Stability issues in Euclidean quantum gravity

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It is known that the action of Euclidean Einstein gravity is not bounded from below and that the metric of flat space does not correspond to a minimum of the action. Nevertheless, perturbation theory about flat space works well. The deep dynamical reasons for this reside in the nonperturbative behavior of the system and have been clarified in part by numerical simulations. Several open issues remain. We treat in particular those zero modes of the action for which $R(x)$ is not identically zero, but the integral of $\sqrt{g(x)}R(x)$ vanishes. [S0556-2821(98)00724-3]

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I. INTRODUCTION

The Euclidean (or ''imaginary time'') formulation of quantum field theory offers several aesthetic and substantial advantages. The practical rules for perturbative computations are simpler than in the Lorentzian case and there is no distinction between upper and lower indices. The only relevant Green function of the linearized equations is the Feynman propagator, and there is no need of formal regularization through the $i\varepsilon$ term. From the non-perturbative point of view, if the Euclidean action S is positive definite, then the functional integral is formally convergent thanks to the exponential of $-S/\hbar$.

When time-independent quantities are computed in the Euclidean theory, the inverse analytical continuation to real time is not necessary. A well known example are the formulas for the static potential [1]. We recall them in some detail in Sec. II.

The physical correspondence between an Euclidean functional integral and a statistical system at the temperature $\Theta = \hbar/k_B$ is immediate. We can also easily visualize the dynamics of the system, after suitable discretization, as a Monte Carlo evolution: starting from a given field configuration, a new configuration is generated through a small random variation; then the system evolves to the new configuration with probability 1 if its action is smaller, or else with probability $\exp(-\Delta S/\hbar)$, and so on.

When the ''bare'' parameters of the action are changed, the ground state of the system, corresponding to the minimum of the action, changes too (''phases'' of the theory). It is possible to insert first some bare parameters into the action, then follow the evolution of the system towards its ground state, and here measure the effective average value of the same parameters. The effective coupling constant, for instance, is usually extracted in this way from the measured potential U.

An interesting application of this method is the quantum Regge calculus by Hamber and Williams [2]. In this case, the physical system under investigation is very peculiar: the Euclidean spacetime, represented by a simplicial manifold and numerically coded in terms of edge lengths and defect angles (see also Sec. V below). In order to obtain from this discretized theory a continuum limit, independent of the details of the discretization procedure, one looks in the parameters space of the system for a second-order transition point, where the range of field correlations diverges.

Does this Euclidean model of the gravitational field constitute a faithful representation of real spacetime, with its complex causal structure distorted by field fluctuations? This question is still unanswered. While we know for sure that for certain curved manifolds the analytical continuation to Euclidean signature is not valid [3], there are no theorems that do allow this continuation in some special case. All we can do is hope that for weak fluctuations with respect to flat space, the Wick rotation of real time to the imaginary axis still makes sense.

Particle physicists do not doubt that the Euclidean Einstein action for weak fields represents a massless spin 2 field correctly and in an unique way. This point of view about gravitation, at variance with the geometrodynamical view of spacetime, has been supported, as is well known, by Feynman, Weinberg and others (see for instance the review by Alvarez [4]). No problems have ever been encountered—apart from the familiar nonrenormalizability of Einstein action—in Euclidean perturbation theory around flat space. The Euclidean formula for the potential works well, too [5].

Nevertheless, a serious problem still affects Euclidean quantum gravity, even in the weak field sector: the nonpositivity of the action. Either if one takes the geometrical point of view (''the Euclidean action is not at a minimum for $R=0$ . . . '') or the particle-physicist point of view (''the quadratic part of the action has undefined sign . . . ''), one ends up in the unpleasant situation of studying perturbatively a system around a configuration that does not appear to be a minimum for the action, but rather a saddle point. The feeling is to control only one part of the dynamics of the system, while the other part—which makes the weak field approximation work—remains elusive.

The nonperturbative Euclidean quantum Regge calculus based upon Einstein action can be helpful under this respect. It represents a geometrical model whose dynamics is entirely under control, at least numerically. So one can use it to throw

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some light on the paradoxes of the continuum Einstein action.

The short-distance regime of Euclidean quantum gravity is to some extent arbitrary, and a strict connection with real spacetime is quite unlikely in that limit. (We recall that due to the dimensional structure of the gravitational action, field fluctuations are stronger at short distances.) In the large distance limit, however, the connection is much more plausible and thus it can be interesting to see how flat space emerges and keeps stable in the Euclidean Regge calculus.

The plan of the paper is the following. In Sec. II we discuss the general formulas which relate, in Euclidean field theory, the static potential of two sources to the vacuum correlations of the field. This also gives us the chance to introduce some basic notations. In Sec. III we deal with the nonpositivity of the Euclidean action, and give explicit examples in the weak field approximation. In Sec. IV we present a novel issue: the zero modes of the integral of the scalar curvature. In Sec. V we give an interpretation of the numerical results of Regge calculus in view of the stability problem. We stress the importance of the sign of the effective cosmological term, which acts as a volume term. In Sec. VI we check the geometrical argument of Sec. V in the continuum, doing a stability analysis of anti–de Sitter space.

II. STATIC POTENTIAL AND EUCLIDEAN VACUUM CORRELATIONS

In Euclidean quantum field theory there is a simple connection between the static potential associated to a bosonic field and the vacuum correlations of the field. This allows signs to be fixed without any ambiguity, which is often crucial in stability issues.

Consider a system comprising a quantum field and an external source J, and denote by $W[J]=\langle 0^+|0^-\rangle_J$ the vacuum-to-vacuum transition amplitude in the presence of the source. The energy of the ground state of the system is given by

$$E_0=-\frac{\hbar}{T}\log W[J], \tag{2.1}$$

where T is the temporal range of the source, which eventually tends to infinity. To check this, let us insert a complete set of energy eigenstates $\{|n\rangle\}$ into the amplitude $W[J]$:

$$\begin{aligned}\langle 0^+|0^-\rangle_J&=\langle 0|e^{-HT/\hbar}|0\rangle\\&=\sum_n \langle 0|e^{-HT/\hbar}|n\rangle\langle n|0\rangle\\&=\sum_n |\langle 0|n\rangle|^2 e^{-E_nT/\hbar}.\end{aligned} \tag{2.2}$$

The smallest eigenvalue among the E_n's corresponds to the ground state, and in the limit $T\to\infty$ its exponential dominates the sum. Thus taking the logarithm of $W[J]$ and multiplying by $(-\hbar/T)$ we obtain the eigenvalue itself. One often considers pointlike sources kept at rest at $\mathbf{x}_1 \ldots \mathbf{x}_N$, namely

$$J(x)=\sum_{j=1}^{N} q_j\,\delta^3(\mathbf{x}-\mathbf{x}_j). \tag{2.3}$$

In this case the ground state energy corresponds, up to a possible additive constant, to the static potential U of the interaction of the sources, and depends on $\mathbf{x}_1 \ldots \mathbf{x}_N$. We have

$$U(\mathbf{x}_1 \ldots \mathbf{x}_N)=-\frac{\hbar}{T}\log\langle\exp(-S_J)\rangle_0, \tag{2.4}$$

where S_J is the term in the action containing the coupling to J, and the average $\langle\,\rangle_0$ is computed through the functional integral, weighing the field configurations with the factor $\exp(-S_0/\hbar)$. For a scalar field ϕ, S_J takes the form

$$S_J=\int d^4x\, J(x)\phi(x) \tag{2.5}$$

and $J(x)$ is as in Eq. (2.3). For a gauge field A_μ, the source term is

$$S_J=\int d^4x\, J_\mu(x)A_\mu(x); \tag{2.6}$$

$$J_\mu(x)=\sum_{j=1}^{N} q_j\,\delta_{0\mu}\delta^3(\mathbf{x}-\mathbf{x}_j). \tag{2.7}$$

Therefore S_J reduces to a sum of one-dimensional integrals computed along temporal lines. When the pointlike sources are only two and the field A_μ vanishes at infinity, $U(\mathbf{x}_1,\mathbf{x}_2)$ can be also expressed in terms of a Wilson loop. For gravity the source term is

$$S_T=\frac{1}{2}\int d^4x\,\sqrt{g(x)}T_{\mu\nu}(x)h_{\mu\nu}(x); \tag{2.8}$$

$$T_{\mu\nu}(x)=\sum_{j=1}^{N} m_j\,\delta_{\mu 0}\,\delta_{\nu 0}\,\delta^3(\mathbf{x}-\mathbf{x}_j). \tag{2.9}$$

The leading order contribution to Eq. (2.4) in the case of two pointlike sources is given in general by an expression of the form

$$U(\mathbf{x}_1,\mathbf{x}_2)=-\frac{\hbar}{T}q_1q_2 \times\int_{-T/2}^{T/2}dt_1\int_{-T/2}^{T/2}dt_2\langle\Phi(t_1,\mathbf{x}_1)\Phi(t_2,\mathbf{x}_2)\rangle, \tag{2.10}$$

where $\langle\,\rangle$ denotes the free propagator and Φ corresponds to the field ϕ in the scalar case and to the components A_0 and h_{00} in the electromagnetic and gravitational cases, respectively (in the latter case, q_1 and q_2 are replaced by the masses m_1 and m_2 of the sources).

The sign of the correlation $\langle\Phi(t_1,\mathbf{x}_1)\Phi(t_2,\mathbf{x}_2)\rangle$ is directly related to that of the potential energy. Some care is needed in order to pick the correct convention for the Euclidean metric. Equations (2.1), (2.2) hold for the Euclidean metric with signature $(-1,-1,-1,-1)$, which is directly connected to the standard Minkowski metric $(1,-1,-1,-1)$ by the transformation $x_0 \leftrightarrow i x_0$. If the Euclidean metric (1,1,1,1) is used, Eq. (2.10) holds with the + sign. This metric is usually preferred and will be employed in the following.

In the scalar and electromagnetic case, the correlation is positive. One finds (apart from positive numerical factors and with $c \equiv 1$)

$$\langle \phi(x_1)\phi(x_2)\rangle \sim \langle A_0(x_1)A_0(x_2)\rangle \sim \frac{\hbar^{-1}}{(x_1-x_2)^2}; \tag{2.11}$$

thus the potential is repulsive if q_1 and q_2 have the same sign, since

$$\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 \frac{1}{(t_1-t_2)^2+(\mathbf{x}_1-\mathbf{x}_2)^2} \sim \frac{T}{|\mathbf{x}_1-\mathbf{x}_2|}. \tag{2.12}$$

In the gravitational case, the correlation is negative:

$$\langle h_{00}(x_1)h_{00}(x_2)\rangle \sim -\frac{\hbar^{-1}G}{(x_1-x_2)^2}; \tag{2.13}$$

m_1 and m_2 being always positive for physical sources, it follows that the potential is always attractive. The negative sign of the correlation (2.13) may look counterintuitive. One should never forget, however, that quantum fields are distributions, and the analogy between a quantum functional integral and classical fields at finite temperature has only a limited validity.

The positivity of the scalar action and of the electromagnetic action in Feynman gauge are evident in the Euclidean theory: one has namely in momentum space

$$S_\phi \sim \int d^4p\, p^2\, \tilde{\phi}^*(p)\tilde{\phi}(p), \tag{2.14}$$

$$S_A \sim \int d^4p\, p^2\, \delta_{\mu\nu}\tilde{A}^*_\mu(p)\tilde{A}_\nu(p) \tag{2.15}$$

and for the propagators

$$\tilde{G}_\phi(p) \sim p^{-2}; \quad \tilde{G}_{A,\mu\nu}(p) \sim \delta_{\mu\nu} p^{-2}. \tag{2.16}$$

[We recall that, still apart from positive numerical factors, $\int d^4p\, e^{ipx} p^{-2} \sim x^{-2}$. Compare Eq. (2.11).]

III. DIFFERENT ASPECTS OF THE SAME PROBLEM: THE ACTION DOES NOT HAVE A MINIMUM

The Hilbert-Einstein action for the gravitational field $g_{\mu\nu}(x)$ is usually written in the form

$$S = -\frac{1}{8\pi G}\int d^4x \sqrt{g(x)} R(x), \tag{3.1}$$

where $R(x)$ is the scalar curvature.

Naively one can observe already at this stage that since R is a quantity which can be positive as well as negative and contains the first and second derivatives of the metric, the integrand does not have a definite sign and can grow in both directions if $g_{\mu\nu}(x)$ varies strongly.

Hawking showed formally several years ago [6] that the Euclidean action is not bounded from below [7]. His argument is important also because it does not make any reference to the weak field approximation. Several possible solutions to the unboundedness problem were proposed later on [8].

Wetterich suggested recently [9] a nonlocal modification of the effective Euclidean action and showed that the phenomenological implications of such a modified action are almost entirely compatible with cosmology. Without entering into this matter, we just quote here his decomposition of the tensor of the metric fluctuations $h_{\mu\nu}(x) = g_{\mu\nu}(x) - \delta_{\mu\nu}(x)$ in terms of irreducible representations of the Euclidean group in d dimensions:

$$h_{\mu\nu}(x) = b_{\mu\nu}(x) + \partial_\mu a_\nu(x) + \partial_\nu a_\mu(x) + \left(\partial_\mu\partial_\nu - \frac{1}{d}\delta_{\mu\nu}\partial^2\right)\chi(x) + \frac{1}{d}\delta_{\mu\nu}\sigma(x), \tag{3.2}$$

where the tensors $b_{\mu\nu}(x)$ and $a_\mu(x)$ satisfy the conditions

$$\partial_\mu a_\mu(x) = 0, \quad \partial_\mu b_{\mu\nu}(x) = 0, \quad \delta_{\mu\nu} b_{\mu\nu}(x) = 0. \tag{3.3}$$

To second order in $h_{\mu\nu}$ one obtains

$$\sqrt{g(x)}R(x) = \frac{1}{4}\partial_\rho b_{\mu\nu}(x)\partial_\rho b_{\mu\nu}(x) - \frac{(d-1)(d-2)}{4d^2} \times \partial_\mu[\sigma(x) - \partial^2\chi(x)]\partial_\mu[\sigma(x) - \partial^2\chi(x)].$$

Therefore for $d>2$ the action becomes negative semidefinite for configurations in which $b_{\mu\nu}(x)$ is zero. It can be shown that the addition of a gauge-fixing term does not change the situation. Wetterich also observes that even though it is possible to make the action positive definite adding short-distance terms (like the R^2 term), the effective action, relevant for large distances, will always keep nonpositive.

In certain cases it can be reasonable to introduce in the theory a cutoff on the momenta, and this will make the scalar curvature bounded. Still, the quadratic part of the action will not be positive-definite. This unpleasant feature does not only concern the small distances sector. It can be exhibited most clearly in harmonic gauge. In this gauge the quadratic part of the Hilbert-Einstein Lagrangian in momentum space is simply given by

$$\tilde{L}^{(2)}(p) \sim -p^2 \tilde{h}^*_{\mu\nu}(p) V_{\mu\nu\alpha\beta}\tilde{h}_{\alpha\beta}(p), \tag{3.4}$$

where $V_{\mu\nu\alpha\beta}$ is a constant tensor which in particular in dimension 4 is equal to

$$V_{\mu\nu\alpha\beta} = \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta}. \tag{3.5}$$

Let us rearrange the 10 independent components of the tensor $\tilde{h}$, to build an array $\tilde{h}_i$ $(i=0,1,\ldots,9)$, as follows: $\tilde{h}_{00}\to\tilde{h}_0,\ldots,\tilde{h}_{33}\to\tilde{h}_3,\tilde{h}_{01}\to\tilde{h}_4,\ldots,\tilde{h}_{23}\to\tilde{h}_9$. We then have

$$\tilde{L}^{(2)}(p)\sim -p^2\tilde{h}_i^*(p)M_{ij}\tilde{h}_j(p), \tag{3.6}$$

where **M** is a block matrix of the form

$$\mathbf{M}=\begin{bmatrix}\mathbf{m}(4\times 4) & \mathbf{0}(4\times 6)\\ \mathbf{0}(6\times 4) & \mathbf{1}(6\times 6)\end{bmatrix} \tag{3.7}$$

and

$$\mathbf{m}=\begin{bmatrix}1 & -1 & -1 & -1\\ -1 & 1 & -1 & -1\\ -1 & -1 & 1 & -1\\ -1 & -1 & -1 & 1\end{bmatrix}. \tag{3.8}$$

One easily checks that $\mathbf{m}^2=4\times\mathbf{1}$, thus the propagator of $\tilde{h}_i$ is given by

$$\tilde{G}_h(p)\sim -p^{-2}\mathbf{M}^{-1}=-p^{-2}\begin{bmatrix}\frac{1}{4}\mathbf{m} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}. \tag{3.9}$$

As anticipated in Sec. II, we see here that the correlation function of h_{00}, like the other ''diagonal'' correlations, is negative.

In order to check that the quadratic part of the gravitational action is not positive-definite, we can also rewrite $\tilde{L}^{(2)}(p)$ in matrix form as

$$\tilde{L}^{(2)}(p)\sim -p^2\{2\,\mathrm{Tr}[\tilde{h}^*(p)\tilde{h}(p)]-|\mathrm{Tr}\,\tilde{h}(p)|^2\}. \tag{3.10}$$

Denoting now by $\tilde{h}_A(p)$ $(A=0,1,2,3)$ the eigenvalues of the symmetric matrix $[\tilde{h}_{\mu\nu}(p)]$, we have

$$\tilde{L}^{(2)}(p)\sim -p^2\left[\sum_A |\tilde{h}_A(p)|^2-\sum_{A\neq B}\tilde{h}_A^*(p)\tilde{h}_B(p)\right], \tag{3.11}$$

or

$$\tilde{L}^{(2)}(p)\sim -p^2\tilde{h}_A^*(p)m_{AB}\tilde{h}_B(p). \tag{3.12}$$

The eigenvalues of **m** are found to be $(2,2,-2,-2)$. Thus the quadratic form (3.12) has no definite sign.

IV. THE ''ZERO MODES'' IN THE INTEGRAL OF *R*

It is known that if the metric has a Lorentzian signature, then Einstein equations in vacuum admit wavelike solutions. The Riemann tensor $R^{\mu}_{\nu\rho\sigma}$ propagates in these solutions, while the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R are identically zero.

We recall that the Einstein equations in the presence of a source $T_{\mu\nu}$ are (with $c\equiv 1$)

$$R_{\mu\nu}(x)-\frac{1}{2}g_{\mu\nu}(x)R(x)=-8\pi G T_{\mu\nu}(x), \tag{4.1}$$

and their trace is

$$R(x)=8\pi G\,\mathrm{Tr}\,T(x). \tag{4.2}$$

From Eq. (4.2) we see that if $T_{\mu\nu}=0$, then $R=0$, as mentioned; therefore gravitational waves are a set of ''zero modes'' of the Hilbert-Einstein *Lagrangian* [10].

There is, however, another peculiar way to obtain zero modes of the gravitational *action*. This is due to the nonpositivity of this action.

Let us consider a solution $g_{\mu\nu}$ of Eq. (4.1) with a (covariantly conserved) source $T_{\mu\nu}$ obeying the additional integral condition

$$\int d^4x\sqrt{g(x)}\,\mathrm{Tr}\,T(x)=0. \tag{4.3}$$

Taking into account Eq. (4.2) we see that the action (3.1) computed for this solution is zero. Condition (4.3) can be satisfied by energy-momentum tensors that are not identically zero, provided they have a balance of negative and positive signs, such that their total integral is zero. Of course, they do not represent any acceptable physical source, but the corresponding solutions of Eq. (4.1) exist nonetheless, and are zero modes of the action.

As an example of an unphysical source which satisfies Eq. (4.3) one can consider the static field produced by a ''mass dipole.'' Certainly negative masses do not exist in nature; here we are interested just in the formal solution of Eq. (4.1) with a suitable $T_{\mu\nu}$, because for this solution we have $\int d^4x\sqrt{g}R=0$. Let us take the following $T_{\mu\nu}$ of a static dipole centered at the origin $(m,m'>0)$:

$$T_{\mu\nu}(\mathbf{x})=\delta_{\mu 0}\delta_{\nu 0}[mf(\mathbf{x}+\mathbf{a})-m'f(\mathbf{x}-\mathbf{a})]. \tag{4.4}$$

Here $f(\mathbf{x})$ is a smooth test function centered at $\mathbf{x}=0$, rapidly decreasing and normalized to 1, which represents the mass density. The range of *f*, say r_0, is such that $a\gg r_0\gg r_{Schw}$, where r_{Schw} is the Schwartzschild radius corresponding to the mass *m*. The mass m' is in general different from *m* and chosen in such a way to compensate a possible small difference, due to the $\sqrt{g}$ factor, between the integrals

$$I^+=\int d^4x\sqrt{g(x)}f(\mathbf{x}+\mathbf{a})$$

and

$$I^-=\int d^4x\sqrt{g(x)}f(\mathbf{x}-\mathbf{a}). \tag{4.5}$$

The procedure for the construction of the zero mode corresponding to the dipole is the following. One first considers Einstein equations with the source (4.4). Then one solves them with a suitable method, for instance in the weak field approximation. Finally, knowing $\sqrt{g(x)}$ one computes the two integrals (4.5) and adjusts the parameter m' in such a way that $(mI^+ - m'I^-)=0$.

Let us implement this procedure to first order. Inside a single mass distribution $mf(\mathbf{x})$, with radius r_0 such that $r_0 \gg r_{Schw}$, the gravitational field satisfies a static equation whose linear approximation is of the form

$$Dh(\mathbf{x}) = m\kappa f(\mathbf{x}), \tag{4.6}$$

where D is a linear partial differential operator and κ denotes, for brevity, $8\pi G$. Let us call $\hat{h}(\mathbf{x})$ the solution of Eq. (4.6) with $m\kappa$ replaced by 1. The solution of the linearized Einstein equations with the source (4.4) is, in the region with positive density, $h^+(\mathbf{x}) = m\kappa \hat{h}(\mathbf{x}+\mathbf{a})$. In the region with negative density the solution is $h^-(\mathbf{x}) = -m'\kappa \hat{h}(\mathbf{x}-\mathbf{a}/2)$. Thus in the region with positive density we have

$$\sqrt{g(x)} \sim 1 + \frac{1}{2} m\kappa \,\mathrm{Tr}\, \hat{h}(\mathbf{x}+\mathbf{a}) \tag{4.7}$$

and in the region with negative density

$$\sqrt{g(x)} \sim 1 - \frac{1}{2} m'\kappa \,\mathrm{Tr}\, \hat{h}(\mathbf{x}-\mathbf{a}). \tag{4.8}$$

The value of the action functional corresponding to this linearized ''virtual dipole'' metric is

$$\begin{aligned} -\frac{1}{\kappa}\int d^4x \sqrt{g(x)} R(x) &= -\int d^4x \sqrt{g(x)}\, \mathrm{Tr}\, T(x) \\ &= -\int d^4x \sqrt{g(x)} [mf(\mathbf{x}+\mathbf{a}) - m'f(\mathbf{x}-\mathbf{a})] \\ &= -\int d^4x \left\{ \left[1+\frac{1}{2}m\kappa\,\mathrm{Tr}\,\hat{h}(\mathbf{x}+\mathbf{a})\right] mf(\mathbf{x}+\mathbf{a}) - \left[1-\frac{1}{2}m'\kappa\,\mathrm{Tr}\,\hat{h}(\mathbf{x}-\mathbf{a})\right] m'f(\mathbf{x}-\mathbf{a}) \right\} \\ &= -\int d^4x [mf(\mathbf{x}+\mathbf{a}) - m'f(\mathbf{x}-\mathbf{a})] + -\frac{1}{2}\kappa \int d^4x [m^2\,\mathrm{Tr}\,\hat{h}(\mathbf{x}+\mathbf{a}) f(\mathbf{x}+\mathbf{a}) \\ &\quad + m'^2\,\mathrm{Tr}\,\hat{h}(\mathbf{x}-\mathbf{a}) f(\mathbf{x}-\mathbf{a})]. \end{aligned} \tag{4.9}$$

Being f normalized, the first integral of Eq. (4.9) gives $-T(m-m')$, where T is the temporal integration range. We then have

$$-\frac{1}{\kappa}\int d^4x \sqrt{g(x)} R(x) = -T(m-m') - \frac{1}{2}\kappa T(m^2+m'^2) \times \int d^3x\, \mathrm{Tr}\, \hat{h}(\mathbf{x}) f(\mathbf{x}). \tag{4.10}$$

The integral on the right-hand side (RHS) is a number of the order of 1 and will be denoted by η. The condition for a zero mode now reads

$$(m-m') + \frac{1}{2}\eta\kappa(m^2+m'^2) = 0 \tag{4.11}$$

and it is satisfied, up to terms of order κ^2, for $m' = m(1 + \eta\kappa m)$.

There is no obstacle, in the functional integral, to the formation of a zero mode like this. It can ''pop up'' at any point in spacetime, or more likely it can be induced by an external localized source, even if weak. The spatial size of the mode can be in principle arbitrarily large.

These modes can develop both in Lorentzian and in Euclidean metric. Sometimes it is argued that in the functional integral with real time and with the oscillating factor $\exp(iS/\hbar)$ the nonpositivity of the action has no importance. But also in that case the zero modes described above can be present.

The only mechanism able to suppress these modes appears to be the presence of an effective volume term with $\Lambda < 0$ (see Sec. V).

V. HOW FLAT SPACE EMERGES FROM THE QUANTUM REGGE LATTICE

It can be helpful to recall briefly here the main features of the quantum Regge calculus technique by Hamber and Williams [2]. In this approach the Euclidean 4D spacetime is approximated by a simplicial manifold and the curvature, all concentrated at the ''hinges,'' is proportional to the defect angle which one finds when a hinge is flattened out. The system is numerically simulated, with the edge lengths as fundamental variables. At the beginning one puts into the action, as ''bare'' parameters, k (inverse of the Newton constant) and a (coefficient of the R^2 term). Then one looks in the phase diagram of the theory for a second-order transition point.

It turns out that the phase diagram is divided in two regions: a ''smooth'' phase, with average curvature small and negative, and fractal dimension close to 4; and a ''rough'' phase, singular, collapsed, with average curvature large and positive and small fractal dimension. It is clear that the sign of the curvature plays a crucial role in the stability of the system. The two phases are separated by a transition line. Approaching this line from the smooth phase, the average curvature $\langle R\rangle$ tends to zero. In this way, flat space is obtained in a dynamical way from the smooth phase, without any need of introducing into the theory a flat background by hand. From here, perturbation theory can somehow start.

A few data can help to complete the picture. The lattice sites are $16\times16\times16\times16=65\,536$, with 1 572 864 simplices. The edge lengths are updated by a straightforward Monte Carlo algorithm. Eventually an ensemble of configurations is generated, distributed according to the Euclidean action. The topology is fixed as a four-torus with periodic boundary conditions. A stable, well behaved ground state is found for $k<k_c\sim0.060$. The system resides in the smooth phase, with fractal dimension four. Six values of k have been investigated: 0.00, 0.01, 0.02, 0.03, 0.04, 0.05.

The static potential is attractive and can be fitted by L^{-1}, with a small Yukawa factor $\exp(-mL)$. The mass extracted this way is consistent with the exponential decay of the correlations of the scalar curvature. The effective Newton constant can be extimated to $G\sim0.14$, in lattice units. More exactly, at the beginning one puts $\lambda=1$ in the action. Then one finds for the average edge length

$$l_0=\sqrt{\langle l^2\rangle}=2.36, \quad \text{i.e.,} \quad l_0=2.36\lambda^{-1/4}. \tag{5.1}$$

The critical value of the bare coupling is

$$k_c\sim0.060, \quad \text{i.e.,} \quad k_c \;\; 0.060\lambda^{1/2} \tag{5.2}$$

and the product Gk_c is independent of l_0 and finite, as hoped. The precision of these data is expected to improve considerably in the next months, thanks to the new dedicated supercomputer AENEAS [11].

As far as stability is concerned, the numerical simulations show as mentioned that in the smooth phase the system evolves toward a stable minimum position with $\langle R\rangle<0$. It is not hard to understand intuitively, from the geometrical point of view, *why* the system is stable in this phase.

The effective action is

$$S_{eff}=\int d^4x\sqrt{g(x)}\left[\frac{\Lambda}{8\pi G}-\frac{R(x)}{8\pi G}\right], \tag{5.3}$$

with $\Lambda\sim\langle R\rangle$. Let us assume that the system is in a configuration with $R(x)=\text{const}=\Lambda$. Now suppose that somewhere a positive fluctuation of R appears. Being proportional to $\exp(-S_{eff})$, the probability of this new configuration is seemingly larger, if we take into account only the second term of the action. The first term, however (the effective cosmological term) can be written as $(\Lambda\mathcal{V})$, where $\mathcal{V}$ is the total volume of the system. This volume is maximum when the manifold is flat and all hinges are completely extended. As soon as a curvature fluctuation appears at some point the total volume decreases, and since Λ is negative, this tends to suppress the fluctuation. The converse happens, of course, in the collapsed phase, where $\Lambda>0$.

Furthermore, this same mechanism will suppress in an even stronger way the zero modes described in Sec. IV, since these modes cause no variation of the integral of R, but only decrease the volume.

The continuum theory is recovered from the lattice only at the transition line, where $\Lambda=0$. This refers, however, to an average computed over all space. More generally, Λ scales with the volume v of the averaged region, according to a power law of the form $|\Lambda|G\sim(v^{-1/4}l_0)^\gamma$.

VI. IS AdS SPACE A STABLE MINIMUM OF THE CONTINUUM ACTION?

As we have seen in the previous section, the discretized Euclidean action appears to be stabilized by a negative cosmological term. This acts as a volume term and opposes the curvature fluctuations, which tend to diminish the volume of the lattice.

Is the geometrical argument offered independent of the Regge lattice regularization? This question suggests a check in the continuum. If Eq. (5.3) really has a stable minimum, then that minimum must be a solution of the Euclidean Einstein equations with a negative cosmological constant. Such a solution is known; this is Euclidean anti–de Sitter (AdS) space.

Therefore, we can do a stability analysis: is AdS space a stable minimum of the action, with only positive modes in the weak-field expansion in this background? Or is it only a saddlepoint, like flat space?

A stability analysis in AdS space is more intricate than in flat space. We must expand the metric with respect to the appropriate background, namely

$$g_{\mu\nu}(x)=g^{AdS}_{\mu\nu}(x)+h_{\mu\nu}(x) \tag{6.1}$$

where $g^{AdS}_{\mu\nu}(x)$ is the solution of the vacuum Euclidean Einstein equations with a negative cosmological term and thus represents a space with constant negative curvature. The form of the metric $g^{AdS}_{\mu\nu}(x)$ depends on the coordinates chosen.

We can formally expand the action as

$$S[g]=S[g^{AdS}]+\frac{\delta S}{\delta g}[g^{AdS}]\times h+\frac{1}{2}\frac{\delta^2 S}{\delta g^2}[g^{AdS}]\times h^2+\cdots. \tag{6.2}$$

The first derivative vanishes at g^{AdS} and to check the stability we must study the sign of the quadratic form $U=(\delta^2S/\delta g^2)[g^{AdS}]$. Remembering that the first variation of S gives the Einstein equations, we obtain

$$U^{\alpha\beta\mu\nu}(x)=\left[\frac{\delta S}{\delta g_{\alpha\beta}}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R-\Lambda g^{\mu\nu}\right)\right]_{g=g^{AdS}(x)}. \tag{6.3}$$

The functional derivative of the first two terms corresponds, up to a gauge fixing, to the usual quadratic form of pure gravity, while the derivative of the cosmological term gives $-\Lambda g^{AdS,\mu\alpha}(x)g^{AdS,\nu\beta}(x)$ [12,13].

The second variation of S at the extremum must take into account the dependence of U on x:

$$\delta^2 S = \frac{1}{2}\int d^4x \sqrt{g^{AdS}(x)}\, h_{\alpha\beta}(x) U^{\alpha\beta\mu\nu}(x) h_{\mu\nu}(x). \tag{6.4}$$

Since the AdS space is homogeneous and we are most interested in localized fluctuations, we could restrict our attention to functions $h_{\mu\nu}(x)$ having support in a small region around the origin. In suitable coordinates we will have $g^{AdS}_{\mu\nu}(x) \sim g^{AdS}_{\mu\nu}(0) = \delta_{\mu\nu}$, but the gauge fixing term constitutes a serious problem, because it must be consistent with the symmetries of the background and with the fact that the fluctuations are localized (compare also Ref. [14], for the de Sitter case and related horizon and infrared problems).

In conclusion, investigating stability along these lines appears to be very hard.

Another possible check concerns the conformal mode. In this case, a weak field expansion is not necessary. For any conformal transformation of the metric of the form

$$g_{\mu\nu} \to g'_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \tag{6.5}$$

the curvature scalar transforms as

$$R(x) \to R'(x) = \Omega^{-2}(x) R(x) - 6\Omega^{-3}(x)\partial^2\Omega(x) \tag{6.6}$$

and the action with cosmological term transforms as (we omit the x dependence)

$$S \to S' = -\frac{1}{8\pi G}\int d^4x \sqrt{g}(\Omega^2 R + 6\partial_\mu\Omega\partial_\nu\Omega g^{\mu\nu} - \Lambda\Omega^4). \tag{6.7}$$

Note that $g_{\mu\nu}(x)$ does not need to be constant, and in our case coincides with the AdS metric.

We see from Eq. (6.7) that for $\Lambda < 0$ the conformal mode is *not* stabilized. On the contrary, it seems that conformal fluctuations can increase the total volume of space and are therefore enhanced. This continuum result is in bold contrast with our intuition of the behavior of the lattice. The contrast could be possibly explained as follows.

(i) It turns out from the numerical simulations that after the simplicial lattice has reached its ground state and has stabilized, the average length l_0 of the links (the ''bones'' of the triangulation) keeps constant and fluctuations are small.

This behavior could be due in part to the R^2 term or to the volumes in phase space, rather than to the Einstein R term; it signals, anyway, that an effective suppression of the conformal modes has occurred.

(ii) If the lengths of the lattice links are approximately constant, then any increase of the curvature implies an increase of defect angles and thus a diminution of the total volume, as argued in the previous section.

This is easily visualized in two dimensions. Let us consider, for instance, an orthogonal pyramid with a regular polygon as its basis (but here we are only interested in the side surface of the pyramid—the 2D volume). Suppose to keep constant the edges of the pyramid—the links. When the height of the pyramid goes to zero, the side surface is maximum and the defect angle δ, associated with the curvature, is zero. (The defect angle is that obtained by ''opening'' the pyramid, as if it were made of paper.) The sharper the pyramid, the larger $R \sim \delta$ and the smaller the side surface. The variation of the side surface is of the order of $\Delta\mathcal{S} \sim -(\Delta \sin\delta) l_0^2$.

The 4D analogue is $\Delta\mathcal{V} \sim -(\Delta \sin \delta) l_0^4$, while the contribution of the curvature to the Einstein action is of the order of $\Lambda(\Delta\delta) l_0^2$. Since in lattice units we have [compare Eq. (5.1)] $l_0 = \sqrt{\langle l^2 \rangle} > 1$ and $|\Lambda| > 1$, the lattice prefers to keep the δ's close to zero.

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[7] In the $R+R^2$ theory the action is positive definite, provided the coefficient a of the R^2 term is such that $4a\lambda - k^2 > 0$ (compare Sec. V). Remembering that the R^2 term also makes the theory renormalizable, it is easy to understand why the $R+R^2$ action has been chosen, instead of the simple Einstein action, for nonperturbative investigations through the Regge discretization. (The unitarity problem does not show up in the nonperturbative approach.) It was found, however, that the emergence in the quantum Regge calculus of a stable ground state with zero average curvature does not strictly require the R^2 term.

[8] See, for instance, J. Greensite, Phys. Lett. B **291**, 405 (1992), and references therein; J. Greensite, Nucl. Phys. **B390**, 439 (1993); E. Mazur and E. Mottola, *ibid.* **B341**, 187 (1990).

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[10] We note in passing that if the signature of the metric is Euclidean, then the equation $\partial^2 f = 0$ does not really admit wave-like solutions. Instead, it has the form of a 4D Poisson equation and its solutions are harmonic functions, which reduce to costants if there are no sources.

[11] The status of the project is described at the URL ⟨http://aeneas.ps.uci.edu/aeneas /index.html⟩.

[12] Setting $g^{AdS}_{\mu\nu}(x)=\text{const}=\delta_{\mu\nu}$ we would obtain the known ''naive'' statement that a cosmological term in weak field approximation amounts to a mass term for the graviton (see for instance [13]). It is clear, however, that this statement is not rigorously true, since in AdS space the metric g^{AdS} which minimizes the action is not flat.

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