Best approximation to a reversible process in black-hole physics and the area spectrum of spherical black holes

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The assimilation of a *quantum* (*finite* size) particle by a Reissner-Nordström black hole inevitably involves an increase in the black-hole surface area. It is shown that this increase can be *minimized* if one considers the capture of the *lightest charged* particle in nature. The unavoidable area increase is attributed to two physical reasons: the *Heisenberg quantum uncertainty principle* and a *Schwinger-type charge emission* (vacuum polarization). The fundamental lower bound on the area increase is $4\hbar$, which is *smaller* than the value given by Bekenstein for neutral particles. Thus this process is a better approximation to a reversible process in black-hole physics. The *universality* of the minimal area increase is further evidence in favor of a *uniformly* spaced area spectrum for spherical quantum black holes. Moreover, this universal value is in excellent agreement with the area spacing predicted by Mukhanov and Bekenstein and independently by Hod. [S0556-2821(99)03002-7]

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I. INTRODUCTION

Can the assimilation of a test particle by a black hole be made *reversible* in the sense that all changes in the black-hole parameters can be undone by another suitable process? This seemingly naive question goes deep into the basic laws of black-hole physics. A *classical* theorem of Hawking's [1] says that black-hole surface area cannot decrease. Hence, any physical process which increases the horizon area is obviously (classically) irreversible. The answer to the above question was given by Christodoulou [2] (later generalized by Christodoulou and Ruffini for the case of charged point particles [3]) almost three decades ago. The assimilation of a (*point*) particle is reversible if it is injected at the *horizon* from a *turning point* of its motion. In such a case the black-hole surface area is left *unchanged* and the changes in the other black-hole parameters (mass, charge, and angular-momentum) can be undone by another suitable (reversible) process.

However, as was pointed out by Bekenstein in his seminal work [4] the limit of a *point* particle is not a legal one in *quantum* theory. As a concession to quantum theory Bekenstein ascribes to the particle a *finite* proper radius b while continuing to assume, in the spirit of Ehrenfest's theorem, that the particle's center of mass follows a classical trajectory. Bekenstein [4] has shown that the assimilation of the finite size neutral particle inevitably causes an increase in the horizon area. This increase is minimized if the particle is captured when its center of mass is at a turning point a proper distance b away from the horizon [4]:

$$(\Delta\alpha)_{min}=2\mu b, \tag{1}$$

where the "rationalized area" α is related to the black-hole surface area A by $\alpha=A/4\pi$ and μ is the rest mass of the particle. For a point particle $b=0$ and one finds $(\Delta\alpha)_{min}=0$. This is Christodoulou's result for a reversible process. However, a quantum particle is subjected to quantum uncertainty. According to Bekenstein's analysis, a relativistic quantum particle cannot be localized to better than its Compton wavelength [This claim certainly is not correct when the particle has (locally measured) energy greater than its mass. In this paper we give a different and rigorous argument which leads to a lower bound on the increase in black-hole surface area]. Thus, b can be no smaller than $\hbar/\mu$. From here one finds a lower bound on the increase in black-hole surface area due to the assimilation of a (neutral) test particle

$$(\Delta\alpha)_{min}=2\hbar. \tag{2}$$

It is easy to check that the reversible process of Christodoulou and Ruffini and the lower bound Eq. (2) of Bekenstein are valid only for *non*-extremal black holes. Thus, for non-extremal black holes there is a *universal* (i.e., independent of the black-hole parameters) minimum area increase as soon as one allows quantum nuances to the problem. This fact is used as one of the major arguments in favor of a *uniformly* spaced area spectrum for quantum black holes [5].

The universal lower bound Eq. (2) derived by Bekenstein is valid only for *neutral* particles [4]. In this paper we analyze the assimilation of a quantum (*finite* size) *charged* particle by a Reissner-Nordström black hole and show that the fundamental lower-bound on the increase in the black-hole surface area is *smaller* than the value given by Bekenstein for neutral particles.

II. ASSIMILATION OF A CHARGED PARTICLE BY A REISSNER-NORDSTRÖM BLACK HOLE

The major goal of this paper is to calculate the (inevitable) *minimal* increase in black-hole surface area caused by the assimilation of a particle of rest mass μ, charge e and proper radius b. We are interested in the area increase ascribable to the *particle* itself, as contrasted with any increase incidental to the *process* of bringing the particle to the black-hole horizon [4]. For example, gravitational radiation emitted by the particle [6] or by any device which might have lowered it into the hole [7] will also cause an increase in the area. In addition, electromagnetic radiation emitted during the process of bringing the charged particle to the horizon [8] will also result in an increase in the black-hole surface area. In this paper, as in the seminal work of Bekenstein [4], we ignore these incidental effects and concentrate on the inevitable increase in the black-hole surface area caused by the captured particle all by itself.

The external gravitational field of a spherically symmetric object of mass M and charge Q is given by the Reissner-Nordström metric

$$ds^2=-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2+\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2+r^2d\Omega^2. \tag{3}$$

The black hole's (event and inner) horizons are located at

$$r_{\pm}=M\pm(M^2-Q^2)^{1/2}. \tag{4}$$

The equation of motion of a charged particle on the Reissner-Nordström background is a quadratic equation for the conserved energy E of the particle [9]

$$r^4E^2-2eQr^3E+e^2Q^2r^2-\Delta(\mu^2r^2+p_\phi^2)-(\Delta p_r)^2=0, \tag{5}$$

where Δ is given by

$$\Delta=r^2-2Mr+Q^2=(r-r_-)(r-r_+). \tag{6}$$

The quantities p_ϕ and p_r are the conserved angular momentum of the particle and its covariant radial momentum, respectively. It is useful to express this last quantity in terms of the physical component (in an orthonormal tetrad) $P\equiv\Delta^{-1/2}rp^r$ [10].

In order to find the change in black-hole surface area caused by an assimilation of a point particle one should first solve Eq. (5) for E and then evaluate it at the horizon $r=r_+$ of the black hole. As was pointed out in Refs. [2] and [3], this increase is minimized (actually *vanishes*) if the particle is captured from a turning point. How would the *non*-zero proper radius b of the particle (which is an *inevitable* feature of the quantum theory) change this scenario? First, we note that in the spirit of the Ehrenfest's theorem we continue to assume that the particle's center of mass follows a classical path. Second, as was pointed out by Bekenstein, regardless of the manner in which the particle arrives at the horizon, it must acquire its parameters (E and p_ϕ) while every part of it is still outside the horizon, i.e., while it is not yet part of the black hole [4]. Thus, the motion of the particle's center of mass at the moment of capture should be described by Eq. (5). Third, in order to generalize the results given in Refs. [2] and [3] to the case of a *finite* size particle one should evaluate E at $r=r_++\delta(b)$, where $\delta(b)$ is determined by [4]

$$\int_{r_+}^{r_++\delta(b)}(g_{rr})^{1/2}dr=b. \tag{7}$$

In other words, $r=r_++\delta(b)$ is a point a proper distance b outside the horizon. Integrating Eq. (7) one finds

$$\delta(b)=(r_+-r_-)\sinh^2\left(\frac{b}{2r_+}\right)[1+O(b/r_+)]. \tag{8}$$

Since we consider the case $b\ll r_+$ we may replace this expression by

$$\delta(b)=(r_+-r_-)\frac{b^2}{4r_+^2}. \tag{9}$$

The conserved energy E of a particle having a physical radial momentum P at $r=r_++\xi$ (where $\xi\ll r_+$) is given by Eq. (5)

$$E=\frac{eQ}{r_+}+\frac{\sqrt{(\mu^2+P^2)r_+^2+p_\phi^2}(r_+-r_-)^{1/2}}{r_+^2}\times\xi^{1/2}\{1+O[\xi/(r_+-r_-)]\}-\frac{eQ}{r_+^2}\xi\{1+O[\xi/(r_+-r_-)]\}. \tag{10}$$

This expression (for $P=0$) is actually the effective potential (gravitational plus electromagnetic plus centrifugal) for given values of μ, e, and p_ϕ. It is clear that it can be *minimized* by taking $p_\phi=0$ (which also minimize the increase in the black-hole surface area. This is also the case for neutral particles [4]). However, P^2 cannot be said to vanish because of Heisenberg quantum uncertainty principle. For $eQ>0$ the effective potential has a *maximum* located at

$$\xi^*=\frac{(r_+-r_-)(\mu^2+P^2)r_+^2}{4e^2Q^2}. \tag{11}$$

The assimilation of the particle results in a change $dM=E$ in the black-hole mass and a change $dQ=e$ in the black-hole charge. Using the first-law of black-hole thermodynamics

$$dM=\Theta d\alpha+\Phi dQ, \tag{12}$$

where $\Theta=\frac{1}{4}(r_+-r_-)/\alpha$ and $\Phi=Qr_+/\alpha$, one finds

$$d\alpha_{min}(s,\mu,e,b)=\frac{4(\mu^2+P^2)^{1/2}r_+}{(r_+-r_-)^{1/2}}\delta(b)^{1/2}-\frac{4eQ}{r_+-r_-}\delta(b), \tag{13}$$

which is the *minimal* area increase for given values of the black-hole parameters r_+ and Q (s stands for these two parameters) and the particle parameters μ, e, b, and P.

In order to be captured by the black hole the particle has to be over the potential barrier. There are two distinct cases that should be treated separately: for particles satisfying the relation $\delta(b)\leqslant\xi^*$ the area increase is *minimized* if bP is minimized. However, the limit $bP\to0$ is not a legal one in the quantum theory. According to Bekenstein's analysis [4] the particle cannot be localized to better than its Compton wavelength $\hbar/\mu$. (We will discuss the validity of this assumption below. In this paper we shall use a more rigorous argument to provide a lower bound on the product bP). On the other hand, particles satisfying the inequality $\delta(b) > \xi^*$ cannot be captured from a turning point of their motion. In order to overcome the potential barrier and be captured by the black hole they must have (at least) an energy $E(\xi^*)$.

Let us consider the first case $\delta(b) \leqslant \xi^*$. Substituting Eq. (9) into Eq. (13) one finds

$$d\alpha_{min}(s,\mu,e,b) = 2(\mu^2 + P^2)^{1/2} b - \frac{eQb^2}{r_+^2}. \tag{14}$$

According to Bekenstein's original analysis [4] one may minimize this expression by minimizing the value of b (and setting $P^2=0$ at the turning point). However, the claim used in [4] that a particle cannot be localized to within less than its Compton wavelength certainly is not correct when the particle has (locally measured) energy greater than its mass. The locally measured energy (by a static observer) of a particle near the horizon of a black hole can be arbitrarily large. In addition, at the turning point the physical radial momentum P^2 cannot be said to vanish, but must be replaced by its uncertainty $(\delta P)^2$ [10]. In other words, according to Heisenberg quantum uncertainty principle the particle's center of mass cannot be placed at the horizon with accuracy better than the radial position uncertainty $\hbar/(2\delta P)$.

Using the restriction $\delta(b) \leqslant \xi^*$ one finds

$$|e| \leqslant \frac{(\mu^2 + P^2)^{1/2} r_+^2}{|Q| b}. \tag{15}$$

Thus, the minimal area increase is given by

$$d\alpha_{min}(s,\mu) = [\mu^2 + (\delta P)^2]^{1/2} b, \tag{16}$$

which, according to Heisenberg quantum uncertainty principle, yields (for $\delta P \gg \mu$).

$$d\alpha_{min}(s,\mu) = \hbar/2. \tag{17}$$

Next, we consider the assimilation of particles which satisfy the relation $\delta(b) > \xi^*$. These particles cannot be captured from a turning point of their motion. In order to be captured by the black hole they must have a minimal energy of

$$E_{min} = E(\xi^*) = \frac{eQ}{r_+} + \frac{(\mu^2 + P^2)(r_+ - r_-)}{4eQ}. \tag{18}$$

Using the first-law of black-hole thermodynamics Eq. (12) one finds that the increase in the black-hole surface area is given by

$$d\alpha_{min}(s,\mu,e) = \frac{(\mu^2 + P^2) r_+^2}{eQ}. \tag{19}$$

What physics prevents us from using particles which make expression (19) as small as we wish? Or, in other words, what physics prevents us from recovering Christodoulou's reversible process $\Delta A = 0$? The answer is *Schwinger-type charge emission* (vacuum polarization) [11]. We must remember that the black-hole may *discharge* itself through a Schwinger-type emission. The critical electric field Ξ_c for pair-production of particles with rest mass μ and charge e is given by [11–13]

$$\Xi_c = \frac{\pi \mu^2}{e\hbar}. \tag{20}$$

This order of magnitude can easily be understood on physical grounds; Schwinger discharge is exponentially suppressed unless the work done by the electric field on the virtual pair of (charged) particles in separating them by a Compton wavelength is of the same order of magnitude (or more) of the particle's mass. Thus, assuming the existence of elementary particles with mass μ and charge e, a spherical black-hole of charge Q and radius r_+ (whose electric field near the horizon is $\Xi_+ = Q/r_+^2$) may be considered as quasistatic only if it satisfies the relation $\Xi_+ \leqslant \Xi_c$, or equivalently

$$|e| \leqslant \frac{\pi \mu^2 r_+^2}{|Q| \hbar}. \tag{21}$$

Substituting this into Eq. (19) one finds (for $\mu^2 \gg P^2$)

$$d\alpha_{min}(s,\mu) = \frac{\hbar}{\pi}, \tag{22}$$

which is now the fundamental lower bound on the increase in black-hole surface area. We note that this lower bound is *universal* in the sense that it is *independent* of the black-hole parameters M and Q.

III. SUMMARY AND DISCUSSION

We have studied the assimilation of a *charged* particle by a Reissner-Nordström black hole. The capture of a particle *necessarily* results in an increase in the black-hole surface area. The minimal area increase equals to $4\hbar$. We note that this value is *smaller* than the value given by Bekenstein for neutral particles. Thus, this process is a better approximation to a *reversible* process in the context of black-hole physics.

As was pointed out by Bekenstein [4] (for neutral particles) the underlying physics which excludes a completely reversible process is the *Heisenberg quantum uncertainty principle*. However, for *charged* particles it must be supplemented by another physical mechanism—*a Schwinger-discharge of the black hole*. Without this mechanism one could have reached the *reversible* limit. It is interesting that the lower bound found here is of the same order of magnitude as the one given by Bekenstein, even though they emerge from *different* physical mechanisms.

The *universality* of the fundamental lower bound (i.e., its independence on the black-hole parameters M and Q) is a further evidence in favor of a *uniformly* spaced area spectrum for spherical quantum black holes (see Ref. [5]). Moreover, the universal value $\Delta A_{min} = 4\hbar$ is in excellent agreement (to within a factor of $ln2$) with the area spacing predicted by Mukhanov and Bekenstein [14,5] and [to within a factor (of order *unity*) of $ln3$] with the area spacing predicted by Hod [15].

It should be recognized that the precise value of the universal lower bound Eq. (22) can be challenged. This lower bound follows from Eq. (20) which can only be interpreted as the critical electric field to within factors of few. Nevertheless, the new and interesting observation of this paper is the role of *Schwinger pair production* in providing an important limitation on the minimal increase in black-hole surface area.

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