Quantum modular group in (2+1)-dimensional gravity

S. Carlip*

Department of Physics, University of California, Davis, California 95616

J. E. Nelson[†]

Dipartimento di Fisica Teorica, Università degli Studi di Torino, via Pietro Giuria 1, 10125 Torino, Italy

The role of the modular group in the holonomy representation of $(2+1)$-dimensional quantum gravity is studied. This representation can be viewed as a ''Heisenberg picture,'' and for simple topologies, the transformation to the ADM ''Schrödinger picture'' may be found. For spacetimes with the spatial topology of a torus, this transformation and an explicit operator representation of the mapping class group are constructed. It is shown that the quantum modular group splits the holonomy representation Hilbert space into physically equivalent orthogonal ''fundamental regions'' that are interchanged by modular transformations. [S0556-2821(99)02302-4]

I. INTRODUCTION

Over the past few years, it has become apparent that $(2+1)$-dimensional general relativity can provide a valuable setting in which to explore some of the fundamental issues of realistic $(3+1)$-dimensional quantum gravity [1]. As a diffeomorphism-invariant theory of spacetime geometry, the $(2+1)$-dimensional model shares the conceptual framework of ordinary $(3+1)$-dimensional gravity. At the same time, however, the reduction in the number of dimensions greatly simplifies the structure: $(2+1)$-dimensional general relativity has only a finite number of physical degrees of freedom, and quantum field theory is effectively reduced to quantum mechanics.

At least fifteen different approaches to quantizing $(2+1)$-dimensional general relativity have been developed over the past decade. Two that have received special attention are reduced phase space quantization, starting with the Arnowitt-Deser-Misner (ADM) formalism and the York time slicing [2–4], and a set of techniques that take Chern-Simons holonomies as the fundamental observables [5–17]. Both approaches to quantization are well understood for the simplest topologies, and in particular for spacetimes with the spatial topology of a torus, $M \approx \mathbf{R} \times T^2$. For these topologies, the two techniques yield complementary information about the quantum behavior, and a comparison has offered valuable insights into both [8,18,19].

One persistent problem has, however, plagued this program. In addition to the usual ''small'' diffeomorphisms, the torus T^2 admits ''large'' diffeomorphisms, diffeomorphisms that cannot be continuously deformed to the identity. In ADM quantization, the natural configuration space is Teichmüller space, and the group of large diffeomorphisms—the modular group—has a well-understood and well-behaved action on this space. As a consequence, standard mathematical results allow us to construct invariant (or more

general ''covariant'') wave functions [9,20]. In the holonomy representation, on the other hand, the modular group does *not* act nicely (i.e., properly discontinuously) on the natural configuration space, and the construction of invariant wave functions is much more problematic [21–23]. Since the two approaches are supposed to be equivalent, this mismatch is a cause for concern.

In this paper, we resolve this problem by explicitly constructing a transformation between the two representations. In the ADM representation, the modular group splits the configuration space into fundamental regions that are interchanged by the action of the group, and an invariant wave function can be defined by giving its value on a single fundamental region. In the holonomy representation, no invariant wave functions exist. But we shall see that the *Hilbert space* now splits into orthogonal ''fundamental regions'' that are interchanged by a unitary action of the modular group. Each of these subspaces is equivalent, and each is equivalent to the ADM Hilbert space of invariant (technically, weight-1/2) wave functions. The choice of one such subspace is the discrete analog of a choice of gauge, and once such a choice is made, the conflict between the two quantizations disappears.

II. TWO QUANTIZATIONS

We start with a very brief review of the two approaches to quantization described in the introduction, focusing on the torus universe $\mathbf{R} \times T^2$. For simplicity, we shall consider only a negative cosmological constant, $\Lambda = -1/\alpha^2$. Details can be found in Refs. [18,19] and [1].

To construct an ADM quantization, we first foliate the spacetime $\mathbf{R} \times T^2$ by time slices of constant mean (extrinsic) curvature k [24]. The fixed value of k on a slice then serves as a time coordinate. The geometry of each T^2 slice is determined up to a conformal factor by a complex modulus $\tau = \tau_1 + i\tau_2$,

$$d\sigma^2 = e^{2\lambda}\tau_2^{-1}|dx + \tau dy|^2. \qquad (2.1)$$

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*Email address: carlip@dirac.ucdavis.edu
[†]Email address: nelson@to.infn.it

It may be shown that the conformal factor is fixed by the Hamiltonian constraint, leaving a physical phase space parametrized by the variables τ_1 and τ_2 and their conjugate momenta p^1 and p^2, or equivalently by complex variables τ and $p = p^1 + ip^2$. Evolution in constant mean curvature time k is generated by an effective Hamiltonian that is just the spatial volume [2,3],

$$H = \int_{T^2} d^2x \sqrt{^{(2)}g} = \frac{1}{\sqrt{k^2 - 4\Lambda}} \widetilde{H}, \quad \widetilde{H} = \tau_2 \sqrt{p\bar{p}}. \quad (2.2)$$

The quantity $\widetilde{H}$ may be recognized as the square of the momentum p with respect to the Poincaré (constant negative curvature) metric

$$d\ell^2 = \tau_2^{-2} d\tau d\bar{\tau}, \quad (2.3)$$

the standard metric on the torus moduli space. The basic Poisson brackets are

$$\{\tau, \bar{p}\} = \{\bar{\tau}, p\} = 2, \quad \{\tau, p\} = \{\bar{\tau}, \bar{p}\} = 0, \quad (2.4)$$

and the reduced Einstein action becomes

$$I_{Ein} = \int dk \left(p^\alpha \frac{d\tau_\alpha}{dk} - H(\tau, p, k) \right). \quad (2.5)$$

The reduction to the variables τ and p eliminates the ''small'' diffeomorphisms, but a group of ''large'' diffeomorphisms, the modular group, remains. One set of generators of this group consists of two transformations S and T, which act classically as

$$S: \tau \rightarrow -\tau^{-1}, \quad p \rightarrow \bar{\tau}^2 p,$$

$$T: \tau \rightarrow \tau + 1, \quad p \rightarrow p \quad (2.6)$$

and satisfy the identities

$$S^2 = 1, \quad (ST)^3 = 1. \quad (2.7)$$

These transformations leave the Hamiltonian (2.2) and Poisson brackets (2.4) invariant.

The reduced phase space action (2.5) is equivalent to that of a finite-dimensional mechanical system with a complicated Hamiltonian. We know, at least in principle, how to quantize such a system: we simply replace the Poisson brackets (2.4) with commutators,

$$[\hat{\tau}_\alpha, \hat{p}^\beta] = i\hbar \delta_\alpha^\beta, \quad (2.8)$$

represent the momenta as derivatives,

$$p^\alpha = \frac{\hbar}{i} \frac{\partial}{\partial \tau_\alpha}, \quad (2.9)$$

and impose the Schrödinger equation

$$i\hbar \frac{\partial \psi(\tau, k)}{\partial k} = \hat{H} \psi(\tau, k), \quad (2.10)$$

where the Hamiltonian $\hat{H}$ is obtained from Eq. (2.2) by some suitable operator ordering.

One fundamental problem is hidden in this last step: it is not at all obvious how one should define $\hat{H}$ as a self-adjoint operator on an appropriate Hilbert space. In particular, $\hat{\tau}_2$ and $\hat{p}$ do not commute, so the operator ordering in $\hat{H}$ is not unique. The simplest choice is that of Eq. (2.2), for which the Hamiltonian becomes

$$\hat{H} = \frac{\hbar}{\sqrt{k^2 - 4\Lambda}} \Delta_0^{1/2}, \quad (2.11)$$

where Δ_0 is the ordinary scalar Laplacian for the constant negative curvature moduli space characterized by the metric (2.3). Other orderings exist, but they are severely restricted by the requirement of diffeomorphism invariance: eigenfunctions of $\hat{H}$ should transform under a unitary representation of the modular group.[1] The representation theory of the modular group has been studied extensively [25–28]; if we restrict our attention to one-dimensional representations, the possible Hamiltonians are all of the form (2.11), but with Δ_0 replaced by[2]

$$\Delta_n = -\tau_2^2 \left(\frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right) + 2in\tau_2 \frac{\partial}{\partial \tau_1} + n(n+1),$$

$$2n \in \mathbf{Z}. \quad (2.12)$$

The operator Δ_n is the weight n Maass Laplacian, and the corresponding eigenfunctions, Maass forms of weight n, have been discussed in considerable detail in the mathematical literature [25–28]. Note that when written in terms of the momentum p of Eq. (2.9), the Δ_n differ from each other by terms of order $\hbar$, as expected for operator ordering ambiguities. Nevertheless, the choice of ordering can have drastic effects on the physics: the spectra of the various Maass Laplacians are very different.

This ambiguity can be viewed as a consequence of the structure of the classical phase space. The torus moduli space is not a manifold, but rather has orbifold singularities, and quantization on an orbifold is generally not unique. Since the space of solutions of the Einstein equations in $3+1$ dimensions has a similar orbifold structure [29], we might expect a similar ambiguity in realistic $(3+1)$-dimensional quantum gravity.

A potentially more serious ambiguity in this approach to quantization comes from the classical treatment of the time slicing. The choice of k as a time variable is rather arbitrary, and it is not at all clear that a different choice would lead to the same quantum theory. The danger of making a ''wrong'' choice is illustrated by the classical solution (3.2), (3.3) de-

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[1]This representation is usually assumed to be one dimensional, but it may be permissible to consider higher-dimensional representations and multicomponent wave functions.

[2]See [9] for details of the required operator orderings.

scribed below: another standard slicing uses $\sqrt{^{(2)}g}$ as time, but it is evident that when $\Lambda < 0$, $\sqrt{^{(2)}g}$ is not even a single-valued function of k.

A possible resolution of this problem is to treat the holonomy representation as fundamental. In this first-order ''frozen time'' approach, the basic observables give a time-independent description of the entire spacetime geometry. There is no Hamiltonian, no time development, and hence no need to choose a time slicing. If we can establish a relationship between the $(\hat{\tau},\hat{p})$ and suitable operators in the first-order formalism, we can convert the problem of time slicing into one of defining the appropriate physical operators. Different choices of slicing would then merely require different operators to represent moduli, and not different quantum theories.

The holonomy representation [10,16] starts with the Chern-Simons formulation of $(2+1)$-dimensional gravity [5,6], and chooses as fundamental variables the traces of the Chern-Simons holonomies around a set of noncontractible curves $\{\gamma_a\}$. For $\Lambda < 0$, the relevant gauge group is a product group $SL(2,\mathbf{R}) \otimes SL(2,\mathbf{R})$ coming from the decomposition of the spinor group of $SO(2,2)$ (the anti–de Sitter group), and one obtains two real, independent sets of traces $R_a^{\pm}$ [11,16].

For the torus, the algebra is simplest if we consider holonomies around three curves: two circumferences γ_1 and γ_2 and a third curve $\gamma_{12} = \gamma_1 \cdot \gamma_2$, where the dot represents composition of curves or multiplication of homotopy classes. The holonomies then satisfy the nonlinear Poisson bracket algebra

$$\{R_1^{\pm}, R_2^{\pm}\} = \mp \frac{1}{4\alpha}(R_{12}^{\pm} - R_1^{\pm}R_2^{\pm})$$

$$\text{and cyclic permutations.} \quad (2.13)$$

The six holonomies $R_{1,2,12}^{\pm}$ provide an overcomplete description of the spacetime geometry of $\mathbf{R} \times T^2$, which is completely characterized by two complex parameters τ and p. To remove this overcompleteness, consider the cubic polynomials

$$F^{\pm} = 1 - (R_1^{\pm})^2 - (R_2^{\pm})^2 - (R_{12}^{\pm})^2 + 2R_1^{\pm}R_2^{\pm}R_{12}^{\pm}. \quad (2.14)$$

These polynomials have vanishing Poisson brackets with all of the traces $R_a^{\pm}$, are cyclically symmetric in the $R_a^{\pm}$, and vanish classically by the $SL(2,\mathbf{R})$ Mandelstam identities; setting $F^{\pm} = 0$ removes the redundancy.

The Poisson algebra (2.13) and its generalization [12] to more complicated spatial topologies can be quantized for any value of the cosmological constant. For a generic topology, one obtains an abstract quantum algebra [11,15]. For genus 1 with $\Lambda < 0$, the quantum theory has been worked out quite explicitly.

There are, in fact, two closely related theories: one can either quantize the algebra and then determine a representation, or first choose a classical representation and then quantize. For the first choice, one replaces the classical Poisson brackets $\{,\}$ by commutators $[,]$,

$$\{x,y\} \to \frac{1}{i\hbar}[x,y], \quad (2.15)$$

and replaces products in Eq. (2.13) by symmetrized products,

$$xy \to \frac{1}{2}(xy + yx). \quad (2.16)$$

The resulting operator algebra is given by

$$\hat{R}_1^{\pm}\hat{R}_2^{\pm}e^{\pm i\theta} - \hat{R}_2^{\pm}\hat{R}_1^{\pm}e^{\mp i\theta} = \pm 2i \, \sin\theta \hat{R}_{12}^{\pm}$$

$$\text{and cyclic permutations} \quad (2.17)$$

with

$$\tan\theta = -\hbar/8\alpha. \quad (2.18)$$

The algebra (2.17) is not a Lie algebra, but it is related to the Lie algebra of the quantum group $SU(2)_q$ [13,16], where $q = \exp 4i\theta$, and where the cyclically invariant q-Casimir is the quantum analog of the cubic polynomial (2.14),

$$\hat{F}^{\pm}(\theta) = \cos^2\theta - e^{\pm 2i\theta}((\hat{R}_1^{\pm})^2 + (\hat{R}_{12}^{\pm})^2)$$

$$- e^{\mp 2i\theta}(\hat{R}_2^{\pm})^2 + 2e^{\pm i\theta}\cos\theta \hat{R}_1^{\pm}\hat{R}_2^{\pm}\hat{R}_{12}^{\pm}. \quad (2.19)$$

It may be checked that traces $\hat{R}_a$ satisfying Eq. (2.17) can be represented by [11,18,19]

$$\hat{R}_1^{\pm} = \sec\theta \, \cosh\frac{\hat{r}_1^{\pm}}{2}, \quad \hat{R}_2^{\pm} = \sec\theta \, \cosh\frac{\hat{r}_2^{\pm}}{2},$$

$$\hat{R}_{12}^{\pm} = \sec\theta \, \cosh\frac{(\hat{r}_1^{\pm} + \hat{r}_2^{\pm})}{2}, \quad (2.20)$$

where the operators $\hat{r}_1^{\pm}$, $\hat{r}_2^{\pm}$ have the commutators

$$[\hat{r}_1^{\pm}, \hat{r}_2^{\pm}] = \pm 8i\theta, \quad [\hat{r}_a^+, \hat{r}_b^-] = 0. \quad (2.21)$$

Alternatively, we could start with a classical representation of the holonomies $R_a^{\pm}$ analogous to the $\hbar \to 0$ limit of Eq. (2.20),

$$R_1^{\pm} = \cosh\frac{r_1^{\pm}}{2}, \quad R_2^{\pm} = \cosh\frac{r_2^{\pm}}{2}, \quad R_{12}^{\pm} = \cosh\frac{(r_1^{\pm} + r_2^{\pm})}{2}, \quad (2.22)$$

which will satisfy the algebra (2.13) provided the parameters $r_a^{\pm}$ satisfy

$$\{r_1^{\pm}, r_2^{\pm}\} = \mp 1/\alpha, \quad \{r_a^+, r_b^-\} = 0. \quad (2.23)$$

In this case the cubic polynomials (2.14) are identically zero. Quantization of Eq. (2.23) then gives

$$[\hat{r}_1^{\pm}, \hat{r}_2^{\pm}] = \mp i\hbar/\alpha. \quad (2.24)$$

From Eq. (2.18), we see that this expression differs from Eq. (2.21) by terms of order $\hbar^3$. For the rest of this paper we will consider only the commutators (2.24); the alternative quantization (2.21) can be obtained by a fairly simple rescaling.

In either approach, the modular group acts both classically and quantum mechanically on the holonomy parameters as

$$S:\hat{r}_1^\pm \rightarrow \hat{r}_2^\pm, \quad \hat{r}_2^\pm \rightarrow -\hat{r}_1^\pm,$$

$$T:\hat{r}_1^\pm \rightarrow \hat{r}_1^\pm + \hat{r}_2^\pm, \quad \hat{r}_2^\pm \rightarrow \hat{r}_2^\pm, \quad (2.25)$$

and satisfies

$$S^2 = -1, \quad (ST)^3 = 1, \quad (2.26)$$

as is appropriate for a spinor representation. The action leaves invariant the Poisson brackets (2.23) and the commutators (2.24).

It will later prove useful to have an explicit representation of the $\hat{r}_a^\pm$ as multiplicative and differential operators, analogous to the representation (2.9) in ADM quantization. An obvious choice is to take the $\hat{r}_2^\pm$ as our configuration space variables, and the $\hat{r}_1^\pm$ as momenta. To simplify future algebra, though, it is useful to pick instead a pair of linear combinations of the $\hat{r}_2^\pm$ to parametrize our configuration space. Let t denote the time coordinate in proper time gauge, related to the York time k by Eq. (3.4) below, and define

$$u = \left(\sin\frac{2t}{\alpha}\right)^{-1/2}(r_2^- e^{it/\alpha} + r_2^+ e^{-it/\alpha}),$$

$$\bar{u} = \left(\sin\frac{2t}{\alpha}\right)^{-1/2}(r_2^- e^{-it/\alpha} + r_2^+ e^{it/\alpha}). \quad (2.27)$$

From the point of view of the holonomy representation, in which the basic variables are time-independent, $\{u(t), \bar{u}(t)\}$ should simply be thought of as a useful one-parameter family of commuting operators. The variables u and $\bar{u}$ satisfy

$$\frac{du}{dt} = -\frac{1}{\alpha}\csc\frac{2t}{\alpha}\,\bar{u},$$

or equivalently $\quad \dfrac{du}{dk} = -\dfrac{\alpha}{4}\sin\dfrac{2t}{\alpha}\,\bar{u}. \quad (2.28)$

In the u representation, the operators $\hat{u}$ and $\hat{u}^\dagger$ will act by multiplication, while suitable linear combinations of the $\hat{r}_1^\pm$ will act by differentiation: from Eq. (2.24),

$$\hat{r}_1^- e^{it/\alpha} + \hat{r}_1^+ e^{-it/\alpha} = -\frac{2\hbar}{\alpha}\left(\sin\frac{2t}{\alpha}\right)^{1/2}\frac{\partial}{\partial u},$$

$$\hat{r}_1^- e^{-it/\alpha} + \hat{r}_1^+ e^{it/\alpha} = \frac{2\hbar}{\alpha}\left(\sin\frac{2t}{\alpha}\right)^{1/2}\frac{\partial}{\partial \bar{u}}. \quad (2.29)$$

III. RELATING REPRESENTATIONS

The ultimate goal of this paper is to relate the quantum theories that arise from the holonomy and ADM representations, in order to investigate the role of the modular group in each theory. To explore this issue, it is necessary to first understand the classical relationship between the two approaches. This requires that we refer back to the space of classical solutions of $(2+1)$-dimensional gravity. For spacetimes with the topology $\mathbf{R}\times T^2$ this space is, fortunately, completely understood.

In the ''proper time gauge'' $N=1$, $N^i=0$, the first-order field equations

$$R^{ab} = d\omega^{ab} - \omega^{ac}\wedge\omega_c{}^b = -\Lambda e^a\wedge e^b$$

$$R^a = de^a - \omega^{ab}\wedge e_b = 0 \quad (3.1)$$

are solved by

$$e^0 = dt$$

$$e^1 = \frac{\alpha}{2}[(r_1^+ - r_1^-)dy + (r_2^+ - r_2^-)dx]\sin\frac{t}{\alpha} \quad (3.2)$$

$$e^2 = \frac{\alpha}{2}[(r_1^+ + r_1^-)dy + (r_2^+ + r_2^-)dx]\cos\frac{t}{\alpha}$$

$$\omega^{12} = 0$$

$$\omega^{01} = -\frac{1}{2}[(r_1^+ - r_1^-)dy + (r_2^+ - r_2^-)dx]\cos\frac{t}{\alpha} \quad (3.3)$$

$$\omega^{02} = \frac{1}{2}[(r_1^+ + r_1^-)dy + (r_2^+ + r_2^-)dx]\sin\frac{t}{\alpha},$$

where x and y each have period 1. It is straightforward to check that the parameters $r_a^\pm$ in Eqs. (3.2), (3.3) are precisely the parameters (2.22) that determine the holonomies. The York time k for this metric is

$$k = -\frac{d}{dt}\ln\sqrt{^{(2)}g} = -\frac{2}{\alpha}\cot\frac{2t}{\alpha}, \quad (3.4)$$

which ranges monotonically from $-\infty$ to ∞ as t varies from 0 to $\pi\alpha/2$, so the slices of constant t are precisely the slices of constant k.

Now, recall that any metric on a constant k slice is diffeomorphic to one of the form (2.1), and that this form defines the ADM variable τ. The modulus can thus be read off from the expression (3.2) for the triad: it is

$$\tau = (r_2^- e^{it/\alpha} + r_2^+ e^{-it/\alpha})^{-1}(r_1^- e^{it/\alpha} + r_1^+ e^{-it/\alpha}). \quad (3.5)$$

The conjugate variable p can be similarly determined from the canonical momenta π^{ij}, which may be computed from Eq. (3.2); one finds that

$$p = \frac{i\alpha}{2}\csc\frac{2t}{\alpha}(r_2^+ e^{it/\alpha} + r_2^- e^{-it/\alpha})^2. \quad (3.6)$$

From Eqs. (3.5), (3.6), the Hamiltonian (2.2) that generates development in k is

$$H = \frac{\alpha}{2\sqrt{k^2 - 4\Lambda}} (r_1^- r_2^+ - r_1^+ r_2^-), \tag{3.7}$$

while from Eq. (3.4), development in coordinate time t is generated by

$$H' = \frac{dk}{dt} H = (k^2 - 4\Lambda) H = \csc\frac{2t}{\alpha} (r_1^- r_2^+ - r_1^+ r_2^-). \tag{3.8}$$

Equations (3.5)–(3.8) give us our desired relationship between the ADM and holonomy representations. Equivalently, in terms of the operators u and $\bar{u}$ defined in the preceding section, we have

$$\hat{\tau} = -\frac{2\hbar}{\alpha} u^{-1} \frac{\partial}{\partial u}, \quad \hat{\tau}^\dagger = \frac{2\hbar}{\alpha} \frac{\partial}{\partial \bar{u}} \bar{u}^{-1} \tag{3.9}$$

and

$$\hat{p} = \frac{i\alpha}{2} \bar{u}^2, \quad \hat{p}^\dagger = -\frac{i\alpha}{2} u^2, \tag{3.10}$$

whereas the Hamiltonians (3.7), (3.8) are

$$\hat{H} = \frac{i\alpha\hbar}{4} \sin\frac{2t}{\alpha} \left(\bar{u} \frac{\partial}{\partial u} + u \frac{\partial}{\partial \bar{u}} \right)$$

$$\hat{H}' = \frac{i\hbar}{\alpha} \csc\frac{2t}{\alpha} \left(\bar{u} \frac{\partial}{\partial u} + u \frac{\partial}{\partial \bar{u}} \right). \tag{3.11}$$

With these orderings, it may be checked that the modulus and momentum satisfy

$$[\hat{\tau}^\dagger, \hat{p}] = [\hat{\tau}, \hat{p}^\dagger] = 2i\hbar, \quad [\hat{\tau}, \hat{p}] = [\hat{\tau}^\dagger, \hat{p}^\dagger] = 0, \tag{3.12}$$

in agreement with Eq. (2.8), by virtue of the commutators (2.24) of the $\hat{r}_a^\pm$. Moreover, their time evolution is given by the standard Heisenberg equations of motion

$$[\hat{p}, \hat{H}'] = i\hbar \frac{d\hat{p}}{dt}, \quad [\hat{\tau}, \hat{H}'] = i\hbar \frac{d\hat{\tau}}{dt}, \tag{3.13}$$

or equivalently,

$$[\hat{u}, \hat{H}'] = i\hbar \frac{d\hat{u}}{dt}, \quad [\hat{\bar{u}}, \hat{H}'] = i\hbar \frac{d\hat{\bar{u}}}{dt}. \tag{3.14}$$

In effect, Eqs. (3.5), (3.6) can be viewed as the general four-parameter solution of the quantum mechanical Heisenberg equations of motion, with the $\hat{r}_a^\pm$ serving as (operator-valued) parameters. The action (2.25) of the classical modular group on the holonomy parameters induces, through Eqs. (3.5) and (3.6), the standard action (2.6) on the torus modulus and momentum, thus confirming consistency. The corresponding quantum action is discussed in the next section.

IV. MODULAR TRANSFORMATIONS IN THE HOLONOMY REPRESENTATION

We have seen that the modular group acts classically on the torus modulus, momentum, and holonomy parameters as

$$S: \tau \to -\tau^{-1}, \quad p \to \bar{\tau}^2 p, \quad r_1^\pm \to r_2^\pm, \quad r_2^\pm \to -r_1^\pm,$$

$$T: \tau \to \tau + 1, \quad p \to p, \quad r_1^\pm \to r_1^\pm + r_2^\pm, \quad r_2^\pm \to r_2^\pm, \tag{4.1}$$

and that the transformation (3.5), (3.6) between representations preserves this action. The goal of this section is to find operators that generate the quantum version of these transformations.

The simplest starting point is the holonomy representation. It is easily checked that the modular transformations of $\hat{r}_a^\pm$ are generated by conjugation with the unitary operators

$$\hat{T}^\pm = \exp\left\{ \pm \frac{i\alpha}{2\hbar} (\hat{r}_2^\pm)^2 \right\}, \tag{4.2}$$

$$\hat{S}^\pm = \exp\left\{ \pm \frac{i\pi\alpha}{4\hbar} [(\hat{r}_1^\pm)^2 + (\hat{r}_2^\pm)^2] \right\}. \tag{4.3}$$

(See the Appendix for a brief description of methods for demonstrating this and similar relations.) The first of these appeared in Ref. [11] in a different notation. The second was calculated independently by the two authors, and appeared in [30] and [31]. The operators $\hat{T}$ and $\hat{S}$ are related to a set of six constants of motion $C_i^\pm$, $i = 1,2,3$, calculated from the holonomies [30]. These global constants of motion were first calculated classically, for $\Lambda = 0$, in terms of the ADM modulus and momentum, in [32]. Explicitly,[3]

$$\hat{T}^\pm = \exp\left\{ \pm \frac{i\alpha}{2\hbar} C_2^\mp \right\}$$

$$\hat{S}^\pm = \exp\left\{ \pm \frac{i\pi\alpha}{4\hbar} (C_1^\mp + C_2^\mp) \right\}. \tag{4.4}$$

We next consider the induced action of $\hat{S}$ and $\hat{T}$ on the modulus and momentum, expressed in the operator ordering given by Eqs. (3.5) and (3.6). Note first that while the classical transformations (2.6) of τ translate easily into operator language, the S transformation of p involves potential ordering ambiguities. The ordering (3.5) that we are considering here corresponds to a transformation

$$S: \hat{p} \to \frac{\hat{\tau}^\dagger}{2} (\hat{\tau}^\dagger \hat{p} + \hat{p} \hat{\tau}^\dagger), \tag{4.5}$$

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[3] The remaining global constants $C_3^\pm = r_1^\mp r_2^\mp$ (see [30]) are related to $C_1^\pm$ and $C_2^\pm$ by $\{C_1^\pm, C_2^\pm\} = \pm(4/\alpha) C_3^\pm$. When quantized, they generate a scaling $r_1^\pm \to e^{-\epsilon} r_1^\pm, r_2^\pm \to e^\epsilon r_2^\pm$. The moduli and momenta scale as $\tau \to e^{-2\epsilon}\tau$, $p \to e^{2\epsilon}p$, leaving the commutators (2.24) and (3.12) and the Hamiltonian (3.7) invariant.

while the choice $\tau = (r_1^- e^{it/\alpha} + r_1^+ e^{-it/\alpha})(r_2^- e^{it/\alpha} + r_2^+ e^{-it/\alpha})^{-1}$, for example, would have led to

$$S: \hat{p} \to (\hat{\tau}^\dagger \hat{p} + \hat{p} \hat{\tau}^\dagger) \frac{\hat{\tau}^\dagger}{2},$$

the cases differing from each other and the classical limit by terms of order $\hbar$. For both orderings the commutators (3.12) are invariant and the identities (2.7) are satisfied.

Quantum mechanically, we can use Eq. (3.6) to reexpress $\hat{T}$ in terms of $\hat{p}$ and its adjoint. We obtain

$$\hat{T} = \hat{T}^+ \hat{T}^- = \exp\left\{ \frac{i}{2\hbar} (\hat{p} + \hat{p}^\dagger) \right\}. \qquad (4.6)$$

Using the commutators (3.12), we easily see that conjugation by this operator generates the transformation (4.1) of $\hat{\tau}$ and $\hat{p}$. The S transformation is more complicated, but the operator (4.3) can also be expressed in terms of the modulus and momentum: using Eqs. (3.5) and (3.6), we eventually obtain

$$\begin{aligned}\hat{S} = \hat{S}^+ \hat{S}^- &= \exp\left\{ \frac{i\pi}{8\hbar} [2(\hat{p}^\dagger + \hat{p}) + \hat{\tau}^\dagger (\hat{\tau}^\dagger \hat{p} + \hat{p} \hat{\tau}^\dagger) \right. \\ &\quad \left. + (\hat{\tau} \hat{p}^\dagger + \hat{p}^\dagger \hat{\tau}) \hat{\tau}] \right\} \\ &= \exp\left\{ \frac{i\pi}{4\hbar} [\hat{p}^\dagger + \hat{p} + (\hat{\tau}^\dagger)^2 \hat{p} + \hat{p}^\dagger (\hat{\tau})^2 \right. \\ &\quad \left. + i\hbar(\hat{\tau}^\dagger - \hat{\tau})] \right\}, \qquad (4.7)\end{aligned}$$

which differs from the classical expression [30] by terms of order $\hbar$. It can be shown, with some difficulty, that the operator (4.7) generates the desired transformations (4.1) for $\hat{\tau}$ and (4.5) for $\hat{p}$ (see Appendix).

Finally, note that in the u representation, the operators (4.6) and (4.7) become

$$\hat{T} = \exp\left\{ \frac{\alpha}{4\hbar} [\bar{u}^2 - u^2] \right\} \qquad (4.8)$$

and

$$\hat{S} = \exp\left\{ -\frac{\pi\alpha}{8\hbar} \left[\frac{4\hbar^2}{\alpha^2} \frac{\partial^2}{\partial u^2} - u^2 - \frac{4\hbar^2}{\alpha^2} \frac{\partial^2}{\partial \bar{u}^2} + \bar{u}^2 \right] \right\}. \qquad (4.9)$$

V. THE TRANSFORMATION BETWEEN REPRESENTATIONS

In Sec. II, we described two quantizations of $(2+1)$-dimensional gravity in the torus universe $\mathbf{R} \times T^2$. At first sight, the ADM representation looks like a standard ''Schrödinger picture'' quantum theory, with time-dependent states whose evolution is determined by a Hamiltonian operator. The holonomy representation is more mysterious, but it resembles a ''Heisenberg picture'' quantum theory, char-

acterized by time-independent states and time-dependent operators. This description suggests that there should be a unitary transformation between the two representations, which could help in the interpretation of both.

One way to construct such a transformation is to start in the Heisenberg picture and diagonalize the generalized position operators $\hat{q}_H(t)$ for all t—that is, to find a family of wave functions $K(x,t)$ such that for any given t, $K(x,t)$ is an eigenfunction of $\hat{q}_H(t)$ with eigenvalue x. (The suffixes H and S stand for ''Heisenberg'' and ''Schrödinger.'') Consider, for example, a free particle of mass m in one spatial dimension. The Heisenberg states are functions $\psi_H(x_0)$ of an initial position x_0, and the position operator is

$$\hat{q}_H(t) = \hat{q}_H(0) + \frac{t}{m} \hat{p}_H(0) = x_0 - \frac{i\hbar t}{m} \frac{\partial}{\partial x_0}. \qquad (5.1)$$

A simple computation shows that the eigenstates

$$\hat{q}_H(t) K(x,t|x_0) = x K(x,t|x_0) \qquad (5.2)$$

are

$$K(x,t|x_0) = \left(\frac{m}{2\hbar \pi t} \right)^{1/2} \exp\left\{ -\frac{im}{2\hbar t} (x - x_0)^2 \right\}. \qquad (5.3)$$

The exponent in Eq. (5.3) is determined by Eq. (5.2); the prefactor is fixed by the normalization requirement that

$$\int dx_0 K^*(x,t|x_0) K(x',t|x_0) = \delta(x - x'). \qquad (5.4)$$

It is now easily checked that the complex conjugate kernel $K^*(x,t|x_0)$ satisfies the free particle Schrödinger equation,

$$-\frac{\hbar^2}{2m} \frac{\partial^2 K^*}{\partial x^2} = i\hbar \frac{\partial K^*}{\partial t} \qquad (5.5)$$

and that a general Schrödinger wave function can be written as a superposition

$$\tilde{\psi}_S(x,t) = \int dx_0 K^*(x,t|x_0) \psi_H(x_0). \qquad (5.6)$$

Equation (5.6) implies that

$$\langle \phi_H | \hat{q}_H(t) | \psi_H \rangle = \int dx \phi_S^*(x,t) x \psi_S(x,t), \qquad (5.7)$$

as required for a transformation between representations. In fact, the kernel $K^*(x,t|x_0)$ is just the standard propagator for a free particle, and Eq. (5.6) is simply the time evolution of the state $\psi_H(x_0)$, considered as an initial state in the Schrödinger picture.

In $(2+1)$-dimensional quantum gravity, the analogous kernel can be obtained by diagonalizing the operators $\hat{\tau}_1$ and $\hat{\tau}_2$, or equivalently $\hat{\tau}$ and $\hat{\tau}^\dagger$. In the u representation of Eqs. (3.9), (3.10), we thus require that

$$\hat{\tau} K(\tau,\bar{\tau},t|u,\bar{u}) = -\frac{2\hbar}{\alpha} u^{-1} \frac{\partial}{\partial u} K(\tau,\bar{\tau},t|u,\bar{u})$$

$$= \tau K(\tau,\bar{\tau},t|u,\bar{u})$$

$$\hat{\tau}^{\dagger} K(\tau,\bar{\tau},t|u,\bar{u}) = \frac{2\hbar}{\alpha} \frac{\partial}{\partial \bar{u}} [\bar{u}^{-1} K(\tau,\bar{\tau},t|u,\bar{u})]$$

$$= \bar{\tau} K(\tau,\bar{\tau},t|u,\bar{u}), \qquad (5.8)$$

where τ and $\bar{\tau}$ are eigenvalues. It is easily checked that the solution is

$$K(\tau,\bar{\tau},t|u,\bar{u}) = \frac{\alpha \tau_2}{2\pi\hbar} \bar{u}(t)$$

$$\times \exp\left\{ -\frac{\alpha}{4\hbar} \tau u(t)^2 + \frac{\alpha}{4\hbar} \bar{\tau}\bar{u}(t)^2 \right\}.$$

$$(5.9)$$

The prefactor in Eq. (5.9) is again determined by normalization: we demand that

$$\int du_1 du_2 K^*(\tau,\bar{\tau},t|u,\bar{u}) K(\tau',\bar{\tau}',t|u,\bar{u})$$

$$= \tau_2^2 \delta(\tau_1 - \tau_1') \delta(\tau_2 - \tau_2'). \qquad (5.10)$$

[The integration measure $du_1 du_2$ is equal to $dr_2^+ dr_2^-$, with no additional Jacobian, so the integral (5.10) is compatible with our original choice of variables in Sec. II. The delta function on the right-hand side of Eq. (5.10) is the one appropriate for the Weil-Petersson metric (2.3) on Teichmüller space.]

By analogy with Eq. (5.6), our candidates for "Schrödinger picture" wave functions are therefore

$$\tilde{\psi}(\tau,\bar{\tau},t) = \int du_1 du_2 K^*(\tau,\bar{\tau},t|u,\bar{u}) \psi(u,\bar{u})$$

$$\psi(u,\bar{u}) = \int \frac{d^2\tau}{\tau_2^2} K(\tau,\bar{\tau},t|u,\bar{u}) \tilde{\psi}(\tau,\bar{\tau},t). \qquad (5.11)$$

These integrals are not yet well defined, however: we have not specified the region of integration, and as we saw in Sec. II, the proper choice of "Heisenberg picture" wave functions $\psi(u,\bar{u})$ requires a better understanding of the action of the modular group. In the next section, we will use Eq. (5.11) to define Heisenberg picture wave functions. For the moment, let us treat Eq. (5.11) as a formal expression.

Our ultimate goal is to understand the modular transformations of $\psi(u,\bar{u})$. An obvious starting point is to investigate the actions of the operators $\hat{S}$ and $\hat{T}$ of the preceding section on $K(\tau,\bar{\tau},t|u,\bar{u})$. In the u representation, $\hat{T}$ acts by multiplication, and it is easy to see from Eqs. (4.8) and (5.9) that

$$\hat{T} K(\tau,\bar{\tau},t|u,\bar{u}) = K(\tau+1,\bar{\tau}+1,t|u,\bar{u}). \qquad (5.12)$$

The T transformations thus act in the standard way on the modulus τ. The action of $\hat{S}$ is rather more complicated to work out, since from Eq. (4.9), $\hat{S}$ is now a differential operator. It is safest to work with real variables $r_2^{\pm}$ and their conjugates, or for simplicity with rescaled variables

$$x = \sqrt{\frac{\alpha}{\hbar}} r_2^+, \qquad -i\frac{\partial}{\partial x} = \sqrt{\frac{\alpha}{\hbar}} r_1^+$$

$$y = \sqrt{\frac{\alpha}{\hbar}} r_2^-, \qquad i\frac{\partial}{\partial y} = \sqrt{\frac{\alpha}{\hbar}} r_1^-, \qquad (5.13)$$

in terms of which, from Eq. (4.9),

$$\hat{S} = \exp\left\{ \frac{\pi i}{4} \left[-\frac{\partial^2}{\partial x^2} + x^2 + \frac{\partial^2}{\partial y^2} - y^2 \right] \right\}. \qquad (5.14)$$

The action of this operator can be studied by means of a simple trick. Note first that

$$e^{ikx} = \sqrt{2\pi} \sum_{n=0}^{\infty} i^n \psi_n(k) \psi_n(x), \qquad (5.15)$$

where the ψ_n are normalized harmonic oscillator wave functions [33]. These wave functions are eigenfunctions of the differential operator in the exponent in Eq. (5.14),

$$\left(\frac{\partial^2}{\partial x^2} - x^2 \right) \psi_n(x) = -(2n+1)\psi_n(x), \qquad (5.16)$$

and thus

$$\hat{S} e^{ikx} = e^{\pi i/4} \sqrt{2\pi} \sum_{n=0}^{\infty} (-1)^n \psi_n(k) \psi_n(x)$$

$$= e^{\pi i/4} \sqrt{2\pi} \sum_{n=0}^{\infty} \psi_n(k) \psi_n(-x)$$

$$= e^{\pi i/4} \sqrt{2\pi} \delta(x+k). \qquad (5.17)$$

Similarly,

$$\hat{S} e^{ik'y} = e^{-\pi i/4} \sqrt{2\pi} \delta(y-k'). \qquad (5.18)$$

Now consider an arbitrary function $F(x,y)$ with a Fourier transform $\tilde{F}(k,k')$:

$$F(x,y) = \frac{1}{2\pi} \int dk dk' e^{ikx} e^{ik'y} \tilde{F}(k,k')$$

$$\tilde{F}(u,v) = \frac{1}{2\pi} \int da db e^{-iau} e^{-ibv} F(a,b). \qquad (5.19)$$

Equations (5.17), (5.18) then imply that

$$(\hat{S}F)(x,y) = \tilde{F}(-x,y) = \frac{1}{2\pi} \int da db e^{iax} e^{-iby} F(a,b). \qquad (5.20)$$

The operator $\hat{S}$ thus acts by Fourier transformation. In retrospect this is perhaps not surprising: by Eq. (2.25), $\hat{S}$ interchanges the observables $r_1^\pm$ with their conjugates $r_2^\pm$, thus acting as a transformation from "position space" to "momentum space."

We can now apply Eq. (5.20) to our kernel $K(\tau,\bar{\tau},t|u,\bar{u})$. A straightforward calculation shows that

$$\hat{S}K(\tau,\bar{\tau},t|u,\bar{u})=-\left(\frac{\tau}{\bar{\tau}}\right)^{1/2}K\left(-\frac{1}{\tau},-\frac{1}{\bar{\tau}},t\bigg|u,\bar{u}\right).$$
(5.21)

Were it not for the phase on the right-hand side, this would be exactly what we would expect from the standard action (2.6) of S on moduli space. The phase makes the transformation "covariant" rather than "invariant." This phase is characteristic of modular forms of weight $-1/2$, which can be viewed as spinors on moduli space [25–28]. Such modular forms have appeared in previous work on $(2+1)$-dimensional gravity with $\Lambda=0$ [8,9], although with a different representation of $\hat{S}$ and $\hat{T}$.

A similar computation shows that the complex conjugate kernel $K^*(\tau,\bar{\tau},t|u,\bar{u})$ transforms as

$$\hat{S}K^*(\tau,\bar{\tau},t|u,\bar{u})=-\left(\frac{\bar{\tau}}{\tau}\right)^{1/2}K^*\left(-\frac{1}{\tau},-\frac{1}{\bar{\tau}},t\bigg|u,\bar{u}\right),$$

$$\hat{T}K^*(\tau,\bar{\tau},t|u,\bar{u})=K^*(\tau-1,\bar{\tau}-1,t|u,\bar{u}),$$
(5.22)

characteristic of a modular form of weight $1/2$. The covariant Laplacian $\Delta_{1/2}$ for modular forms of weight $1/2$ is the Maass Laplacian (2.12) [25–28],

$$\Delta_{1/2}=-\tau_2^2\left(\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial^2}{\partial\tau_2^2}\right)+i\tau_2\frac{\partial}{\partial\tau_1}+\frac{3}{4}.$$
(5.23)

As noted in Sec. II, this operator differs from the ordinary Laplacian Δ_0 by terms of order $\hbar$, and can thus be viewed as a different operator ordering of the standard Laplacian. A straightforward computation now shows that

$$\left(\frac{i\alpha}{2}\sin\frac{2t}{\alpha}\frac{\partial}{\partial t}\right)^2 K^*(\tau,\bar{\tau},t|u,\bar{u})=(\Delta_{1/2}-1)K^*(\tau,\bar{\tau},t|u,\bar{u}).$$
(5.24)

Up to a constant order $\hbar$ correction, this is the square of the reduced phase space Schrödinger equation (2.10) with an operator ordering appropriate for a form of weight $1/2$, and it serves as a check that our kernel $K^*(\tau,\bar{\tau},t|u,\bar{u})$ behaves as it ought to. In particular, Eq. (5.24) implies that the "Schrödinger picture" wave functions of Eq. (5.11) will satisfy a similar Klein-Gordon-like equation.

It is also interesting to consider the action of the "Heisenberg picture" Hamiltonian (3.8) on $K^*(\tau,\bar{\tau},t|u,\bar{u})$. From Eq. (3.11), we see that

$$\hat{H}'K^*(\tau,\bar{\tau},t|u,\bar{u})=-i\hbar\frac{\partial}{\partial t}K^*(\tau,\bar{\tau},t|u,\bar{u}).$$
(5.25)

Equations (5.24) and (5.25) imply that, in some sense, $\hat{H}'\sim(\Delta_{1/2})^{1/2}$, i.e., that the first-order holonomy-based quantum theory is a "square root" of the second-order ADM theory. A similar phenomenon was noted earlier in the theory with $\Lambda=0$, although with different variables [8]. Whether this relation can be made more rigorous remains an open question. The basic problem is that the square root of a Laplacian is highly nonunique: it can be defined mode by mode in a spectral decomposition, but the sign of the square root can be chosen arbitrarily for each mode. It is not clear which, if any, of this infinite number of square roots should be associated with $\hat{H}'$.

VI. MODULAR TRANSFORMATIONS OF HOLONOMY WAVE FUNCTIONS

We are now ready to use the results of the preceding section to analyze the behavior of the "Heisenberg picture" wave function $\psi(u,\bar{u})$ under modular transformations. Our strategy will be to use the well-understood properties of the "Schrödinger picture" wave function $\tilde{\psi}(\tau,\bar{\tau},t)$, along with the transformation (5.11) between representations.

We begin with the second equation in (5.11),

$$\psi(u,\bar{u})=\int_\mathcal{F}\frac{d^2\tau}{\tau_2^2}K(\tau,\bar{\tau},t|u,\bar{u})\tilde{\psi}(\tau,\bar{\tau},t).$$
(6.1)

Since $K(\tau,\bar{\tau},t|u,\bar{u})$ is, roughly speaking, a modular form of weight $-1/2$, as implied by Eq. (5.21), we might expect $\tilde{\psi}(\tau,\bar{\tau},t)$ to be a form of weight $1/2$, that is, a function invariant under the transformations

$$\hat{S}\tilde{\psi}(\tau,\bar{\tau},t)=-\left(\frac{\bar{\tau}}{\tau}\right)^{1/2}\tilde{\psi}\left(-\frac{1}{\tau},-\frac{1}{\bar{\tau}},t\right),$$

$$\hat{T}\tilde{\psi}(\tau,\bar{\tau},t)=\tilde{\psi}(\tau+1,\bar{\tau}+1,t).$$
(6.2)

We shall see below that this is indeed the case.

Our first task is to determine the range of integration $\mathcal{F}$ in Eq. (6.1). This can be fixed by the requirement that $\psi(u,\bar{u})$ be properly normalized:

$$\int d^2u |\psi(u,\bar{u})|^2$$

$$= \int d^2u \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \int_{\mathcal{F}} \frac{d^2\tau'}{\tau_2'^2} K(\tau,\bar{\tau},t|u,\bar{u})$$

$$\times K^*(\tau',\bar{\tau}',t|u,\bar{u}) \tilde{\psi}(\tau,\bar{\tau},t) \tilde{\psi}^*(\tau',\bar{\tau}',t)$$

$$= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \int_{\mathcal{F}} \frac{d^2\tau'}{\tau_2'^2} \tau_2'^2 \delta^2(\tau-\tau') \tilde{\psi}(\tau,\bar{\tau},t) \tilde{\psi}^*(\tau',\bar{\tau}',t)$$

$$= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} |\tilde{\psi}(\tau,\bar{\tau})|^2, \qquad (6.3)$$

where we have used the orthonormality relation (5.10). But we understand the normalization of ''Schrödinger picture'' wave functions $\tilde{\psi}(\tau,\bar{\tau},t)$: the right-hand side of Eq. (6.3) will be unity when $\mathcal{F}$ is a fundamental region for the action (2.6) of the modular group on Teichmüller space.

With this choice of integration region, we take Eq. (6.1) as the *definition* of $\psi(u,\bar{u})$. Let us now consider the action of the operators $\hat{S}$ and $\hat{T}$ of Sec. IV on this wave function. From Eq. (5.22), it is easy to see that

$$\hat{T}\psi(u,\bar{u}) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} (\hat{T}K)(\tau,\bar{\tau},t|u,\bar{u}) \tilde{\psi}(\tau,\bar{\tau},t)$$

$$= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} K(\tau+1,\bar{\tau}+1,t|u,\bar{u}) \tilde{\psi}(\tau,\bar{\tau},t)$$

$$= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} K(\tau+1,\bar{\tau}+1,t|u,\bar{u}) \tilde{\psi}(\tau+1,\bar{\tau}+1,t),$$

$$(6.4)$$

where the invariance of $\tilde{\psi}$ under the transformations (6.2) has been used in the last line. Changing integration variables to $\tau+1$ and $\bar{\tau}+1$, we see that

$$\hat{T}\psi(u,\bar{u}) = \int_{T^{-1}\mathcal{F}} \frac{d^2\tau}{\tau_2^2} K(\tau,\bar{\tau},t|u,\bar{u}) \tilde{\psi}(\tau,\bar{\tau},t), \quad (6.5)$$

where $T^{-1}\mathcal{F}$ is the new fundamental region obtained from $\mathcal{F}$ by a T^{-1} transformation. A similar argument shows that

$$\hat{S}\psi(u,\bar{u}) = \int_{S^{-1}\mathcal{F}} \frac{d^2\tau}{\tau_2^2} K(\tau,\bar{\tau},t|u,\bar{u}) \tilde{\psi}(\tau,\bar{\tau},t), \quad (6.6)$$

provided $\tilde{\psi}(\tau,\bar{\tau},t)$ is a modular form of weight 1/2, invariant under the transformations (6.2). [The extra phase factor in Eq. (6.2) is needed to cancel the phase in the transformation (5.21), as anticipated.]

Now, the kernel $K(\tau,\bar{\tau},t|u,\bar{u})$ is *not* modular invariant, and the shift of integration region in Eqs. (6.5) and (6.6)

matters: the wave function $\psi(u,\bar{u})$ is not invariant under the action of the mapping class group.[4] Indeed, there is a sense in which $\psi(u,\bar{u})$ and (for example) $\hat{T}\psi(u,\bar{u})$ differ maximally—they are, in fact, orthogonal. To see this, we can repeat the calculation of Eq. (6.3); from the orthonormality of $K(\tau,\bar{\tau},t|u,\bar{u})$, we now obtain

$$\langle \psi | \hat{T}\psi \rangle = \int_{T^{-1}\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \int_{\mathcal{F}} \frac{d^2\tau'}{\tau_2'^2} \tau_2'^2 \delta^2(\tau-\tau')$$

$$\times \tilde{\psi}(\tau,\bar{\tau},t) \tilde{\psi}^*(\tau',\bar{\tau}',t). \qquad (6.7)$$

But the regions $\mathcal{F}$ and $T^{-1}\mathcal{F}$ are disjoint except on a set of measure zero, so the delta function in Eq. (6.7) is identically zero.

A similar argument shows that if g is any nontrivial modular transformation, then

$$\langle \psi | \hat{g}\psi \rangle = 0. \qquad (6.8)$$

In fact, this conclusion can be strengthened. Let $\psi_1(u,\bar{u})$ and $\psi_2(u,\bar{u})$ be two different wave functions defined by integrals of the form (6.1) over the same fundamental region $\mathcal{F}$. Repeating the computation of Eq. (6.7), we now see that

$$\langle \psi_1 | \hat{g}\psi_2 \rangle = 0 \qquad (6.9)$$

for any nontrivial modular transformation g.

In accord with the results of Refs. [21–23], our ''Heisenberg picture'' wave functions are not modular invariant. But the ''maximal noninvariance'' of Eq. (6.9) is almost as good. Pick a fundamental region $\mathcal{F}$, and consider the set of wave functions defined by Eq. (6.1). These will form a subspace $\mathcal{H}_{\mathcal{F}}$ of the Hilbert space $\mathcal{H}$ of square-integrable functions of (u_1,u_2) or (r_2^+, r_2^-). A modular transformation g maps this subspace into an orthogonal subspace $\mathcal{H}_{g^{-1}\mathcal{F}}$, which is obtained by integrals of the form (6.1) over the translated fundamental region $g^{-1}\mathcal{F}$. In fact, the modular group splits the space $\mathcal{H}$ into an infinite set of orthogonal subspaces.

These subspaces are physically equivalent. Indeed, let $\hat{\mathcal{O}}$ be an arbitrary modular invariant operator on $\mathcal{H}$. There is no reason to expect the orthogonal subspaces to be superselected—that is, if $\psi \in \mathcal{H}_{\mathcal{F}}$, it need not be the case that $\hat{\mathcal{O}}\psi \in \mathcal{H}_{\mathcal{F}}$—so if we wish to restrict ourselves to a single subspace, we must appropriately project $\hat{\mathcal{O}}$ to that subspace. Let $\hat{P}_{\mathcal{F}}$ denote the standard Hilbert space projector onto the

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[4]This fact was first pointed out to one of the authors (S.C.) by Jorma Louko.

subspace $\mathcal{H}_{\mathcal{F}}$, and define $\hat{\mathcal{O}}_{\mathcal{F}} = \hat{P}_{\mathcal{F}}\hat{\mathcal{O}}\hat{P}_{\mathcal{F}}$. Then if $\psi_1, \psi_2 \in \mathcal{H}_{\mathcal{F}}$, it follows that

$$
\begin{aligned}
\langle \psi_1 | \hat{\mathcal{O}}_{\mathcal{F}} | \psi_2 \rangle &= \langle \psi_1 | \hat{P}_{\mathcal{F}} \hat{\mathcal{O}} \hat{P}_{\mathcal{F}} | \psi_2 \rangle \\
&= \langle \psi_1 | \hat{g}^{-1} \hat{P}_{g\mathcal{F}} \hat{g} \hat{\mathcal{O}} \hat{g}^{-1} \hat{P}_{g\mathcal{F}} \hat{g} | \psi_2 \rangle \\
&= \langle g\psi_1 | \hat{P}_{g\mathcal{F}} \hat{\mathcal{O}} \hat{P}_{g\mathcal{F}} | g\psi_2 \rangle \\
&= \langle g\psi_1 | \hat{\mathcal{O}}_{g\mathcal{F}} | g\psi_2 \rangle,
\end{aligned}
\tag{6.10}
$$

where we have used the modular invariance of $\hat{\mathcal{O}}$ and the fact that $\hat{g}^{-1}\hat{P}_{g\mathcal{F}}\hat{g} = \hat{P}_{\mathcal{F}}$. Matrix elements can thus be computed in any of the subspaces $\mathcal{H}_{g\mathcal{F}}$, and the appropriate restrictions of modular invariant operators will give the same physics.

The modulus $\hat{\tau}$, of course, is not an invariant operator, and its matrix elements will depend on the choice of subspace. But this is not surprising, since the same is true classically. One can build invariant operators from $\hat{\tau}$, whose matrix elements satisfy Eq. (6.10). One example is the operator version of the modular function $J(\tau)$ of Dedekind and Klein [27],

$$
J(\tau) = \frac{(60G_4(\tau))^3}{(60G_4(\tau))^3 - 27(140G_6(\tau))^2},
\tag{6.11}
$$

where the $G_{2k}(\tau)$ are Eisenstein series,

$$
G_{2k}(\tau) = \sum_{m,n \in \mathbf{Z}}' \frac{1}{(m+n\tau)^{2k}}.
\tag{6.12}
$$

(The prime means that the value $m = n = 0$ is excluded from the sum.) It may be shown that any meromorphic modular function is a rational function of $J(\tau)$. Such functions are certainly less familiar than trigonometric functions, but in principle they are no more extraordinary. Since $J(\tau)$ depends only on the modulus and not the momentum, there are no ordering ambiguities, and Eq. (6.11) may be taken to be an operator expression.

What we have discovered is a ''quantum mechanical fundamental region'' for the modular group. As several authors have pointed out [21–23], the modular group does not act nicely (that is, properly discontinuously) on the configuration space of the first-order formalism. But we now see that the modular group *does* act nicely on the corresponding Hilbert space, which is all that is required for a sensible quantum theory.

VII. CONCLUSION

The phase space of $(2+1)$-dimensional gravity with $\Lambda < 0$ on a manifold $\mathbf{R} \times \Sigma$ has two natural descriptions: as the cotangent bundle of the Teichmüller space of Σ, and as a space of $SL(2,\mathbf{R}) \otimes SL(2,\mathbf{R})$ holonomies. Classically, the

two descriptions are equivalent,[5] and can be viewed as different coordinate choices for a single space. On the *phase space*, the mapping class group has a properly discontinuous action in either set of coordinates, and invariant functions are well defined.

To quantize such a phase space, however, one must choose a polarization, that is, a distinction between ''positions'' and ''momenta.'' Therein lies the root of the problem discussed in Refs. [21–23]. While the mapping class group acts nicely on the phase space, there is no guarantee that it does so on the configuration space, and hence no assurance that one can find invariant wave functions. When Σ is a torus, this is precisely what goes wrong: the modular group fails to act properly discontinuously on a ''configuration space'' of holonomies, and the definition of invariant wave functions becomes highly problematic.

One could, of course, evade this issue by choosing a different polarization [34]. But the polarization for which the problems arise is a natural one, and it seems implausible that a perfectly good choice of classical coordinates should lead to such disastrous consequences for the quantum theory.

In this paper, we have solved this problem. By constructing the exact transformation between the ADM and holonomy states, we have shown that the modular group *does* have a nice, albeit unexpected, action on the holonomy states. There are, indeed, no invariant wave functions in the holonomy representation. Instead, the modular group acts on the Hilbert space in much the same way that it acts on Teichmüller space—it splits the Hilbert space into physically equivalent orthonormal ''fundamental regions,'' each one of which is equivalent to the Hilbert space that arises from ADM quantization. In the course of our argument, we have also derived a collection of explicit operator representations of the torus mapping class group.

The splitting of the Hilbert space described in Sec. VI relies on the transformation (5.11) between representations, and thus refers back to the ADM quantum theory. It would be desirable to have a description that depended only on the intrinsic properties of the Hilbert space in the holonomy representation. We do not yet have such a description, but we see no reason why one should not exist.

APPENDIX

The generators (4.2), (4.3) and (4.6), (4.7) of modular transformations act by conjugation, and to compute their action, one must evaluate expressions of the form

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[5]Strictly speaking, one must restrict the holonomies to ensure that the metric is nonsingular and that the slices Σ are spacelike [21].

$$e^A B e^{-A} = B + [A,B] + \frac{[A,[A,B]]}{2!} + \cdots . \qquad (A1)$$

In this appendix, we briefly describe two ways to evaluate such expressions, one based on explicit summation and a second based on a trick that converts the problem to one of solving differential equations.

As an example of the explicit calculation, write the generator $\hat{S}$ of Eq. (4.7) as

$$\hat{S} = \exp\left\{ \frac{\pi i}{8}(\hat{a} + \hat{a}^\dagger) \right\}, \qquad (A2)$$

where

$$\hbar \hat{a} = 2\hat{p}^\dagger + \hat{\tau}\hat{p}^\dagger\hat{\tau} + \hat{p}^\dagger\hat{\tau}^2,$$

$$\hbar \hat{a}^\dagger = 2\hat{p} + \hat{\tau}^\dagger\hat{p}\hat{\tau}^\dagger + (\hat{\tau}^\dagger)^2\hat{p} \qquad (A3)$$

and

$$[\hat{a}, \hat{a}^\dagger] = 0. \qquad (A4)$$

To use Eq. (A1) to evaluate the transformation of $\hat{p}^\dagger$, for example, one must compute the multiple commutators

$$[A,[A,[A, \ldots \ldots [A,\hat{p}^\dagger] \ldots]]] \qquad (A5)$$

where $A = -(\pi i/8)\hat{a}$. Note first that by Eq. (3.12),

$$[\hat{a}, \hat{\tau}] = 4i(1 + \hat{\tau}^2), \quad [\hat{a}, \hat{p}^\dagger] = -8i\hat{p}^\dagger\hat{\tau} - 4\hbar. \qquad (A6)$$

Direct computation shows that the odd commutators are all proportional:

$$[A,[A,[A, \ldots \ldots [A,\hat{p}^\dagger] \ldots]]]_{2n+1} = (-\pi^2)^n [A,\hat{p}^\dagger]. \qquad (A7)$$

Similarly, the even commutators can be computed to be

$$[A,[A,[A, \ldots \ldots [A,\hat{p}^\dagger] \ldots]]]_{2n}$$
$$= (-\pi^2)^{n-1}[A,[A,\hat{p}^\dagger]]$$
$$= \frac{(-1)^{n-1}\pi^{2n}}{4}(\hbar\hat{a} - 4\hat{p}^\dagger). \qquad (A8)$$

It follows that the sum of the odd commutators in Eq. (A1) is

$$[A,\hat{p}^\dagger]\sum_{n=0}^{\infty} \frac{(-\pi^2)^n}{(2n+1)!} = \frac{1}{\pi}[A,\hat{p}^\dagger]\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{\pi}[A,\hat{p}^\dagger]\sin \pi = 0, \qquad (A9)$$

whereas the sum of the even commutators, starting from $[A,[A,\hat{p}^\dagger]]$, is

$$\left(\frac{\hbar\hat{a}}{4} - \hat{p}^\dagger\right)\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\pi^{2n}}{(2n)!}$$

$$= \left(\frac{\hbar\hat{a}}{4} - \hat{p}^\dagger\right)(1 - \cos \pi) = \frac{\hbar\hat{a}}{2} - 2\hat{p}^\dagger. \qquad (A10)$$

Thus from Eq. (A1),

$$e^A\hat{p}^\dagger e^{-A} = \hat{p}^\dagger + \left(\frac{\hbar\hat{a}}{2} - 2\hat{p}^\dagger\right) = \frac{1}{2}(\hat{p}^\dagger\hat{\tau}^2 + \hat{\tau}\hat{p}^\dagger\hat{\tau}), \qquad (A11)$$

in agreement with Eq. (4.5), as required.

We next give an alternative method for calculating the S transformations of $\hat{\tau}$ and $\hat{p}$. Let

$$F(s) = e^{-is\hat{a}}\hat{\tau}e^{is\hat{a}}, \quad G(s) = e^{-is\hat{a}}\hat{p}^\dagger e^{is\hat{a}}. \qquad (A12)$$

By Eq. (A2), the transformed values of $\hat{\tau}$ and $\hat{p}^\dagger$ are simply $F(\pi/8)$ and $G(\pi/8)$. But by differentiating $F(s)$ and $G(s)$ with respect to s and using the commutators (A6), we can reduce the problem to one of solving a pair of differential equations,

$$\frac{dF}{ds} = 4(1 + F^2), \quad \frac{dG}{ds} = -8GF + 4i\hbar. \qquad (A13)$$

The first equation in (A13) has the solution

$$F(s) = \tan 4(s - s_0), \qquad (A14)$$

with initial conditions

$$F(0) = -\tan 4s_0 = \hat{\tau}. \qquad (A15)$$

Hence

$$F(\pi/8) = \cot 4s_0 = -\hat{\tau}^{-1}, \qquad (A16)$$

yielding the correct transformation (4.1) for $\hat{\tau}$. To calculate the corresponding transformation of $\hat{p}^\dagger$, observe that by Eq. (A13),

$$\frac{d}{ds}[G(1 + F^2)] = -8GF(1 + F^2) + 4i\hbar(1 + F^2) + 2GF\frac{dF}{ds}$$

$$= 4i\hbar(1 + F^2) = i\hbar\frac{dF}{ds}, \qquad (A17)$$

and thus

$$G(s)(1 + F(s)^2) - G(0)(1 + F(0)^2) = i\hbar(F(s) - F(0)). \qquad (A18)$$

Setting $s = \pi/8$ and using Eq. (A16), we find that

$$G(\pi/8) = \hat{p}^\dagger\hat{\tau}^2 - i\hbar\hat{\tau} = \frac{1}{2}(\hat{p}^\dagger\hat{\tau}^2 + \hat{\tau}\hat{p}^\dagger\hat{\tau}), \qquad (A19)$$

recovering Eq. (4.5).

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