N}=4\,U(N)$ super-Yang-Mills theory. Through an analysis, presented in Sec. II, of anomalous three point functions of the R current and the stress-energy tensor, we are able to reproduce this number on the field theory side.

In Sec. III we compute the spectrum of the scalar laplacian on the T^{pq} manifolds. Typically, dimensions of operators are irrational as well. In the case of T^{11}, only part of this spectrum can be related to chiral primaries and their descendants; most of the others again have irrational dimensions. There is no reason to suppose that these dimensions do not flow as couplings are changed, but this makes their finite values in the limit of supergravity's validity all the more remarkable: they provide an example of operators whose dimensions are not protected but nevertheless tend to finite values in the strong coupling limit.

II. VOLUME AND CURVATURE OF T^{pq}

From the basic AdS-CFT setup as enunciated in Refs. [2, 3], one can conclude that the central charge of the conformal field theory is inversely proportional (in the large N limit at least) to the volume of the compact five-dimensional Einstein manifold M_5. To see this, consider the geometry

$$ds^2=\frac{L^2}{z^2}\,(d\mathbf{x}^2+dz^2)+L^2 ds^2_{M_5},$$
$$F=\frac{N\sqrt{\pi}}{2\,\mathrm{Vol}\,M_5}\,(\mathrm{vol}_{M_5}+\mathrm{vol}_{\mathrm{AdS}}), \tag{1}$$

[1] Since one needs a stress tensor to define a CFT and hence a graviton in the bulk spectrum, it seems that a theory containing gravity is necessary in the bulk to make the conjecture reasonable. If anything more than a large N limit of the boundary theory is desired, the bulk theory must be quantum mechanical. It is for this reason that we consider string theory or M theory the only candidates for the bulk theory.

where the metric $ds^2_{M_5}$ is taken to have curvature $R_\alpha{}^\beta = 4\delta_\alpha^\beta$. The volume Vol M_5 and the five-forms vol_{M_5} and $\mathrm{vol}_{\mathrm{AdS}}$ refer to the metrics $ds^2_{M_5}$ and $(1/z^2)(d\mathbf{x}^2+dz^2)$ (i.e., without the powers of L which appear in the full ten-dimensional metric). N is the integer three-brane charge. The Einstein equation

$$R_M{}^N = \frac{\kappa^2}{6} F_{MP_1P_2P_3P_4} F^{NP_1P_2P_3P_4} \tag{2}$$

is satisfied if one takes

$$L^4 = \frac{\sqrt{\pi}}{2} \frac{\kappa N}{\mathrm{Vol}\ M_5}. \tag{3}$$

After compactification on M_5, type-II B supergravity can be summarized by the action

$$S = \frac{\mathrm{Vol}\ M_5}{2\kappa^2} L^8 \int d^5x \sqrt{g}[R + 12 - \tfrac{1}{2}(\partial\phi)^2 + \cdots]. \tag{4}$$

In Eq. (4), the metric is taken to be $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (1/z^2)(d\mathbf{x}^2+dz^2)$: we have rescaled out all factors of L from the integrand. In view of Eq. (3), the prefactor on S in Eq. (4) is $\pi N^2/(8\ \mathrm{Vol}\ M_5)$. All correlators calculated via the holographic prescription of Refs. [2,3] include an overall factor 1/Vol M_5 from this prefactor.

Indeed, one of the first checks made in Ref. [2] was to see that the central charge of the $\mathcal{N}=4$ theory came out correctly from the two-point function of the stress-energy tensor, calculated essentially via Eq. (4) with $M_5 = S^5$. In a more involved calculation, the authors of Ref. [12] succeeded in establishing

$$\langle T^\alpha{}_\alpha \rangle_{g_{\mu\nu}} = -aE_4 - cI_4, \tag{5}$$

where

$$E_4 = \frac{1}{16\pi^2}(R^2_{ijkl} - 4R^2_{ij} + R^2),$$
$$I_4 = -\frac{1}{16\pi^2}\left(R^2_{ijkl} - 2R^2_{ij} + \frac{1}{3}R^2\right) \tag{6}$$

from the holographic prescription applied to an arbitrary boundary metric $g_{\mu\nu}$ in the conformal class of (compactified) Minkowski space. The calculation again follows from Eq. (4) (improved by boundary terms and regulated), so a and c carry an inverse factor of Vol M_5. The coefficient c is the same as what we have called the central charge (namely, the normalization of the two-point function of the stress-energy tensor), so the first term of Eq. (5) follows from what was already worked out in Ref. [13]. The derivation of Ref. [12] shows that $a = c$ is a consequence of holography plus the product space structure of the spacetime (thus restricting the types of theories one could hope to construct holographically from product geometries). It was also reported there that the agreement between the holographic and free field values of a and c persists to orbifolded theories such as those considered in Ref. [4]. The free field counting is done using Table I [14] (all other entries follow from the first three).

TABLE I. Trace anomaly coefficients in free field theory.

Field	a	c
real scalar	1/360	1/120
complex Weyl fermion	11/720	1/40
vector boson	31/180	1/10
$\mathcal{N}=1$ chiral multiplet	1/48	1/24
$\mathcal{N}=1$ vector multiplet	3/16	1/8
$\mathcal{N}=2$ hyper multiplet	1/24	1/12
$\mathcal{N}=2$ vector multiplet	5/24	1/6
$\mathcal{N}=4$ vector multiplet	1/4	1/4

Counting the free field content is a reliable method for computing the anomaly coefficients as long as there is at least $\mathcal{N}=1$ supersymmetry, since then $T^\mu{}_\mu$ and $\partial_\mu R^\mu$ are superpartners and the Adler-Bardeen theorem may be applied. In fact, the results of Refs. [15,16] imply that the free field counting is valid to leading order in large N even for nonsupersymmetric theories obtained by orbifolding the $\mathcal{N}=4$ theory. Indeed, all the agreements obtained so far are strictly leading large N limits: the exact result $a = c = (N^2-1)/4$ for $\mathcal{N}=4$ SU(N) gauge theory apparently includes a one-loop contribution from supergravity.

To learn the central charge of the T^{pq} theories, we must compute the volume of these manifolds. We will use the methods of Ref. [17]. There are a few important differences between the conventions used here and those of Refs. [17] and [11], and we will write out formulas explicitly enough to disambiguate them. To form the quotient space T^{pq} we start with SU(2)×SU(2) generated by $i\sigma_k$ and $i\tau_k$, where k runs from 1 to 3, and divide out by the U(1) generated by $\omega = p\sigma_3 + q\tau_3$. Let us write the generators as

$$i\sigma_l, \quad i\tau_s, \quad Z = qi\sigma_3 - pi\tau_3, \quad \omega = pi\sigma_3 + qi\tau_3, \tag{7}$$

where l and s run over 1,2. Referring to these anti-Hermitian generators collectively as t_a, we define the structure constants and Killing metric so that

$$[t_a, t_b] = C^c_{ab} t_c,$$
$$\gamma_{ab} = -\frac{1}{8} C^d_{ac} C^c_{bd}. \tag{8}$$

The normalization of γ_{ab} is chosen so that this is the same metric as the natural one on $S^3\times S^3$ where S^3 is the unit three-sphere. In this metric, the volume of SU(2)×SU(2) is

$(2\pi^2)^2$, and the length of the orbit of ω is $2\pi\sqrt{p^2+q^2}$. So before the symmetric rescaling,

$$\text{Vol } T^{pq} = \frac{2\pi^3}{\sqrt{p^2+q^2}}. \tag{9}$$

The symmetric rescaling referred to in Ref. [11] and detailed in Ref. [17] is a replacement $e^a \to r(a)e^a$, where e^a is the vielbein for the coset space generated by $i\sigma_l$, $i\tau_s$, and Z. In a (hopefully) obvious notation, let us choose

$$r(l) = \sqrt{a}\gamma, \quad r(s) = \sqrt{b}\gamma, \quad r(Z) = \gamma. \tag{10}$$

The volume in the new metric, which we shall call the squashed metric, is

$$\text{Vol } T^{pq} = \frac{2\pi^3}{\sqrt{p^2+q^2}} \frac{1}{ab\gamma^5}. \tag{11}$$

The squashed metric still has an isometry group SU(2)×SU(2)×U(1). Furthermore, with a special choice of a and b, the metric can be made Einstein. In Ref. [17] the following expression is given for the Riemann tensor (up to a convention-dependent factor of 2):[2]

$$\begin{aligned} R^a{}_{bcd} &= \frac{1}{2} C^a_{bc} C^c_{de} \begin{pmatrix} a & b \\ & c \end{pmatrix} + C^a_{b\omega} C^\omega_{de} r(d) r(e) \\ &+ \frac{1}{4} C^a_{cd} C^c_{be} \begin{pmatrix} a & c \\ & d \end{pmatrix} \begin{pmatrix} b & c \\ & e \end{pmatrix} \\ &- \frac{1}{4} C^a_{ce} C^c_{bd} \begin{pmatrix} a & c \\ & e \end{pmatrix} \begin{pmatrix} b & c \\ & d \end{pmatrix}, \end{aligned} \tag{12}$$

$$\begin{pmatrix} a & b \\ & c \end{pmatrix} \equiv \frac{r(a)r(c)}{r(b)} + \frac{r(b)r(c)}{r(a)} - \frac{r(a)r(b)}{r(c)}.$$

From the Einstein requirement $R_a{}^b = \Lambda \delta^b_a$ one now derives Eq. (2.5) of Ref. [11]

$$\frac{\Lambda}{\gamma^2} = 4a - 2a^2y^2 = 4b - 2b^2x^2 = 2a^2y^2 + 2b^2x^2, \tag{13}$$

where $x = p/\sqrt{p^2+q^2}$ and $y = q/\sqrt{p^2+q^2}$. Note that for given x and y, the determination of a and b reduces to solving a cubic.

To sum up the discussion at the beginning of this section, the ratio of the central charge c in the conformal field theory "defined" by an Einstein manifold M_5 to the central charge $c_0 = N^2/4$ of $\mathcal{N}=4$ super-Yang-Mills with gauge group U(N) should be the same as the ratio of Vol S^5 to Vol M_5 computed with respect to Einstein metrics with the same cosmological constant. Using Eq. (13) and the fact that the unit S^5 has Vol $S^5 = \pi^3$ and $\Lambda = 4$, one arrives at

TABLE II. Parameters characterizing T^{pq} manifolds.

T^{pq}	x	y	a	b	γ	c/c_0
T^{01}	0	1	1	1/2	$\sqrt{2}$	$\sqrt{2}$
T^{11}	$1/\sqrt{2}$	$1/\sqrt{2}$	4/3	4/3	$3/\sqrt{8}$	27/16
T^{34}	3/5	4/5	25/18	25/27	$\frac{9}{5}\sqrt{\frac{2}{5}}$	$\frac{243}{25}\sqrt{\frac{2}{5}}$

$$\frac{c}{c_0} = 2\sqrt{2}\,\frac{ab(p^2+q^2)^3}{(q^2a^2+p^2b^2)^{5/2}}. \tag{14}$$

The results for the three simplest examples are quoted in Table II. The value listed for γ is the one which makes $\Lambda = 4$. This normalization will be useful in the next section.

The higher T^{pq} have very complicated a and b, and it does not seem worth the space to quote their central charges; suffice it to say that c/c_0 is typically an irrational involving irreducible square and cube roots. One can show using Eqs. (13) and (14) that $c/c_0 \geqslant (\sqrt{pq})$ (a better bound may be possible).

The anomaly coefficients in field theory. In view of the construction [10] of the T^{11} theory from a relevant deformation of the $\mathcal{N}=2$ Z_2 orbifold theory, one would expect to be able to derive the central charge in Table II from a field theory analysis along the lines of Ref. [18]. This is quite a nontrivial test of holography because it tests a nonperturbative field theoretic effect involving a renormalization group (RG) flow from a simple UV theory to a IR theory which lies deep inside the conformal window. (Because $N_f = 2N_c$ for each of the gauge groups separately, the IR theory is not close to either edge of the conformal window [19] $3N_c/2 \leqslant N_f \leqslant 3N_c$. Perturbation theory in the electric or magnetic representation, respectively, can be used if $N_f = 3N_c - \epsilon$ or $N_f = 3N_c/2 + e$.) In an $\mathcal{N}=1$ superconformal theory, the anomaly coefficients a and c can be read off from anomalous three point functions $\partial_\mu \langle T_{\alpha\beta} T_{\gamma\delta} R^\mu \rangle$ and $\partial_\mu \langle R_\alpha R_\beta R^\mu \rangle$, where R_μ is the R current: the former is proportional to $a-c$, while the latter is proportional to $5a-3c$.

The ultraviolet theory, in $\mathcal{N}=1$ language, has $2N_c^2$ vector multiplets filling out the adjoints of the two U(N_c) gauge groups, plus $6N_c^2$ chiral multiplets filling out an adjoint for each gauge group ($2N_c^2$) plus bifundamental matter ($4N_c^2$). The R current descends from one of the $\mathcal{N}=4$ SU(4) R currents. The R charge of the $N_\lambda = 2N_c^2$ gluino fields is 1, whereas the R charge of the $N_\chi = 6N_c^2$ quark fields is $-\frac{1}{3}$. There is no gravitational anomaly in $\langle TTR \rangle$ because the U(1)$_R$ generator is traceless. So $a_{\rm UV} - c_{\rm UV} = 0$. For the $\partial_\mu \langle R_\alpha R_\beta R^\mu \rangle$ anomaly, we use the position space analysis

[2]In Eq. (12) the Riemann tensor has been defined from the spin connection via $R^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu} dy^\mu \wedge dy^\nu = d\omega + \omega\wedge\omega$, whereas in Ref. [17] the factor of $\frac{1}{2}$ is omitted. In Ref. [11] the normalization seems to be the same as used here.

summarized in Ref. [18]: for a single Majorana spinor with axial current $J_\mu=\frac{1}{2}\bar{\psi}\gamma_\mu\gamma^5\psi$,[3]

$$\frac{\partial}{\partial z^\mu}\langle J_\alpha(x)J_\beta(y)J^\mu(z)\rangle=-\frac{1}{12\pi^2}\epsilon_{\alpha\beta\gamma\delta}\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial y^\delta}\times\delta^4(x-z)\delta^4(y-z)\equiv\frac{9}{16}\mathcal{A}_{\alpha\beta}(x,y,z). \tag{15}$$

By counting U(1) charges, one obtains

$$\partial_\mu\langle R_\alpha R_\beta R^\mu\rangle=(5a_{\rm UV}-3c_{\rm UV})\mathcal{A}_{\alpha\beta}=\frac{9}{16}\left[N_\lambda+\left(-\frac{1}{3}\right)^3N_\chi\right]\mathcal{A}_{\alpha\beta}=N_c^2\mathcal{A}_{\alpha\beta}. \tag{16}$$

So far, this has just verified the known result from free field counting with Table I that $a_{\rm UV}=c_{\rm UV}=N_c^2/2$.

The relevant deformation discussed in Ref. [10] gives a mass to the chiral multiplets in the adjoints of the gauge groups, leaving behind $4N_c^2$ chiral multiplets with a quartic superpotential, plus $2N_c^2$ vector multiplets as before. The anomalous dimension of mass operators of the form tr AB was found in the infrared [10] from the exact beta function [20,21] to be $\gamma_{\rm IR}=-\frac{1}{2}$. After the adjoint chirals have been made massive and integrated out, the R current which is the superpartner of the stress-energy tensor is no longer conserved, in part due to internal anomalies proportional to the beta function. A nonanomalous U(1) current [22] is $S_\mu=R_\mu+\frac{1}{3}(\gamma_{\rm IR}-\gamma)K_\mu$, where K_μ is the Konishi current, which assigns charge 1 to quarks and 0 to gluinos.[4] The external gauge and gravitational anomalies of S_μ are independent of scale. Since $S_\mu=R_\mu$ at the IR fixed point, the strategy is to evaluate the external anomalies of S_μ at a high scale where $\gamma\to0$, and identify the answers as the IR fixed point anomalies of R_μ. When $\gamma\to0$, the S charges of the fermion fields which remain massless are 1 for the $N_\lambda=2N_c^2$ gluinos and $-\frac{1}{2}$ for the $N_\chi=4N_c^2$ quarks. As before, there is no gravitational anomaly because the $\mathrm{U}(1)_S$ generator is traceless. So $a_{\rm IR}-c_{\rm IR}=0$. On the other hand,

$$\partial_\mu\langle R_\alpha R_\beta R^\mu\rangle=(5a_{\rm IR}-3c_{\rm IR})\mathcal{A}_{\alpha\beta}=\frac{9}{16}\left[N_\lambda+\left(-\frac{1}{2}\right)^3N_\chi\right]\mathcal{A}_{\alpha\beta}=\frac{27}{32}N_c^2\mathcal{A}_{\alpha\beta}. \tag{17}$$

Thus $a_{\rm IR}/a_{\rm UV}=c_{\rm IR}/c_{\rm UV}=\frac{27}{32}$, exactly the ratio of Vol $S^5/\mathbf{Z}_2$ to Vol T^{11}. Of course, the agreement with supergravity is only to leading order in large N. A more careful analysis of the field theory would have the U(1) parts of the gauge groups decoupling, changing a and c by a factor $1-O(1/N^2)$ which is invisible in classical supergravity.

III. THE LAPLACIAN ON T^{pq}

To study the spectrum of chiral primaries in the T^{pq} CFT (insofar as we grant that the theory exists and is defined in the large N limit by the supergravity), the first step is to find the spectrum of the scalar Laplacian on T^{pq}. A convenient parametrization of these coset spaces descends from Euler angles on SU(2)×SU(2). Let us write a group element of SU(2) as $g=e^{i\alpha J_z}e^{i\beta J_y}e^{i\gamma J_z}$. The standard unit sphere metric on SU(2) is, in these variables,

$$ds^2=\frac{1}{4}(d\alpha^2+d\beta^2+d\gamma^2+2\cos\beta\, d\alpha d\gamma). \tag{18}$$

The angles are allowed to vary over the ranges

$$\alpha\in(0,2\pi),\quad \beta\in(0,\pi),\quad \gamma\in(0,4\pi). \tag{19}$$

The reason that α is not allowed to range over $(0,4\pi)$ is to avoid a double cover of SU(2): $e^{2\pi iJ_z}=-1$.

The convenience of Euler angles is that we can represent the coset space T^{pq} as the hypersurface $p\gamma_1+q\gamma_2=0$ in the Euler angle coordinatization $(\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\gamma_2)$ of SU(2)×SU(2). Writing $\gamma_1=-q\alpha_3/\sqrt{p^2+q^2}$, $\gamma_2=p\alpha_3/\sqrt{p^2+q^2}$, and ordering our coordinates for T^{pq} as $(\beta_1,\beta_2,\alpha_1,\alpha_2,\alpha_3)$, one arrives at a metric whose inverse is

$$g^{ab}=4\gamma^2\begin{pmatrix} a & 0 & 0 & 0 & 0\\ 0 & b & 0 & 0 & 0\\ 0 & 0 & a\csc^2\beta_1 & 0 & ya\cot\beta_1\csc\beta_1\\ 0 & 0 & 0 & b\csc^2\beta_2 & -xb\cot\beta_2\csc\beta_2\\ 0 & 0 & ya\cot\beta_1\csc\beta_1 & -xb\cot\beta_2\csc\beta_2 & 1+y^2a\cot^2\beta_1+x^2b\cot^2\beta_2\end{pmatrix} \tag{20}$$

[3]There is a small typo in Eq. (4.10) of Ref. [18] which is corrected in Eq. (15).

[4]

after squashing. One can now verify Eq. (11) by integrating $\sqrt{g}=\sqrt{\det g_{ab}}=ab\gamma^5 \sin\beta_1 \sin\beta_2$ over the allowed ranges for the variables

$$\beta_1,\beta_2\in(0,\pi),\quad \alpha_1,\alpha_2\in(0,2\pi),\quad \alpha_3\in(0,4\pi/\sqrt{p^2+q^2}). \tag{21}$$

The Laplacian has a fairly simple form:

$$\begin{aligned}\Box\phi=4\gamma^2\bigg[&a\,\frac{1}{\sin\beta_1}\frac{\partial}{\partial\beta_1}\sin\beta_1\frac{\partial}{\partial\beta_1}\\&+b\,\frac{1}{\sin\beta_2}\frac{\partial}{\partial\beta_2}\sin\beta_2\frac{\partial}{\partial\beta_2}+a\csc^2\beta_1\frac{\partial^2}{\partial\alpha_1^2}\\&+b\csc^2\beta_2\frac{\partial^2}{\partial\alpha_2^2}+2ya\cot\beta_1\csc\beta_1\frac{\partial^2}{\partial\alpha_1\partial\alpha_3}\\&-2xb\cot\beta_2\csc\beta_2\frac{\partial^2}{\partial\alpha_2\partial\alpha_3}+(1+y^2a\cot^2\beta_1\\&+x^2b\cot^2\beta_2)\frac{\partial^2}{\partial\alpha_3^2}\bigg]\phi=-E\phi.\end{aligned} \tag{22}$$

Amusingly enough, this equation can be solved completely by separation of variables even though T^{pq} is not metrically a product space: writing

$$\phi=\phi_1(\beta_1)\phi_2(\beta_2)\exp\left(i\sum_{j=1}^{3}m_j\alpha_j\right) \tag{23}$$

we arrive at

$$E=4\gamma^2(aE_1+bE_2+m_3^2), \tag{24}$$

where E_1 and E_2 are determined by the ordinary differential equations

$$\begin{aligned}\bigg[&\frac{1}{\sin\beta_i}\frac{\partial}{\partial\beta_i}\sin\beta_i\frac{\partial}{\partial\beta_i}\\&-(m_3y_i\cot\beta_i+m_i\csc\beta_i)^2\bigg]\phi_i=-E_i\phi_i,\end{aligned} \tag{25}$$

where $y_1=y$ and $y_2=-x$. For $y_i=0$ and 1, we recognize Eq. (25) as the differential equation that determines eigenvalues of the Laplacian on S^2 and S^3, respectively: in these cases one would have $E_i=l(l+1)$ or $l(l+2)/4$.

The next step is to determine, given specified η and χ, the values of E for which

$$\left[\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\sin\beta\frac{\partial}{\partial\beta}-(\eta\cot\beta+\chi\csc\beta)^2\right]\phi=-e\phi \tag{26}$$

admits a regular solution on the interval $\beta\in[0,\pi]$. Introducing a new variable $z=\cos^2(\beta/2)$, one can reduce Eq. (26) to a hypergeometric equation. Solutions of the form

$$\phi=z^F(1-z)^G F(A,B;C;z) \tag{27}$$

with suitable A, B, C, F, and G are smooth in the interior of the interval, and have a behavior at the end points which can be determined using the formula [24]

$$\begin{aligned}F(A,B;C;z)=&\frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}\\&\times F(A,B;A+B-C+1;1-z)\\&+(1-z)^{C-A-B}\frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)}\\&\times F(C-A,C-B;C-A-B+1;1-z).\end{aligned} \tag{28}$$

In brief, the solutions are regular when they can be expressed in terms of a hypergeometric function which is a polynomial. This is so when

$$\frac{1}{2}-\sqrt{\frac{1}{4}+e+\eta^2}+\max\{|\eta|,|\chi|\}\in\mathbf{Z}^-\equiv\{0,-1,-2,-3,\ldots\}. \tag{29}$$

One can verify Eq. (29) in four cases, taking $\eta+\chi$ and $\eta-\chi$ positive or negative. The eigenfunctions ϕ in fact vanish at the end points except when $|\eta|=|\chi|$. The case $\eta=\chi=0$ leads to the Legendre polynomials. Let us go through the derivation of Eq. (29) for the case $\eta-\chi\leqslant0$ and $\eta+\chi\geqslant0$, and leave the other cases as an exercise for the reader. Choosing $F=-(\eta-\chi)/2$ and $G=(\eta+\chi)/2$, we are led to

$$A=\frac{1}{2}+\chi-\sqrt{\frac{1}{4}+e+\eta^2},$$

$$B=\frac{1}{2}+\chi+\sqrt{\frac{1}{4}+e+\eta^2},$$

$$C=1+\chi-\eta.$$

Regularity at $z=0$ is guaranteed by our choice of the sign on F. Because of the choice of sign on G, regularity at $z=1$ depends on having A or B vanish, so that the second term in Eq. (28) is absent. This occurs when $A\in\mathbf{Z}^-$ or $B\in\mathbf{Z}^-$. The latter is impossible because $\chi\geqslant|\eta|$; so we are left with $A\in\mathbf{Z}^-$, which indeed reduces to Eq. (29).

[4] S. P. de Alwis has shown [23] how one can define the current S_μ in the full $\mathcal{N}=2\mathbf{Z}_2$ orbifold theory instead of first integrating out the chiral adjoint fields, as we have done here. Equation (17) is unchanged because the S charge of the fermions in those chiral adjoints is 0 in the ultraviolet.

Writing $l=k+\max\{|\eta|,|\chi|\}$ where $k\in\mathbf{Z}^+\equiv\{0,1,2,3,...\}$, one finds the simple expression $e=l(l+1)-\eta^2$. Returning to Eq. (24), the final expression for the eigenvalues of the Laplacian on T^{pq} is

$$E=4\gamma^2[al_1(l_1+1)+bl_2(l_2+1)+m_3^2(1-ay^2-bx^2)]. \tag{30}$$

In Eq. (30), $l_i=k_i+\max\{|m_3y_i|,|m_i|\}$, $k_i\in\mathbf{Z}^+$, $m_i\in\mathbf{Z}-m_3y_i$, $m_3\in[\sqrt{(p^2+q^2)}/2]\mathbf{Z}$, and $i=1,2$. The shift in the allowed values of m_i is necessary to ensure that Eq. (23) is single valued. Note that m_3y_i is always either integer or half-integer, and that E is completely specified by m_3 [essentially, the U(1) charge] and the spins l_i under the two SU(2)'s. The expression of Eq. (30) as a linear combination of the quadratic Casimirs for the symmetry group SU(2)×SU(2)×U(1) is precisely the form expected for a coset manifold.[5]

As in the case of S^5, the dimension of the scalar operator in the conformal field theory to which a given mode of the dilaton ϕ couples is

$$\Delta=2+\sqrt{4+E}, \tag{31}$$

where now γ must be chosen in Eq. (30) so that $\Lambda=4$. In fact, these modes may be complexified by replacing ϕ by a complex scalar B which also includes the axion. Comparing Eqs. (2.33) and (2.53) of Ref. [25], one arrives at the conclusion that the operators coupling to the modes of h_α^α and $a_{\alpha\beta\gamma\delta}$ (which can be mixed together in two different ways) have dimensions

$$\Delta=\begin{cases}-2+\sqrt{4+E},\\ 6+\sqrt{4+E}.\end{cases} \tag{32}$$

These dimensions can be read off directly from the eigenvalues of $\Box$ because the relevant fields have a mode expansions purely in terms of the scalar eigenfunctions on T^{pq}. In Eqs. (31) and (32) we have summarized the spectrum of operators which in Ref. [25] corresponded to the first, third, and sixth Kaluza-Klein towers of scalars. To obtain the rest of the bosonic spectrum, one must deal with the analogues of the vector, symmetric tensor, and antisymmetric tensor spherical harmonics appearing in Eq. (2.20) of Ref. [25]; for the fermionic spectrum, a study of the eigenfunctions of the Dirac operator is also required. We leave these more involved studies for future work.

Inspection of the spectrum of dimensions on all the T^{pq} reveals a rather uninteresting sequence of numbers, mostly complicated irrationals. However, T^{11} exhibits a fascinating feature which finds its explanation in the superconformal algebra. Operators of algebraically protected dimension are typically those whose dimension is the lowest possible given a certain R charge, or else descendants of such operators derived by a series of supersymmetry transformations. The R symmetry in this context is the U(1) part of the isometry group of T^{11}, which acts by shifting α_3. The integer R charge k is related to m_3 by $m_3=k/\sqrt{2}$. Without loss of generality let us take $k\geq 0$. Using Eqs. (30), (31), and (32), one finds that the smallest possible value of Δ is $\Delta=3k/2$, and corresponds to a mode of h_α^α and $a_{\alpha\beta\gamma\delta}$ with $l_1=l_2=k/2$ and $|m_1|,|m_2|\leq k/2$. Thus we find a set of operators filling out a $(\mathbf{k+1},\mathbf{k+1})_k$ multiplet of SU(2)×SU(2)×U(1), where $(\mathbf{d}_1,\mathbf{d}_2)_r$ indicates R-charge r and SU(2) representations of dimensions d_1 and d_2. Indeed, $\Delta=3k/2$ saturates the algebraic bound on Δ following from the superconformal algebra [26]. This part of the spectrum was anticipated on field theory grounds in Ref. [10], and it was argued there that the form of the operators is $\mathrm{tr}(AB)^k$. The related operators in the third and sixth towers are descendants which presumably have the form $\mathrm{tr}\,F_1^2(AB)^k+\mathrm{tr}\,F_2^2(BA)^k$ and $\mathrm{tr}\,F_1^4(AB)^k+\mathrm{tr}\,F_2^4(BA)^k$, where F^4 is the special Lorentz contraction $F_{\mu_1}{}^{\mu_2}F_{\mu_2}{}^{\mu_3}F_{\mu_3}{}^{\mu_4}F_{\mu_4}{}^{\mu_1}-\frac{1}{4}(F_{\mu_1}{}^{\mu_2}F_{\mu_2}{}^{\mu_1})^2$.[6]

The supergravity predicts in addition a spectrum of operators with at least one SU(2) spin larger than the R charge. These operators far outnumber the chiral primaries $\mathrm{tr}(AB)^k$: there are $N_\chi(\Delta)\sim\frac{8}{81}\Delta^3$ such chiral primaries with dimension less than Δ, versus a number $N(\Delta)\sim\frac{4}{405}\Delta^5$ of operators with larger SU(2) spins, as follows from Weyl's law for the growth of eigenvalues of $\Box$: $N(\Delta)\sim(\mathrm{Vol}\,M_5/60\pi^3)\Delta^5$. The dimensions of these nonchiral operators are in general irrational (square roots of integers). However, there are special nonnegative integer values of n_1 and n_2 such that for $l_1=n_1+k/2$ and $l_2=n_2+k/2$, the dimensions of the operators in the $(\mathbf{2l_1+1},\mathbf{2l_2+1})_k$ multiplet are again integer or half-integer: $E=[2(n_1+n_2+1)+\frac{3}{2}k]^2-4$, so $\Delta=2(n_1+n_2+2)+\frac{3}{2}k$ for relatives of the dilaton and $\Delta=2(n_1+n_2)+\frac{3}{2}k$ or $2(n_1+n_2+4)+\frac{3}{2}k$ for relatives of h_α^α and $a_{\alpha\beta\gamma\delta}$. The n_i for which this occurs are solutions to the Diophantine equation $n_1^2+n_2^2-4n_1n_2-n_1-n_2=0$: that is, consecutive terms in the sequence $\{0,0,1,5,20,76,285,...\}$.[7]

If these higher-dimension operators are, in some exotic sense, algebraic descendants of chiral primaries, then since the SU(2) spins for given R charge are larger than for the

[5]Thanks to E. Witten for a discussion on this point.

[6]Thanks to I. Klebanov for a discussion on this point.

[7]Let us briefly indicate the solution of the Diophantine equation. Define

$$f\pm(n)=\frac{1+4n\pm\sqrt{1+12n+12n^2}}{2}$$

By construction, any pair $(n_1,n_2)=[f_-(n),n]$ solves the equation, and any solution must have this form up to interchange of n_1 and n_2. Also, $f_+^{(i)}(0)$ is always a nonnegative integer, where the superscript denotes iterative application of $f+$. To see that $(n_1,n_2)=\{f_-[f_+^{(i)}(0)],f_+^{(i)}(0)\}$ where $i=0,1,2,3,\ldots$, are the only solutions, note that $f_-[f_+(n)]=n$ for nonnegative n and $f_+[f_-(n)]=n$ for positive n. Furthermore, $f_-(n)\leq n$ for nonnegative integer n, with equality if $n=0$; and $f_-(n)=0$ only for $n=0$ or 1. Starting with any putative solution $[f_-(n),n]$, one can find successively smaller solutions by repeated application of f_- to n until one reaches $n=0$. The properties of $f_\pm$ now guarantee that n is a member of the sequence $\{f_+^{(i)}(0)\}_{i=0}^\infty=\{0,1,5,20,76,285,\ldots\}$.

chiral primaries, the relevant algebra must involve the SU(2)'s. But, in contrast to the U(1) R-symmetry group, the SU(2)'s do not participate in the usual $\mathcal{N}=1$ superconformal algebra. Besides, the nilpotency of supersymmetry generators implies that the dimension of a descendant cannot be arbitrarily larger than the dimension of its chiral primary parent. So we do not see any reason why these peculiar series of operators with integer or half-integer dimensions should be protected.

IV. CONCLUSIONS

It is an open question whether a quantum field theory constructed holographically from compactified string theory must include a local gauge invariance. Supergravity, and by extension closed strings, see only the gauge invariant observables. One can argue that if the compactification is supported by Ramond-Ramond charge, then it should be realizable in string theory as a D-brane configuration, and gauge invariance emerges from the dynamics of open strings attached to the branes. In light of this reasoning it would be interesting to find singular six-manifolds to which the T^{pq} spaces are related in the same way T^{11} is related to the conifold $z_1^2+z_2^2+z_3^2+z_4^2=0$ [10]. An analysis of D-branes on such manifolds should reveal a weak coupling gauge theory version of the conformal field theories defined holographically in this paper.

Perhaps the most general statement that can be made about supersymmetric theories constructed holographically from a product space $\mathrm{AdS}_5\times M_5$ is that the R current must be free of gravitational anomalies: $\partial_\mu\langle T_{\alpha\beta}T_{\gamma\delta}R^\mu\rangle=0$. This statement is nontrivial for $\mathcal{N}=1$ and 2 supersymmetric theories, where the R symmetry is U(1) or U(2); for $\mathcal{N}=4$ the R-symmetry group is SU(4), which is simple and hence automatically free of gravitational anomalies.

The original correspondence [1–3] between $\mathrm{AdS}_5\times S^5$ and $\mathcal{N}=4$ super-Yang-Mills theory has been questioned because so much of what it predicts is either nonverifiable (for instance the coefficient on the $q\bar{q}$ potential [27,28]) or else largely a consequence of the large supergroup apparent on both sides of the duality (for example scaling dimensions of chiral operators). Already, holography's successes go beyond the constraints of symmetry in predicting Green's functions [29,30] and in elucidating a geometric picture of confinement [31] and of baryons [32,33]. Hopefully the example of $D3$-branes on conifolds, described for large N by the T^{11} manifold, will serve as further evidence that holography captures not only supergroup theory but gauge theory dynamics. The verification of $\gamma_{\mathrm{IR}}=-\frac{1}{2}$ and $c_{\mathrm{IR}}/c_{\mathrm{UV}}=\frac{27}{32}$ seems particularly nontrivial, since both relations rely on properties of the RG flow from the $S^5/\mathbf{Z}_2$ theory to the T^{11} theory.

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