Quantum Newtonian dynamics on a light front

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We recall the special features of quantum dynamics on a light front (in an infinite momentum frame) in string and field theory. The reason this approach is more effective for string than for fields is stressed: the light-front dynamics for string is that of a true Newtonian many particle system, since a string bit has a fixed Newtonian mass. In contrast, each particle of a field theory has a variable Newtonian mass P^+; so the Newtonian analogy actually requires an infinite number of species of elementary Newtonian particles. This complication substantially weakens the value of the Newtonian analogy in applying light-front dynamics to nonperturbative problems. Motivated by the fact that conventional field theories can be obtained as infinite tension limits of string theories, we propose a way to recast field theory as a standard Newtonian system. We devise and analyze some simple quantum mechanical systems that display the essence of the proposal, and we discuss prospects for applying these ideas to large N_c QCD. [S0556-2821(99)01302-8]

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I. INTRODUCTION

The possibility of developing the quantum dynamics of a relativistic system in light-front form has been under occasionally active investigation since Dirac first suggested the idea 50 years ago [1]. In the 1960s simplifications in the derivation of current algebra sum rules occurred [2,3] in an infinite momentum frame, the light front in another guise. Studying Feynman diagrams in such a frame, Weinberg [4] discovered much simplified rules in which the energies in the denominators of old-fashioned perturbation theory took the non-relativistic form $(\mathbf{p}^2+\mu^2)/2P_L$ where $\mathbf{p}$ is the momentum transverse to the longitudinal momentum P_L taken to infinity. Later Susskind [5] systematized these simplifications by identifying a Galilei invariance on this transverse space in which the longitudinal momentum played the role of Newtonian mass. Thus the essentially Newtonian character of light-front dynamics was recognized.

Another study of light-front quantum dynamics [6] was inspired by the physics of deep inelastic lepton scattering, which probes the light-cone singularities of current correlators. This line of thought leads to an intuitively appealing description of the parton wave functions [7]. For nonperturbative purposes, the utility of these ideas in field theory, beyond conceptual clarifications, was limited by the fact that the ''Newtonian mass'' P_L was actually a continuous variable ranging from 0 to ∞. Furthermore, the standard vertices of Feynman diagrams gave nonzero amplitudes for a ''particle'' of Newtonian mass P_L to transform into several ''particles'' with masses P_L^k as long as $\Sigma_k P_L^k = P_L$. Thus the Newtonian analogy was imperfect: a continuously infinite number of species of Newtonian particles, which could transmute into each other, was required. A further annoyance is that field modes with $P_L=0$ have no Newtonian interpretation at all and must be explicitly removed, either by deleting them or by ''integrating them out,'' before the Newtonian analogy can be exploited. Neither procedure is without controversy.

It was not until these ideas were applied [8] to the Nambu-Goto relativistic string [9] that the full power of the Newtonian analogy was realized. With light-front time $\tau = x^+ = (x^0+x^1)/\sqrt{2}$ taken as the analogue of Newtonian time and with the points on the string parametrized so that the density of longitudinal momentum $P^+ \equiv (P^0+P^1)/\sqrt{2}$ is constant, the dynamics of relativistic string is identical to that of ordinary elastic non-relativistic string moving and vibrating in the transverse space, described by coordinates x^k, $k=2,\ldots,D-1$. In this description all information about the motion of string in the remaining direction x^- is redundant save for its conjugate momentum P^+ which measures the total Newtonian mass of the string. From this point of view, the continuous variability of the Newtonian mass simply reflects the property that string is made up of continuous material. It is natural to suppose that in reality, just as with a violin string, relativistic string is not continuous but made up of tiny constituents [10], string-bits [11]. With this proposal, the dynamics of fully interacting string can be formulated as those of a standard Newtonian system.

As we have noted above, the light-front description of an ordinary quantum field theory requires the introduction of Newtonian ''particles'' with every possible value of the mass. This is not necessary with string because variation in Newtonian mass is naturally achieved by the breaking and joining of pieces of string containing various numbers of string-bits. Long ago in pursuit of a connection between field theory and string theory, we showed that light-front field theory can be made more ''Newtonian'' by discretizing the $P^+ \rightarrow Mm$ each field quantum can carry [12]; see also [13].[1] Thus instead of a continuous infinity of species of particles,

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[1]Since these early proposals, a major industry, known as discrete light cone quantization (DLCQ), has developed from them, starting with [14]. The literature in this field is now enormous and can be tapped by consulting the recent review article in [15].

there is only a discrete infinity, one species for each number M of fundamental mass units m. Field theoretic interactions would then occur in two fundamentally different ways: (1) There could be Newtonian-like potentials, either ''contact'' delta function potentials, due to quartic local terms in the original Hamiltonian, or non-local potentials induced by integrating out $P^+=0$ modes and/or constrained gauge fields, and (2) transition interactions in which mass is redistributed either through exchange in a 2 to 2 vertex or through fission in 1 to 2 and 1 to 3 vertices or through fusion in 2 to 1 or 3 to 1 vertices. Indeed, the light-front Hamiltonian P^- of the field theory is precisely that of a second-quantized many-body system, which includes terms that do not conserve particle number even though Newtonian mass is conserved. The difficulties of dealing with such a Hamiltonian are comparable to those of dealing with the standard time-like (in Dirac's language ''instant'') form of the Hamiltonian, which is why the Newtonian analogy has been less useful in this situation.

An important inspiration for this work is the new optimism about the tractability of 't Hooft's large N_c limit of QCD [16] generated by the intriguing conjecture that large N_c gauge theories are equivalent to classical string theories on certain anti–de Sitter (AdS) backgrounds [17–19]. Even as these ideas are being vigorously pursued, we think it is important to reconsider earlier efforts to connect large N_c gauge theory to string theory. This is especially true since the status of the conjecture at finite 't Hooft coupling $N_c g^2$ is problematic; so alternative ideas might yield useful insight on this score. Some 20 years ago, following a suggestion by 't Hooft [16], we sought to identify the sum of planar diagrams, parametrized on a light front, with the path integral over a light-front parametrized world sheet [12,20].[2] We found that such an identification made sense only in a certain large 't Hooft coupling limit, $N_c g^2 \to \infty$, which enforced a ''wee parton'' approximation. Interestingly, this is also the limit in which the Maldacena conjecture has the strongest support, because then the problematic string theory on a curved AdS background can be replaced by its well understood low energy supergravity limit. Away from this limit it was also clear from our earlier work that the light-front approach to large N_c field theory dictated several physical modifications of the minimal Nambu-Goto dynamics, including summing over ''holes'' or ''tears'' in the world sheet and also over the contribution of ''valence'' partons carrying a finite fraction of the string momentum. The first complication can be neatly handled by simply replacing a harmonic nearest neighbor wee parton interaction with a short range attractive potential [22,23]. However, we offered no such efficient way of including valence partons except by brute force summation. We are therefore motivated to ask whether valence partons can be effectively included in the context of a conventional Newtonian many body system made up of wee partons only.

Thus, the purpose of this article is to explore the possibility that underlying the light-front form of quantum field theory is a completely standard Newtonian system of ''bits'' residing on the transverse space. The fact that perturbative string dynamics *is* Newtonian in this pure sense and, in the infinite tension limit, can be described by an effective quantum field theory implies that such an underlying system should, at least in theory, exist: first obtain string theory from a string-bit model and then take its infinite tension limit. Whether it is possible to spell out its dynamics in a useful way, and whether its existence is any help in dealing with interesting non-perturbative issues, such as quark confinement in QCD, are issues we will not address. Our aim here is the more modest one of examining the features such an underlying theory must possess and using some simple quantum mechanical models to illustrate how the mechanisms can work.

Our basic proposal is that just as string can be regarded as a polymeric bound state of string bits, a field quantum can be regarded as a very tightly bound state of bits. The quantity of P^+ such a quantum carries is just proportional to the number of bits it contains. If such an interpretation is successful, string theory and quantum field theory would be effective low energy descriptions of a single kind of underlying theory. From a pragmatic standpoint, rephrasing complicated dynamical issues in quantum field theory, such as quark confinement, into a question about the properties of various kinds of Newtonian many-body systems could lead to new insights as well as to new quantitative results.

In the next section we recall how field theoretic interactions look on the light front by examining a cubic scalar field interaction. We then go on in Sec. III to study how the ideas sketched above play out for a simple 2-bit truncation of the scalar field model. We exhibit and solve a simple two particle potential model which serves as the underlying Newtonian model for the truncated field theory. The final section is devoted to a discussion of the prospects for applying these ideas to full-fledged field theory models, especially large N_c QCD.

II. CUBIC VERTEX IN SCALAR FIELD THEORY

Let us begin by reviewing the light-front description of a scalar field. It can be summarized by writing

$$\phi(\mathbf{x},x^-)=\int_0^\infty dP^+\,\frac{a(\mathbf{x},P^+)e^{-iP^+x^-}+a^\dagger(\mathbf{x},P^+)e^{iP^+x^-}}{\sqrt{4\pi P^+}} \tag{2.1}$$

where $[a(\mathbf{x},P^+),a^\dagger(\mathbf{y},Q^+)]=\delta(\mathbf{x}-\mathbf{y})\,\delta(P^+-Q^+)$. The free field Hamiltonian is just

$$H_0=P_0^-=\int_0^\infty d\mathbf{x}dP^+a^\dagger(\mathbf{x},P^+)\,\frac{(-\nabla^2+\mu^2)}{2P^+}a(\mathbf{x},P^+). \tag{2.2}$$

A typical field theoretic interaction, a $g\phi^3/6$ term, has the light front presentation

[2] For a gauge theory such as QCD the validity of such an identification [21] was clouded by the uncertainty of how to effectively deal with the $P^+=0$ singularities of light-front gauge. We hope that the ideas advanced in this article will lead to a clarification of such issues.

$$V_3=\frac{g}{8\sqrt{\pi}}\int d\mathbf{x}\int_0^\infty dP^+dQ^+\frac{a^\dagger(\mathbf{x},P^++Q^+)a(\mathbf{x},P^+)a(\mathbf{x},Q^+)+a^\dagger(\mathbf{x},P^+)a^\dagger(\mathbf{x},Q^+)a(\mathbf{x},P^++Q^+)}{\sqrt{P^+Q^+(P^++Q^+)}}. \tag{2.3}$$

We would like to explore the possibility that $a^\dagger(P^+)$ creates not an elementary quantum with Newtonian mass P^+, but rather a tightly bound state of infinitely many bits whose total Newtonian mass is P^+. Begin with a discretization of $P^+=Mm$, where m is the Newtonian mass of an elementary bit. Then $a(\mathbf{x},P^+)$ is replaced by $a_M(\mathbf{x})/\sqrt{m}$, so that $[a_M(\mathbf{x}),a_N^\dagger(\mathbf{y})]=\delta_{MN}\delta(\mathbf{x}-\mathbf{y})$. Then the preceding equations reduce to

$$\phi(\mathbf{x},x^-)=\sum_{M=1}^{\infty}\frac{1}{\sqrt{4\pi M}}(a_M(\mathbf{x})e^{-iMmx^-}+a_M^\dagger(\mathbf{x})e^{iMmx^-}) \tag{2.4}$$

with the free field Hamiltonian

$$H_0=P_0^-=\frac{1}{m}\int d\mathbf{x}\sum_{M=1}^{\infty}a_M^\dagger(\mathbf{x})\frac{(-\nabla^2+\mu^2)}{2M}a_M(\mathbf{x}) \tag{2.5}$$

and the cubic interaction

$$V_3=\frac{g}{8m\sqrt{\pi}}\int d\mathbf{x}\sum_{M,N=1}^{\infty}\frac{1}{\sqrt{MN(M+N)}}\times[a_{M+N}^\dagger(\mathbf{x})a_M(\mathbf{x})a_N(\mathbf{x})+a_M^\dagger(\mathbf{x})a_N^\dagger(\mathbf{x})a_{M+N}(\mathbf{x})]. \tag{2.6}$$

Note that our discretization includes a prescription for regulating the notorious $P^+=0$ singularities of light cone quantization: the $M=0$ terms are simply deleted. We therefore implicitly assume that any physical phenomena involving $P^+=0$ are adequately described as a limit from $P^+>0$. This might, of course, require that the modes with $P^+=0$ be ‘‘integrated out,’’ inducing new interactions among the modes with $P^+\neq 0$. In cases where the $P^+=0$ problems cannot be dealt with in this way (see, for example, [24]), the Newtonian analogy would fall short in an important respect, and the more far-reaching aspects of our proposal of a perfect Newtonian analogy would not apply.

If $a_M^\dagger$ creates a bound state, rather than an elementary quantum, the interaction (2.6) is to be regarded as a term in an effective Hamiltonian, which reproduces a transition process in the underlying theory in the limit where the size of the composite state is negligible compared to the wavelengths characterizing the transition. The factors $1/\sqrt{MN(M+N)}$, crucial for Poincaré invariance, must arise as properties of the bound system and are not automatic. For example, in the case of the discretized bosonic string, it was shown in [10] that the square root is, in generic transverse dimensionality d, replaced by the fractional power $d/48$. This leads to the critical dimensionality $d=24$. Although string theory provides an existence proof for an appropriate binding mechanism, we are suggesting that the phenomenon could be more general.

The free Hamiltonian (2.5) includes a term giving the free particle energy for each value of M. For $g=0$ the M dependence displayed is required by Lorentz invariance. If each quantum is in fact a composite, the energy is given by the binding dynamics and cannot be put in by hand. However, the coefficient of $-\nabla^2$ is guaranteed by the underlying Galilei invariance of this dynamics: the term just gives the center of mass kinetic energy. The M dependence of the term $\mu^2/2mM$ is *not* guaranteed *a priori* and represents a limitation on the binding dynamics. In the case of string viewed as a polymer of string-bits, this dependence arises for large M due to the one-dimensionality of the bound system (so the length of the system is proportional to M) and the universal 1/length dependence of phonon energies. Notice, for example, that an ordinary *elastic* p-brane would have a linear size proportional to $M^{1/p}$ and therefore an incorrect M dependence unless $p=1$. However, when a relativistic membrane is viewed on a light front, the restoring energies are of order $(\partial x)^{2p}$, giving a classical energy estimate of order $(1/\text{size})^p$ restoring, at least superficially, the correct M dependence.

III. TWO-BIT MODEL

In order to illustrate the manner in which an effective ‘‘elementary’’ quantum with $M\neq 1$ may be regarded as a bound state of quanta with $M=1$ only, we turn to an admittedly highly rarefied truncation of the scalar field theory described in the previous section. We specify this model by restricting the scalar field theory to the sector with $M=2$. That is we have only two classes of Fock states: those with two quanta with $M=1$ and those with a single quantum with $M=2$. A general state in this sector therefore has the representation

$$|\psi,\chi\rangle=\int d\mathbf{x}_1d\mathbf{x}_2a_1^\dagger(\mathbf{x}_1)a_1^\dagger(\mathbf{x}_2)|0\rangle\psi(\mathbf{x}_1,\mathbf{x}_2)+\int d\mathbf{x}a_2^\dagger(\mathbf{x})|0\rangle\chi(\mathbf{x}). \tag{3.1}$$

With this truncation the cubic vertex reduces to only two terms:

$$V_3^{trunc}=\frac{g}{8m\sqrt{2\pi}}\int d\mathbf{x}[a_2^\dagger(\mathbf{x})a_1(\mathbf{x})a_1(\mathbf{x})+a_1^\dagger(\mathbf{x})a_1^\dagger(\mathbf{x})a_2(\mathbf{x})]. \tag{3.2}$$

The time independent Schrödinger equation for this system is thus a coupled pair of differential equations:

$$\frac{1}{2m}[-\nabla_1^2-\nabla_2^2+2\mu^2-2mE]\psi(\mathbf{x}_1,\mathbf{x}_2)+\frac{g}{8m\sqrt{2\pi}}\delta(\mathbf{x}_1-\mathbf{x}_2)\chi(\mathbf{x}_1)=0$$

$$\frac{1}{4m}[-\nabla^2+\mu_2^2-4mE]\chi(\mathbf{x})+2\frac{g}{8m\sqrt{2\pi}}\psi(\mathbf{x},\mathbf{x})=0. \tag{3.3}$$

Notice that we have allowed for the $M=2$ quantum to have a ''bare'' Lorentzian mass μ_2 different from that of the $M=1$ quantum μ. This is because the Lorentzian mass of the $M=2$ quantum is obviously renormalized by the interactions unlike that of the $M=1$ quantum.

By Galilei invariance we may work in the center of mass system for which ψ is a function $f(\mathbf{x})$ only of the relative coordinate $\mathbf{x}\equiv\mathbf{x}_1-\mathbf{x}_2$ and χ is a constant. Then the second equation can be trivially solved for χ, which can then be substituted back into the first equation to give the single particle Schrödinger equation

$$[-\nabla^2+\mu^2-mE]f(\mathbf{x})-\frac{g^2\delta(\mathbf{x})}{8\pi(\mu_2^2-4mE)}f(\mathbf{0})=0. \tag{3.4}$$

The delta function potential is of course singular in most transverse dimensionalities; so we need to regulate it. A convenient regularization is to specify that $\delta(\mathbf{x})$ be replaced by a radial delta function $\delta(|\mathbf{x}|-a)/a^{d-1}\Omega_d$ displaced a distance a from the origin on s-waves and be zero on all other states. Here $\Omega_d=2\pi^{d/2}/\Gamma(d/2)$ is the volume of a unit $(d-1)$-sphere. Then Eq. (3.4) gives non-trivial dynamics on s-waves where it reduces to the radial equation $[f_{s\text{-}wave}\equiv R(r)]$

$$\left[-\frac{d^2}{dr^2}-\frac{d-1}{r}\frac{d}{dr}+\mu^2-mE\right]R(r)-\frac{g^2}{8\pi\Omega_d a^{d-1}(\mu_2^2-4mE)}\delta(r-a)R(a)=0. \tag{3.5}$$

This simple model can be completely solved. In order that the $M=2$ quantum have the same Lorentzian mass as the $M=1$ quantum, we require that there be a discrete s-wave energy eigenstate with $E=\mu^2/4m$. This condition will determine the bare mass μ_2. Putting $\kappa\equiv\sqrt{\mu^2-mE}=\sqrt{3}\mu/2$, the solutions of Eq. (3.5) for $r\neq a$ are the Bessel functions $I_\nu(\kappa r),K_\nu(\kappa r)$ with $\nu=(d-2)/2$ for $r<a$, $r>a$ respectively. Continuity at $r=a$ and the discontinuity in first derivatives implied by the delta potential lead to the relation

$$\mu_2^2=\mu^2+\frac{g^2}{8\pi\Omega_d a^{d-2}}K_\nu(\kappa a)I_\nu(\kappa a). \tag{3.6}$$

Of course μ_2 diverges as $a\to0$, for $d\geqslant2$.

Now we turn to the continuous part of the spectrum with $\mu^2-mE\equiv-k^2<0$. Then the solutions are the ordinary Bessel functions $J_\nu(kr),N_\nu(kr)$. The s-wave phase shift is then determined by the matching conditions at $r=a$ of the two forms

$$R(r)=J_\nu(kr), \qquad r<a,$$

$$R(r)=J_\nu(ka)\frac{\cot\delta J_\nu(kr)-N_\nu(kr)}{\cot\delta J_\nu(ka)-N_\nu(ka)}, \qquad r>a. \tag{3.7}$$

Solving these conditions gives

$$\cot\delta=\frac{N_\nu(ka)}{J_\nu(ka)}+\frac{16\Omega_d a^{d-2}(\mu^2-4mE)}{g^2J_\nu(ka)^2}+\frac{2}{\pi}\frac{K_\nu(\kappa a)I_\nu(\kappa a)}{J_\nu(ka)^2}. \tag{3.8}$$

Recalling that $a\neq0$ was a temporary regulator, we now take the limit $a\to0$, which exists at fixed g for $\nu<1$ corresponding to $d<4$:

$$\cot\delta\to\cot\pi\nu-\csc\pi\nu\left(\frac{\kappa}{k}\right)^{2\nu}-\Gamma(1+\nu)^2\frac{2^{2\nu+6}\Omega_d(k^2+\kappa^2)}{g^2k^{2\nu}}. \tag{3.9}$$

We notice that this is just the d-dimensional effective range approximation:

$$k^{2\nu}\cot\delta=k^{2\nu}\cot\pi\nu-\kappa^{2\nu}\csc\pi\nu-\Gamma(1+\nu)^2\frac{2^{2\nu+6}\Omega_d(k^2+\kappa^2)}{g^2}. \tag{3.10}$$

The familiar case of $d=3$ corresponds to $\nu=1/2$, whence the effective range formula is

$$k\cot\delta=-\frac{1}{a_s}+\frac{1}{2}r_{eff}k^2$$

with a_s the scattering length and r_{eff} the effective range. Thus the r_{eff} is *negative* for this system. Since d is transverse dimensionality, it would actually be 2 in our 3 dimensional world, corresponding to $\nu=0$. In that case our general formula reduces to

$$\cot\delta=\frac{2}{\pi}\ln\frac{k}{\kappa}-\frac{128\pi(k^2+\kappa^2)}{g^2}\Gamma(1+\nu)^2. \tag{3.11}$$

For this simple system, the question we pose in this article is whether the system can be equally well described by a two $M=1$ particle system, without the explicit introduction of a new species of particle with $M=2$. To be more precise, we do not mean to ''integrate out'' the $M=2$ field as in Eq. (3.4), which gives an effective two particle dynamics. The presence of E dependence in the potential term is the tip-off that a degree of freedom has been eliminated, and this is what we want to avoid. In other words, we seek a two particle potential *independent* of E which reproduces the same physics as Eq. (3.4). Since the effective range approximation is a universal low energy behavior for potential scattering, we expect that there are many potentials that do the trick. However, since a negative effective range is perhaps unfamiliar, we think it illuminating to exhibit a particular sample potential which yields the desired behavior.

To avoid the usual positive sign of the effective range, it is essential to use a potential that is not monotonic. A simple tractable choice which does the job is a potential of the form

$$V(r)=-\gamma\delta(r-b)+\lambda\,\delta(r-a),$$
$$0<b<a \quad \text{and} \quad \gamma,\lambda>0. \tag{3.12}$$

This is an idealized version of a more generic potential of the shape shown in Fig. 1

The important qualitative features here are an attractive potential to produce a bound state to simulate the $M=2$ particle, and a repulsive barrier to suppress the coupling of this bound state to the two particle state unless the two particles are within a distance of $O(a)$ from each other. In the limit $a\to 0$, the couplings can be tuned so that the physics of Eq. (3.4) is reproduced.

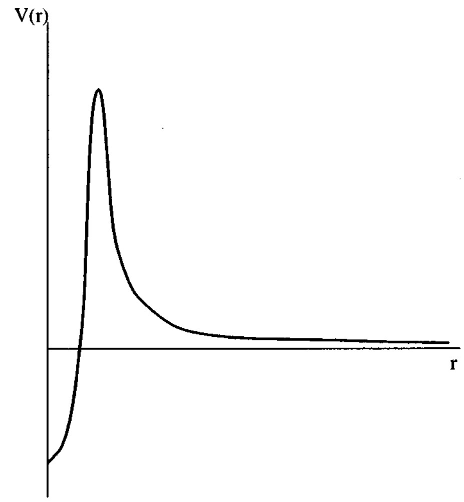

FIG. 1. Potential energy function for the two bit model.

Here is a sketch of the calculation. The s-wave radial wave function is given in the three regions by

$$R(r)=J_\nu(kr), \quad r<b,$$

$$R(r)=J_\nu(kb)\frac{\cot\phi J_\nu(kr)-N_\nu(kr)}{\cot\phi J_\nu(kb)-N_\nu(kb)}, \quad b<r<a,$$

$$R(r)=J_\nu(kb)\frac{\cot\phi J_\nu(ka)-N_\nu(ka)}{\cot\phi J_\nu(kb)-N_\nu(kb)}\,\frac{\cot\delta J_\nu(kr)-N_\nu(kr)}{\cot\delta J_\nu(ka)-N_\nu(ka)}, \quad r>a. \tag{3.13}$$

The jump condition for R' at $r=b$ and $r=a$ can be solved for $\cot\phi$ and $\cot\delta$:

$$\cot\phi=\frac{N_\nu(kb)}{J_\nu(kb)}+\frac{1}{\hat{\gamma}J_\nu^2(kb)}$$

$$\cot\delta=\frac{N_\nu(ka)}{J_\nu(ka)}+\frac{N_\nu(ka)-J_\nu(ka)\cot\phi}{J_\nu(ka)[\hat{\lambda}\{J_\nu^2(ka)\cot\phi-J_\nu(ka)N_\nu(ka)\}-1]}, \tag{3.14}$$

where, to reduce clutter, we have defined $\hat{\lambda}=\pi m\lambda a/2$ and $\hat{\gamma}=\pi m\gamma b/2$. Eliminating $\cot\phi$ in the second of these equations and rearranging factors slightly leads to

$$\cot\delta=\frac{N_\nu(ka)}{J_\nu(ka)}-\frac{1}{J_\nu^2(ka)}\left[\hat{\lambda}+\frac{1}{J_\nu^2(ka)}\left(\frac{N_\nu(ka)}{J_\nu(ka)}-\frac{N_\nu(kb)}{J_\nu(kb)}-\frac{1}{\hat{\gamma}J_\nu^2(kb)}\right)^{-1}\right]^{-1}, \tag{3.15}$$

To compare with Eq. (3.9), we need to study the low energy behavior of the phase shift; i.e., we take $ka\ll 1$. The small argument behaviors of the Bessel functions yield

$$\frac{1}{J_\nu^2(z)}=\Gamma(1+\nu)^2\left(\frac{z}{2}\right)^{-2\nu}\left[1+\frac{z^2}{2(1+\nu)}+O(z^4)\right]$$

$$\frac{N_\nu(z)}{J_\nu(z)}=\cot\,\pi\nu-\frac{\Gamma(1+\nu)^2}{\pi\nu}\left(\frac{z}{2}\right)^{-2\nu}\left[1-\frac{\nu z^2}{2(1-\nu^2)}+O(z^4)\right]. \tag{3.16}$$

For definiteness we restrict our low energy analysis to ν in the range $0<\nu<1$ which will cover the case $d=3$ and the case $d=2$ as a limit. Then inspection shows that the first term of Eq. (3.15) has a singular behavior as $a\to 0$ whose cancellation requires that the quantity within square brackets approach $-\pi\nu$. Further, in order to yield a nontrivial phase shift in the limit, this value must be approached as the power $a^{2\nu}$:

$$\hat{\lambda}+\frac{1}{J_\nu^2(ka)}\left(\frac{N_\nu(ka)}{J_\nu(ka)}-\frac{N_\nu(kb)}{J_\nu(kb)}-\frac{1}{\hat{\gamma}J_\nu^2(kb)}\right)^{-1}=-\pi\nu+O(a^{2\nu}). \tag{3.17}$$

This can be achieved by tuning the a dependence of $\hat{\gamma},\hat{\lambda}$. Putting in the small argument expansion for the Bessel functions in Eq. (3.17) gives, with $\eta\equiv b/a$,

$$\hat{\lambda}+[1+O(a^2)]\left[\frac{\eta^{-2\nu}}{\pi\nu}\left(1-\frac{\nu k^2b^2}{2(1-\nu^2)}\right)-\frac{1}{\pi\nu}\left(1-\frac{\nu k^2a^2}{2(1-\nu^2)}\right)-\frac{\eta^{-2\nu}}{\hat{\gamma}}\left(1+\frac{k^2b^2}{2(1+\nu)}\right)\right]^{-1}=-\pi\nu+O(a^{2\nu}). \tag{3.18}$$

In order to have k dependence in the limit, $\hat{\gamma}$ must be tuned so that quantity in the denominator of the second term vanishes as the power $a^{1-\nu}$, so that the quadratic terms in k will contribute the requisite power $a^{2\nu}$. Thus put

$$\frac{1}{\pi\nu}(\eta^{-2\nu}-1)-\frac{1}{\hat{\gamma}}\eta^{-2\nu}=-\xi a^{1-\nu}\frac{1-(a/l)^{2\nu}}{\nu}, \tag{3.19}$$

where the extra factor ensures the proper behavior at $\nu=0$. Then Eq. (3.18) becomes

$$\begin{aligned}&\hat{\lambda}+[1+O(a^2)]\left(-\xi a^{1-\nu}\frac{1-(a/l)^{2\nu}}{\nu}+(ka)^2\left[\frac{1-\eta^{2-2\nu}}{2\pi(1-\nu^2)}-\frac{\eta^{2-2\nu}}{2\hat{\gamma}(1+\nu)}\right]\right)^{-1}\\&\sim\hat{\lambda}-\left[\frac{\nu}{\xi a^{1-\nu}[1-(a/l)^{2\nu}]}+O(a^{1+\nu})\right]\left(1+\frac{k^2a^{1+\nu}\nu}{\xi[1-(a/l)^{2\nu}]}\left[\frac{1-\eta^{2-2\nu}}{2\pi(1-\nu^2)}-\frac{\eta^{2-2\nu}-\eta^2}{2\pi\nu(1+\nu)}\right]\right)\\&=-\pi\nu+O(a^{2\nu}),\end{aligned} \tag{3.20}$$

where, in the second line, we have substituted the limiting form for $1/\hat{\gamma}$ in the coefficient of k^2a^2, and we have also approximated the reciprocal by the first two terms of the Taylor series. We can now easily read off

$$\hat{\lambda}=\frac{\nu}{\xi a^{1-\nu}[1-(a/l)^{2\nu}]}-\frac{\pi\nu}{[1-(a/l)^{2\nu}]}-\frac{C\nu^2a^{2\nu}}{[1-(a/l)^{2\nu}]^2} \tag{3.21}$$

and thence the s-wave phase shift

$$\cot\,\delta=\cot\,\pi\nu-\frac{\Gamma(1+\nu)^2}{\pi\nu}\left(\frac{kl}{2}\right)^{-2\nu}-\frac{\Gamma(1+\nu)^2}{\pi^2}\left(\frac{k}{2}\right)^{-2\nu}\left[C+\frac{k^2}{\xi^2}\left(\frac{1-\eta^{2-2\nu}}{2\pi(1-\nu^2)}-\frac{\eta^{2-2\nu}-\eta^2}{2\pi\nu(1+\nu)}\right)\right]. \tag{3.22}$$

Notice that the coefficient of k^2 will be negative as in Eq. (3.9) if the quantity

$$f(\eta^2)\equiv\frac{\nu(1-\eta^{2-2\nu})}{(1-\nu)}-\eta^{2-2\nu}+\eta^2$$

is positive. To see when this occurs, note that $f'=1-\eta^{-2\nu}<0$ for $0<\eta^2<1$, and $f(0)=\nu/(1-\nu)$, $f(1)=0$. It follows that f is positive in this interval which is when $b<a$. We have been careful to set things up so that the case of $d=2$ is properly described by the singular limit $\nu\to 0$.

We conclude this section by stating, for this baby two-bit model, how our results realize the goals set out in the Introduction. The underlying ''microscopic'' theory of the model is the two particle system described by the potential (3.12). The parameters of the microscopic theory are the couplings λ,g and the distance scales a,b. The effective baby field theory is described by the pair of equations (3.3), with g the ''bare'' cubic coupling and μ,μ_2 the ''bare'' Lorentzian masses. This effective field theory has ultraviolet divergences which require a regulator. After removing the regulator, keeping measurable parameters fixed and tuned so that the ''renormalized'' Lorentzian masses are the same for different values of M, one obtains the scattering phase shift (3.9). The phase shift of the underlying microscopic model (3.15) shows a lot of structure at the microscopic scale $k \sim a$. However, at low energies $ka \ll 1$ it shows the same behavior (3.22) as the baby field theory.

Comparing Eq. (3.22) to Eq. (3.9) relates the effective field theoretic coupling g to the microscopic parameters:

$$g^2 = \xi^2 \frac{16(2\pi)^3 \Omega_d \nu(1-\nu^2)}{\nu - \eta^{2(1-\nu)} + (1-\nu)\eta^2}. \tag{3.23}$$

Notice that weak field theoretic coupling $g \to 0$ corresponds at fixed a,b to the height of the barrier going to infinity $\hat{\lambda} \to \infty$ while the coefficient of the attractive component of the potential goes to a finite limit $\hat{\gamma} \to \pi\nu/(1-\eta^{2\nu})$. The opposite limit $g \to \infty$ corresponds to vanishing $\hat{\gamma}$ and $\hat{\lambda}$ approaching a finite negative constant. Thus in this latter limit the barrier disappears.

IV. DISCUSSION

Our crude two bit model illustrates the mechanism we have in mind for dealing with the variable P^+ carried by lines in light-front Feynman diagrams. Instead of explicitly summing over each P^+, it is hoped that the tight-binding part of the interaction potential will cause a collection of M ''bits'' with Newtonian mass m to behave as a single particle with Newtonian mass Mm. Of course, for this to really work, the many body bound states must exhibit many consistency conditions embodied in the fact that they must act as a component of a relativistic field.

For M larger than 2, it is not at all clear for a generic field theory that a restriction to only two body interactions will afford enough flexibility to meet these conditions. For example, a one-dimensional many particle system with the same attractive delta function interaction between each pair is exactly soluble but has entirely the wrong scaling behavior with large M.

However, for matrix field theories at large N_c the prospects are brighter. This is because, as shown in [22], the dynamics of the the large N_c limit can be mapped onto those of a linear chain on the light front. The field quanta or partons are in this limit ordered around a ring and only nearest neighbors on the ring interact. In string-bit models of fundamental string, all partons are ''wee'' and nearest neighbors interact via a potential of the shape shown in Fig. 2. As is well-known [11], this sort of dynamics leads to precisely the Nambu-Goto string. For a confining field theory like QCD, however, the chain dynamics includes processes in which the gluon quanta fuse and fission so that the number of gluons is not conserved.

Let us recall how large N_c gluon dynamics was formulated in [22]. After discretizing P^+ in the usual way, we can consider the dynamics of a glueball carrying M units of P^+.

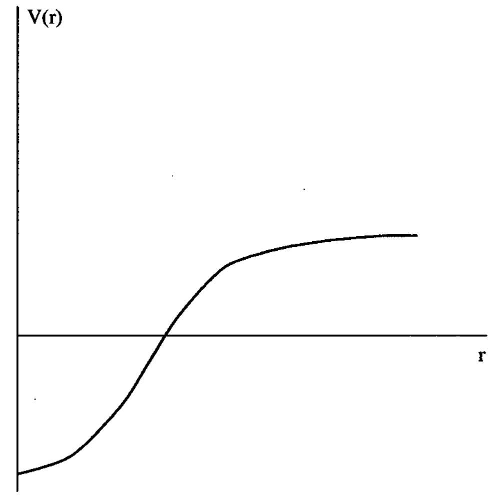

FIG. 2. Potential energy function for fundamental string.

Then a state of the glueball can be described by an M component wave function, the kth component of which describes a system of k gluons and therefore depends on the transverse coordinate, polarization, and number of P^+ units of each gluon and is cyclically symmetric:

$$\begin{aligned}\Psi_k(\mathbf{x}_1, i_1, M_1; \ldots; \mathbf{x}_k, i_k, M_k) \\ = \Psi_k(\mathbf{x}_k, i_k, M_k; \mathbf{x}_1, i_1, M_1; \ldots; \mathbf{x}_{k-1}, i_{k-1}, M_{k-1}),\end{aligned} \tag{4.1}$$

with $\Sigma M_k = M$. Gluon dynamics is then formulated as a set of M coupled Schrödinger equations of the schematic form

$$\begin{aligned}\left(\sum_{l=1}^{k} \frac{\mathbf{p}_l^2}{2mM_k} + V_{k,k} - E \right) \Psi_k \\ = -V_{k,k-2}\Psi_{k-2} - V_{k,k-1}\Psi_{k-1} \\ - V_{k,k+1}\Psi_{k+1} - V_{k,k+2}\Psi_{k+2}.\end{aligned} \tag{4.2}$$

The term $V_{k,k}$ is a sum of nearest neighbor interaction potentials among the k gluons described by Ψ_k. It is actually a matrix differential operator because gluon spin and P^+ can be exchanged between the two neighbors. The coupling terms on the right-hand side (RHS) take into account the possibility of a change in the number of gluons. In each case these number changes respect the cyclic ordering. For example, by virtue of the cubic Yang-Mills vertex, a pair of nearest neighbors can convert into a single gluon, and that gluon occupies the same spot on the chain as the original pair. Similarly, if a single gluon on the chain converts to a pair, that pair's chain location is the same as that of the original gluon. Because of this nearest neighbor pattern, the processes just described are not unlike those in the baby field theory described in Sec. III. Just as we eliminated the $M = 2$ component in that case, we could imagine eliminating all of the Ψ_k for $k < M$, ending up with a horrific single equation for the ''wee parton'' component Ψ_M. Such a procedure looks hopelessly intractable.

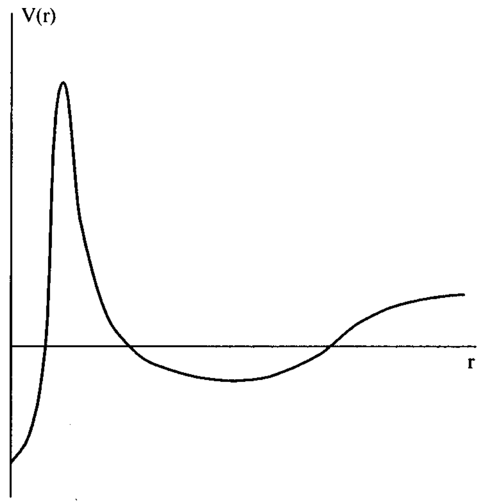

FIG. 3. Potential energy function for confining field theory.

Instead, we are suggesting in this article that by modifying the terms in $V_{M,M}$ to have a potential shape indicated in Fig. 3, one might do away with all the components Ψ_k, $k<M$ accounting for their effects as tunneling processes described by the new Schrödinger equation for Ψ_M. The long distance attractive potential well accounts for the stringy (confinement) behavior of a gluon chain and the short distance attraction and barrier enable long-lived tightly bound clusters of wee gluons which, we hope, act like valence gluons. To explore further this possibility, it is probably not a good idea to try to derive this new potential from gluon dynamics.

This is because the new potential describes an underlying theory different from QCD: the scale a is the scale at which gluons show compositeness; for instance, it could be the distance scale of fundamental string. Rather, one should directly explore the underlying theory and try to test whether it can reproduce QCD physics. Among these tests would be to see whether the right M dependence can come out of the gluon number changing transitions arising from tunneling through the short range barrier. The nearest neighbor interaction pattern of the large N_c limit provides a natural similarity between the conversion of a cluster of varying size into two smaller clusters: the tunneling process only involves the single link between the two clusters regardless of the cluster size. Of course larger clusters will have more inertia, so that the transition amplitudes *will* depend on cluster size. Another favorable circumstance is that the nearest neighbor pattern will naturally make the clusters polymeric and therefore stringy scaling laws are more likely.

Although we have not taken into account the many body aspects of this scheme for dealing with large N_c QCD, we can at least roughly understand why the limit of large 't Hooft coupling entails a wee parton approximation. Referring back to our baby field theory, it is the coupling g^2 that plays the role of $N_c g^2$. We have seen that, in the large g limit, the barrier of Fig. 3 disappears. Thus the nearest neighbor interaction reverts to a simple potential well as in Fig. 2 which wipes out the clustering effects responsible for valence partons.

Finally, to bring this discussion full circle, we would like to note that there is similar physics lurking in the AdS-QCD connection or, more precisely, in Polyakov's ''confining string'' proposal [25]. He suggests that the coefficient $a(\phi)$ of $(\partial x)^2$ in the usual world sheet action should depend on the Liouville field ϕ. Here $a(\phi)$ then has the interpretation of a dynamical tension. In [18] ϕ is just the ''fifth'' dimension of AdS_5. When such a world sheet dynamics is referred to the light front, one finds the Hamiltonian

$$P^- = \int_0^{P^+} d\sigma \frac{1}{2}[\mathcal{P}^2 + a^2(\phi)\mathbf{x}'^2 + a(\phi)(\Pi_\phi^2 + \phi'^2)]. \tag{4.3}$$

With σ discretized, the significance of $a^2(\phi)$ becomes a dynamical spring constant, and the quantum dynamics of ϕ can be interpreted as a certain average over variable spring constants. For harmonic oscillators, averaging over variable spring constants is equivalent (dual) to averaging over masses. But averaging over masses is what is accomplished by our clustering of wee partons into valence partons. What is not at all clear, of course, is whether the weightings of these averages have anything to do with one another.

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