Finite temperature nonlocal effective action for quantum fields in curved space

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Massless and massive scalar fields and massless spinor fields are considered at arbitrary temperatures in four dimensional ultrastatic curved spacetime. Scalar models under consideration can be either conformal or nonconformal and include self-interaction. The one-loop nonlocal effective action at finite temperature and free energy for these quantum fields are found up to the second order in background field strengths using the covariant perturbation theory. The resulting expressions are free of infrared divergences. Spectral representations for nonlocal terms of high temperature expansions are obtained. [S0556-2821(98)00524-4]

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I. INTRODUCTION

Finite temperature field theory has been developed in a series of seminal papers [1–3]. Nowadays it is an actively growing branch of theoretical physics [4]. Thermodynamical properties of thermal quantum fields in the presence of background fields are very important for a large number of applications in high energy physics, astrophysics, and cosmology. However most of these studies are devoted to the situation when background fields are constant (homogeneous) [5,6]. This particular form of the effective action, the effective potential [7,8], when large background fields are taken into account nonperturbatively, is useful for study of phase transitions in the early Universe or quark-gluon plasma. For a long time, the opposite situation, when background fields are small but rapidly fluctuating, lacked investigation even in zero temperature field theory. Traditional tools of quantum field theory, like the short proper time Schwinger-DeWitt expansion [9–11], are intrinsically local; hence, they miss nonlocal contributions. As a consequence of this deficiency artificial infrared divergences appear in the perturbative effective action for massless fields, and perturbation theory breaks down. Finite temperature effects also contribute to infrared divergences [4], and methods of diagram summations have been developed to improve the perturbation series [12,13].

To deal with massless field theories properly, such as gauge field theories or quantum gravity, Vilkovisky suggested a new powerful method [14] which is known as the covariant perturbation theory [15–18]. In these papers it was shown that infrared divergences are artificial and brought into existence by a mode of calculation rather than by a field theory. They disappear after summation of terms with infinite number of derivatives acting on background fields, which results in nonlocal terms entering the effective action [15]. Such a summation can only be performed in a given order in background field strengths.

Thermodynamics of an ensemble of quantum fields in equilibrium in static curved spacetimes is well defined, and most properties of the system can be derived from its free energy [19–21]. In this paper we consider ensembles of scalar and spinor fields in the presence of external ultrastatic gravitational field. The scalar models may have an arbitrary interaction potential and an arbitrary coupling to gravity. We employ the method of covariant perturbation theory to find the finite temperature effective action and the corresponding free energy of these quantum fields on highly inhomogeneous gravitational backgrounds. An example of the situation when finite temperature effects on curved background are important, and, thus, nonlocal effective action is needed, is the Hawking radiation by black holes [22,23].

The paper is organized as follows. In the next section we describe how to obtain nonlocal free energy at finite temperature with the help of the covariant perturbation theory. In Sec. III we derive the free energy of interacting massless scalar fields and study its high temperature behavior. Massive scalar fields at high temperatures are treated in Sec. IV. In Sec. V we derive free energy for massless spinor fields at finite and high temperatures. The conclusion and a discussion of possible applications and extensions of obtained results can be found in Sec. VI. We place the necessary complicated computations into Appendixes A and B.

II. ONE-LOOP EFFECTIVE ACTION AND FREE ENERGY OF QUANTUM FIELDS IN ULTRASTATIC SPACETIMES

Let us consider fields φ described by the classical action $S(\varphi)$ and the corresponding canonical Hamiltonian in a generic curved static spacetime. Statistical free energy F_S of the ensemble is defined as the trace of logarithm of eigenvalues of the normal-ordered Hamiltonian. In canonical quantization scheme, ultraviolet divergencies are tradition-

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ally subtracted from F_S by the normal ordering prescription [21,24]. On the other hand, in the imaginary time formalism of Matsubara [1] the problem of finding free energy of the system in equilibrium reduces to the computation of the path integral of the Euclidean quantum field theory and the corresponding effective action $W^\beta[g]$:

$$\mathrm{e}^{-W^\beta[g]}=\int D[\varphi]\,\mathrm{e}^{-S[\varphi,g]}. \tag{1}$$

The temperature $T=1/(k_B\beta)$ enters the calculation via the condition of (anti)periodicity in the Euclidean time τ imposed on quantum fields with (Fermi) bose statistics

$$\varphi(x,\tau)=\pm\varphi(x,\tau+\beta) \tag{2}$$

(the Boltzmann's constant $k_B=1$ everywhere).

The canonical free energy F_S and the thermal renormalized Euclidean effective action W^β are closely related to each other and differ [19,24] only by terms $\tilde{F}$ that are independent of temperature

$$\frac{1}{\beta}W^\beta[\phi]=F_S^\beta[\phi]+\tilde{F}[\phi], \tag{3}$$

where $\phi=\langle\varphi\rangle$ are mean fields. The effective action is usually regularized using covariant methods, e.g., zeta function [25,26], dimensional [10], etc., while the canonical free energy is regularized via normal ordering of operators. The difference $\tilde{F}[\phi]$ is related to different ways of taking into account vacuum energy contributions in covariant and canonical regularization schemes. The covariant approach is more appropriate to our problems since it is consistent with calculations of the stress tensor and vacuum polarization effects in external fields. In any case, it is easy to compute $\tilde{F}(\phi)$ which is temperature independent and local [24]. Henceforth, we restrict our consideration to calculation of the one-loop Euclidean effective action W^β and the corresponding covariant Euclidean free energy

$$F^\beta\equiv\frac{1}{\beta}W^\beta. \tag{4}$$

We calculate the free energy of quantum fields on static background fields which include mean field ϕ and static gravitational field. The Tolman temperature of such a field system in equilibrium is not constant throughout the static space. It is more convenient to perform calculations of temperature effects in the Euclidean ultrastatic (optical) spacetimes,

$$\mathrm{d}s^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=\mathrm{d}\tau^2+g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{5}$$

where local temperature is constant throughout the space. Ultrastatic and static spacetimes are related to each other by a conformal transformation of the metric. Conformal properties of the effective action have been studied in detail [27], and applied to free energy calculations by Dowker and Schofield [28,29]. Using scaling properties of the finite temperature zeta functions it was shown that the difference of free energies in two conformally related spaces does not depend on temperature. Then, all temperature dependent terms can be found from the free energy in an ultrastatic space, where solution of the problem simplifies significantly. This difference can be found by integrating the conformal anomaly, but the method of Ref. [28] works for generic nonconformal operators as well.

The brief outline of our research program is to calculate free energy in an ultrastatic space, and then using a relation between free energies in static and ultrastatic spaces to express the final result in terms of quantities defined in a physical (static) spacetime. In this paper we implement the first and most complicated step of obtaining W^β and F^β on the ultrastatic metric (5).

Let us consider quantum n-component scalar field $\varphi\equiv\varphi_A$, $A=1,\ldots,n$, which satisfies the equation

$$\left[\left(\Box-\frac{1}{6}R\right)\hat{1}+\hat{P}(\phi)\right]\varphi=0. \tag{6}$$

Our notations correspond to those of Refs. [10,16]: the Laplacian $\Box=g^{\mu\nu}\nabla_\mu\nabla_\nu$ is constructed of covariant derivatives which are characterized by the commutator curvature

$$(\nabla_\mu\nabla_\nu-\nabla_\nu\nabla_\mu)\varphi=\hat{\mathcal{R}}_{\mu\nu}\varphi. \tag{7}$$

This quantity is, of course, zero for scalar fields, but we will need it in Sec. IV where spinor fields are considered. The potential $\hat{P}$ may depend on the metric and mean field ϕ, which is a part of classical background. Thus, this class of models includes self-interacting fields. The vanishing potential $\hat{P}=0$ corresponds to the case of free conformal scalar fields, and $\hat{P}=\hat{1}R/6$ to the minimally coupled free scalar fields. The overhat symbol indicates the matrix structures, $\hat{P}=P^A{}_B$, and term $R/6$ in Eq. (6) is explicitly singled out for convenience. The three field strengths $R_{\mu\nu}$, $\hat{\mathcal{R}}_{\mu\nu}$, $\hat{P}$ will be also referred to as curvatures. This massless field theory will be generalized to the case of massive fields in Sec. IV.

The one-loop Euclidean effective action W^β is defined in terms of the functional trace of the heat kernel,

$$W^\beta=-\frac{1}{2}\int_0^\infty\frac{\mathrm{d}s}{s}\mathrm{Tr}K^\beta(s), \tag{8}$$

where the heat kernel $\hat{K}^\beta(s)$ is the periodic in Euclidean time solution of the problem

$$\left\{\frac{\mathrm{d}}{\mathrm{d}s}-\left[\hat{1}\left(\Box-\frac{1}{6}R(x)\right)+\hat{P}(x)\right]\right\}\hat{K}^\beta(s|x,x')$$
$$=\hat{1}\,\delta(s)\,\delta(x,x'), \tag{9}$$

$$\hat{K}^\beta(s|\tau,\mathbf{x};\tau',\mathbf{x}')=\hat{K}^\beta(s|\tau+\beta,\mathbf{x};\tau',\mathbf{x}'). \tag{10}$$

The functional trace is understood as

$$\mathrm{Tr}K(s)=\int\mathrm{d}^Dx\,\mathrm{tr}\,\hat{K}(s), \tag{11}$$

with the tr standing for the matrix trace, e.g., $\mathrm{tr}\hat{1} = \delta^A{}_A$, $\mathrm{tr}\hat{P} = P^A{}_A$.

The thermal (periodic in the Euclidean time) heat kernel K^β can be expressed as an infinite sum of zero temperature (vacuum) heat kernels [5,30]

$$\hat{K}^\beta(s|\tau,\boldsymbol{x};\tau',\boldsymbol{x}') = \sum_{n=-\infty}^{\infty} \hat{K}(s|\tau,\boldsymbol{x};\tau'+\beta n,\boldsymbol{x}'). \tag{12}$$

This image sum is equivalent to summation over Matsubara frequencies in a momentum space representation in thermal field theory. The image sum in the context of Casimir energy calculations was introduced in Ref. [31].

Temperature effects are inherently connected to the imaginary time. It is convenient to factorize the heat kernel into temporal and spatial $K^{(3)}$ parts,

$$\hat{K}(s|\tau,\boldsymbol{x};\tau',\boldsymbol{x}') = \frac{1}{(4\pi s)^{1/2}} \exp\left(-\frac{(\tau-\tau')^2}{4s}\right)\hat{K}^{(3)}(s|\boldsymbol{x};\boldsymbol{x}'), \tag{13}$$

which is possible to do in ultrastatic spacetimes. Then, the trace of the heat kernel takes a form [19],

$$\mathrm{Tr}K^\beta(s) = \theta_3(0,\mathrm{e}^{-(\beta^2/4s)})\frac{\beta}{(4\pi s)^{1/2}}\int \mathrm{d}^3x\, \mathrm{tr}\,\hat{K}^{(3)}(s|\boldsymbol{x},\boldsymbol{x}), \tag{14}$$

when expressed in terms of the Jacobi theta function [32], which is defined in a usual way,

$$\theta_3(a,b) \equiv \sum_{n=-\infty}^{n=\infty} \mathrm{e}^{2nai} b^{n^2}. \tag{15}$$

The free energy of quantum fields in static spacetime F^β is defined via the finite temperature Euclidean effective action W^β and can be written in the form,

$$F^\beta = -\frac{1}{2\beta}\int_0^\infty \frac{\mathrm{d}s}{s}\mathrm{Tr}K^\beta(s). \tag{16}$$

The vacuum mode $n=0$ in the infinite sum (14),(15) corresponds to the zero temperature effective action which suffers ultraviolet divergencies [9,10,26]. Fortunately, this is the only divergent term of the sum [19], so it is convenient to treat it separately. We subtract the zero temperature ($\beta=\infty$) free energy F^∞ from F^β and renormalize it with the use of the zeta function regularization [25,26,33],

$$W^\infty_{\mathrm{ren}} = -\frac{1}{2}\frac{\partial}{\partial\epsilon}\left[\frac{\mu^{2\epsilon}}{\Gamma(\epsilon)}\int_0^\infty \frac{\mathrm{d}s}{s^{1-\epsilon}}\mathrm{Tr}K(s)\right]_{\epsilon=0}, \tag{17}$$

where μ is a masslike regularization parameter and Γ is the gamma function. F^∞_{ren} will be combined with $n\neq 0$ terms at the end of our derivations. Therefore, we compute

$$F^\beta_{\mathrm{ren}} - F^\infty_{\mathrm{ren}} = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}s}{s}(\theta_3(0,\mathrm{e}^{-(\beta^2/4s)})-1)\frac{1}{(4\pi s)^{1/2}} \times \int \mathrm{d}^3x\, \mathrm{tr}\hat{K}^{(3)}(s|\boldsymbol{x};\boldsymbol{x}). \tag{18}$$

The heat kernel $K^{(3)}(s)$ is defined as a solution of Eq. (9) with the three dimensional operator,

$$\left\{\frac{\mathrm{d}}{\mathrm{d}s} - \left[\hat{1}\left(\triangle - \frac{1}{6}R(\boldsymbol{x})\right) + \hat{P}(\boldsymbol{x})\right]\right\}\hat{K}^{(3)}(s|\boldsymbol{x};\boldsymbol{x}) = \hat{1}\,\delta(s)\,\delta(\boldsymbol{x},\boldsymbol{x}'). \tag{19}$$

In this case the three dimensional Laplacian $\triangle$, and potential $\hat{P}(\boldsymbol{x})$ and the curvature $R_{ij}(\boldsymbol{x})$ are defined on a three dimensional hypersurface $\tau=$const of the ultrastatic spacetime.

Many methods have been developed for calculation of the trace of the heat kernel [9,34]. Most of them (see reviews [11,10]) reduce to various representations of its small s expansion,

$$\mathrm{Tr}K(s) = \frac{1}{(4\pi s)^{D/2}}\int \mathrm{d}x^D g^{1/2}(x)\sum_{n=0}^{\infty} s^n \mathrm{tr}\hat{a}_n(x,x), \quad s\to 0. \tag{20}$$

However, as soon as the inverse temperature β is finite, the behavior of the heat kernel at large values of proper time s becomes very important [19]. Therefore, expansion (20) is not suitable for our task of finding the free energy at finite temperature. Besides, Schwinger-DeWitt coefficients a_n are local functions of background fields, henceforth, nonlocal free energy cannot be derived using Eq. (20). To solve the problem of obtaining nonlocal free energy at finite temperature we have to resort to the covariant perturbation theory [15–18]. There is no need to repeat derivations of the covariant perturbation theory here because an expression for $\mathrm{Tr}\,K(s)$ is already known in arbitrary D dimensions [16,18]. In this paper we will take it up to terms quadratic in curvatures,

$$\begin{aligned}\mathrm{Tr}K(s) = \frac{1}{(4\pi s)^{D/2}}\int \mathrm{d}^Dx\, g^{1/2} &\times \mathrm{tr}\{\hat{1} + s\hat{P} + s^2[R_{\mu\nu}f_1(-s\Box)R^{\mu\nu}\hat{1} \\ &+ Rf_2(-s\Box)R\hat{1} + \hat{P}f_3(-s\Box)R \\ &+ \hat{P}f_4(-s\Box)\hat{P} + \hat{\mathcal{R}}_{\mu\nu}f_5(-s\Box)\hat{\mathcal{R}}^{\mu\nu}]\} \\ &+ \mathrm{O}[\mathfrak{R}^3].\end{aligned} \tag{21}$$

Analytic functions f_i (*form factors*) have the dimensionless argument $s\Box$. (The appearance of nonlocal form factors in the momentum space representation of the effective action originates in the classical paper of Schwinger [35].) The form factors act on tensor invariants constructed of the set of field strengths $R^{\alpha\beta}$, $\hat{P}$, $\hat{\mathcal{R}}_{\mu\nu}$ characterizing background. The collective notation $\mathfrak{R}$ will be used for these curvatures.

The first two terms of the sum (21) are purely local and coincide with the first two coefficients of the short proper time expansion (20). Formally, the expansion (21) is valid only in an asymptotically flat Euclidean spacetime with the topology R^D. All background curvatures $\Re$ are supposed to vanish at spacetime infinity [16]. Since we use a perturbation theory, all calculations are carried out with accuracy $O[\Re^n]$, i.e., up to terms of nth and higher power in the curvatures $\Re$. The very structure of this curvature expansion restricts its validity to background fields satisfying the relation,

$$\nabla\nabla\Re \gg \Re^2. \tag{22}$$

Physically it means that gravitational fields are small in magnitude but quickly oscillate.

All form factors in Eq. (21) can be expressed in terms of one basic form factor

$$f(-s\Box)=\int_0^1 d\alpha\, e^{\alpha(1-\alpha)s\Box}. \tag{23}$$

Their explicit form reads [16]

$$f_1(-s\Box)=\frac{f(-s\Box)-1-\frac{1}{6}s\Box}{(s\Box)^2}, \tag{24}$$

$$f_2(-s\Box)=\frac{1}{8}\left[\frac{1}{36}f(-s\Box)-\frac{1}{3}\frac{f(-s\Box)-1}{s\Box}-\frac{f(-s\Box)-1-\frac{1}{6}s\Box}{(s\Box)^2}\right], \tag{25}$$

$$f_3(-s\Box)=\frac{1}{12}f(-s\Box)-\frac{1}{2}\frac{f(-s\Box)-1}{s\Box}, \tag{26}$$

$$f_4(-s\Box)=\frac{1}{2}f(-s\Box), \tag{27}$$

$$f_5(-s\Box)=\frac{1}{2}\frac{f(-s\Box)-1}{s\Box}. \tag{28}$$

Even though, in the following consideration general covariance is broken because of the presence of temperature, we will refer to this curvature expansion as to the covariant perturbation theory. In spatial dimensions the covariance remains explicit.

A few words about validity of this approximation are in order. Since we consider quantum fields at some fixed temperature, one can say that the field system in question represents a canonical ensemble. To define a canonical ensemble rigorously we have to assume that the fields are in some cavity of a finite volume, as it is usually assumed in the presence of a black hole [21]. This assumption should be reconciled, however, with our method of computation described above, which in the present form works only for asymptotically flat spacetimes and requires vanishing background fields at spacetime infinity. It is important to note that background field strengths, sources of vacuum polarization, have a compact support on a manifold, thus, providing an effective volume cutoff. In regard to gravitational field this property is due to the presence of the Ricci tensor rather than the Riemann tensor [16].

III. FREE ENERGY OF MASSLESS SCALAR FIELDS

Let us now compute free energy (18) of massless scalar fields at finite temperature. This case was briefly reported in our paper [36]. After introducing a new variable $y=\beta^2/4s$, first two terms of the trace of the heat kernel (21) take the form of the integral,

$$\int_0^\infty dy[\theta_3(0,e^{-y})-1]\,y^{a-1}=2\,\zeta(2a)\,\Gamma(a), \tag{29}$$

where ζ is the Riemann zeta function, Γ is the gamma function. When a is taking values 2 and 1, expression (29) gives for the zeroth and first curvature orders coefficients $\pi^4/45$ and $\pi^2/3$ correspondingly. These local contributions to the free energy are well known [19,37] and coincide with the first two terms of high temperature expansion. Since all information about temperature is separated from tensor invariants, we can write down an anticipated form of free energy up to second order in the field strengths,

$$F_{\rm ren}^{\beta}-F_{\rm ren}^{\infty}=-\int d^3x\, g^{1/2}\,{\rm tr}\left\{\frac{\pi^2}{90\beta^4}\hat{1}+\frac{1}{24\beta^2}\hat{P}+\frac{1}{32\pi^2}[R_{ij}\gamma_1^{\beta}(-\triangle)R^{ij}+R\gamma_2^{\beta}(-\triangle)R+\hat{P}\gamma_3^{\beta}(-\triangle)R+\hat{P}\gamma_4^{\beta}(-\triangle)\hat{P}\,]+O[\Re^3]\right\}. \tag{30}$$

Then, the problem with the second curvature order is reduced now to calculation of the thermal form factors,

$$\gamma_i^{\beta}(-\triangle)\equiv\gamma_i(\beta\sqrt{-\triangle})=\int_0^\infty\frac{ds}{s}[\theta_3(0,e^{-(\beta^2/4s)})-1]f_i(-s\triangle), \tag{31}$$

where f_i are given by Eqs. (24)–(27). We show how to compute Eq. (31) when $f_i(-s\triangle)$ is the basic form factor (23). After substituting Eq. (23) into Eq. (31) and writing down the theta function (15) explicitly we get

$$\gamma(\beta\sqrt{-\triangle})=2\sum_{n=1}^{\infty}\int_0^1 d\alpha\int_0^\infty\frac{dy}{y}\times\exp\left(-yn^2-\frac{1}{4y}\alpha(1-\alpha)\beta^2(-\triangle)\right). \tag{32}$$

Integration over y produces the modified Bessel function of the second kind

$$\gamma(z)=4\sum_{n=1}^{\infty}\int_0^1 d\alpha K_0(nz\sqrt{\alpha(1-\alpha)}), \tag{33}$$

where $z=\beta\sqrt{-\triangle}$. Change of variables, $x=2\sqrt{\alpha(1-\alpha)}$, allows us to express Eq. (33) in terms of the exponential integrals:

$$\int_0^1 \mathrm{d}x\,\frac{x}{\sqrt{1-x^2}}K_0\left(\frac{nzx}{2}\right) = \frac{1}{nz}\left[\mathrm{Ei}\left(\frac{nz}{2}\right)\mathrm{e}^{-nz/2}-\mathrm{Ei}\left(-\frac{nz}{2}\right)\mathrm{e}^{nz/2}\right]. \quad (34)$$

Now we can use for the right hand side of Eq. (34) its standard form in terms of elementary functions [32] and obtain

$$\gamma(z)=4\int_0^\infty \mathrm{d}\,t\sum_{n=1}^{\infty}\frac{\sin(t)}{t^2+n^2z^2/4}. \quad (35)$$

The sum over n can be evaluated [32],

$$\sum_{n=1}^{\infty}\frac{1}{t^2+n^2z^2/4}=\frac{1}{2}\left[\frac{2\pi}{zt}\;\frac{1}{\mathrm{th}(2\pi t/z)}-\frac{1}{t^2}\right], \quad (36)$$

and the resulting expression reads

$$\gamma(z)=2\int_0^\infty \mathrm{d}t\,\sin(t)\left[\frac{2\pi}{zt}\;\frac{1}{\mathrm{th}(2\pi t/z)}-\frac{1}{t^2}\right]. \quad (37)$$

As can be seen from Eqs. (24)–(27), there are two other types of basic thermal form factors (with one and with two subtractions). Their derivations can be found in Appendix A. Applying results (37), (A8), and (A17) to the table of form factors we obtain for all thermal form factors the following expression:

$$\gamma_i(\beta\sqrt{-\triangle}) = \int_0^\infty \mathrm{d}t\, g_i(t)\left[\frac{2\pi}{\beta\sqrt{-\triangle}\,t}\;\frac{1}{\mathrm{th}(2\pi t/(\beta\sqrt{-\triangle}))}-\frac{1}{t^2}\right], \quad (38)$$

and g_i $(i=1,\ldots,4)$ are simple combinations of elementary functions

$$g_1(t)=-\frac{1}{2}\left(\frac{\sin(t)}{t^2}+3\left[\frac{\cos(t)}{t^3}-\frac{\sin(t)}{t^4}\right]\right), \quad (39)$$

$$g_2(t)=\frac{1}{48}\left(\frac{1}{3}\sin(t)+2\frac{\cos(t)}{t}+\frac{\sin(t)}{t^2} +9\left[\frac{\cos(t)}{t^3}-\frac{\sin(t)}{t^4}\right]\right), \quad (40)$$

$$g_3(t)=\frac{1}{2}\left(\frac{1}{3}\sin(t)+\frac{\cos(t)}{t}-\frac{\sin(t)}{t^2}\right), \quad (41)$$

$$g_4(t)=\sin(t). \quad (42)$$

The final result for renormalized free energy at finite temperature F^{β}_{ren} is presented by a sum of Eqs. (30), (38)–(42) and renormalized free energy at zero temperature $F^{\infty}_{\mathrm{ren}}$. After the zeta regularization (17), the latter one takes the form

$$F^{\infty}_{\mathrm{ren}}=-\frac{1}{32\pi^2}\int \mathrm{d}^3x\, g^{1/2}\mathrm{tr}\{R_{ij}\gamma_1(-\triangle)R^{ij} +R\gamma_2(-\triangle)R+\hat{P}\gamma_3(-\triangle)R+\hat{P}\gamma_4(-\triangle)\hat{P} +\mathrm{O}[\mathfrak{R}^3]\}, \quad (43)$$

where zero temperature form factors $\gamma_i(-\triangle)$, $i=1,\ldots,4$, are

$$\gamma_1(-\triangle)=\frac{1}{60}\left[-\ln\left(-\frac{\triangle}{\mu^2}\right)+\frac{46}{15}\right], \quad (44)$$

$$\gamma_2(-\triangle)=\frac{1}{180}\left[\ln\left(-\frac{\triangle}{\mu^2}\right)-\frac{97}{30}\right], \quad (45)$$

$$\gamma_3(-\triangle)=-\frac{1}{18}, \quad (46)$$

$$\gamma_4(-\triangle)=\frac{1}{2}\left[-\ln\left(-\frac{\triangle}{\mu^2}\right)+2\right]. \quad (47)$$

This expression differs from the one obtained using dimensional regularization only by unessential additive constants [16].

Formulas (30), (38)–(47), we have obtained, are valid at arbitrary finite temperature. Now we would like to study asymptotic behavior of the free energy in high temperature regime, the most interesting and the best studied limit. In the framework of perturbation theory, the problem boils down to finding $\beta\to 0$ asymptotic of thermal form factors (38). We have to be careful while dealing with mutually compensating singularities. After relatively straightforward calculations the outcome for Eq. (37) is

$$\gamma(\beta\sqrt{-\triangle})=\frac{2\pi^2}{\beta\sqrt{-\triangle}}+2\left[\ln\left(\frac{\beta\sqrt{-\triangle}}{4\pi}\right)-1+\mathrm{C}\right] -\frac{\zeta(3)}{24\pi^2}\beta^2(-\triangle)+\frac{\zeta(5)}{640\pi^4}\beta^4(-\triangle)^2 +\mathrm{O}[\beta^6],\quad \beta\to 0, \quad (48)$$

where C is Euler's constant and ζ is the Riemann zeta function.

Now, expressions for the vacuum free energy (43) and the high temperature expansion of Eq. (30) match, and can be combined into a single formula. The resulting $T\to\infty$ expansion of the renormalized one loop free energy takes a form,

$$F^{\beta}_{\rm ren}=-\int \mathrm{d}^3x\, g^{1/2}\,\mathrm{tr}\Bigg\{\frac{\pi^2}{90\beta^4}\hat{1}+\frac{1}{24\beta^2}\hat{P}+\frac{1}{32\beta}\Bigg[\frac{1}{16}R_{ij}\frac{1}{\sqrt{-\triangle}}R^{ij}-\frac{25}{1152}R\frac{1}{\sqrt{-\triangle}}R-\frac{1}{12}\hat{P}\frac{1}{\sqrt{-\triangle}}R+\hat{P}\frac{1}{\sqrt{-\triangle}}\hat{P}\Bigg]$$
$$+\frac{1}{16\pi^2}\left(\ln\left(\frac{\beta\mu}{4\pi}\right)+\mathrm{C}\right)\left[\frac{1}{60}R_{ij}R^{ij}-\frac{1}{180}RR+\frac{1}{2}\hat{P}\hat{P}\right]+\beta^2\frac{\zeta(3)}{128\pi^4}\left[\frac{1}{840}R_{ij}\triangle R^{ij}-\frac{1}{3780}R\triangle R+\frac{1}{180}\hat{P}\triangle R+\frac{1}{12}\hat{P}\triangle\hat{P}\right]$$
$$+\beta^4\frac{3\zeta(5)}{1024\pi^6}\left[\frac{1}{15120}R_{ij}\triangle^2R^{ij}+\frac{1}{1260}\hat{P}\triangle^2R+\frac{1}{120}\hat{P}\triangle^2\hat{P}\right]+\mathrm{O}[\mathfrak{R}^3]+\mathrm{O}[\beta^6]\Bigg\},\qquad \beta\to 0. \tag{49}$$

All local terms of this result perfectly reproduce those of Refs. [19,28]. The combination of quadratic in curvatures terms at the logarithm is just the trace of the second Schwinger-DeWitt coefficient a_2, taken with the Riemann curvature expressed via the Ricci one [9,10,16]. Higher powers of β in Eq. (49) are also quadratic in curvatures parts of a_3 and a_4 Schwinger-DeWitt coefficients [34,38].

We obtained the explicit form of all nonlocal terms of the second order in curvatures. They are proportional to $1/\beta$, and were known to exist [28]. The general structure of nonlocal terms is $\mathfrak{R}(1/\sqrt{-\triangle})\mathfrak{R}$, and, therefore, techniques based on local (small s) expansions could not generate them. Terms of higher orders in curvatures [18,38] will also give nonlocal contribution linear in temperature.

The meaning of nonlocal structures can be understood from spectral representations in terms of massive Green functions [39,17,18]. For this particular form we have the following spectral formula:

$$\frac{1}{\sqrt{-\triangle}}=\frac{2}{\pi}\int_0^\infty \mathrm{d}m\,\frac{1}{m^2-\triangle}. \tag{50}$$

A remarkable property of the expression (49) is that it contains the only kind of nonlocality, Eq. (50). All logarithmic nonlocalities $\ln(-\triangle)$, that are present in $F^{\infty}_{\rm ren}$ and $F^{\beta}_{\rm ren}$, have mutually canceled, leaving logarithm temperature dependence in the form of $\ln(\beta\mu)$, This local combination is well known in both flat [5,40] and curved [19] space thermal field theory. The $\ln(-\triangle)$ disappearance is still being analyzed in a different physical language and in a different setting [41].

Of course, we are not completely satisfied with the integral representation for the free energy at finite temperature (30). Although, it admits a closed form, we would prefer to see F^{β} expressed entirely in terms of analytical and special functions. Indeed, it is possible to obtain such a form after applying the Poisson resummation [32],

$$\sum_{n=-\infty}^{\infty}\mathrm{e}^{-(\beta^2/4s)n^2}=\frac{\sqrt{4\pi s}}{\beta}\sum_{k=-\infty}^{\infty}\mathrm{e}^{-(4\pi^2 s/\beta^2)k^2}. \tag{51}$$

Then, the following identity holds:

$$\theta_3(0,\mathrm{e}^{-(\beta^2/4s)n^2})-1=\frac{\sqrt{4\pi s}}{\beta}\left(\sum_{k=-\infty}^{\infty}\mathrm{e}^{-(4\pi^2 s/\beta^2)k^2}-\int_{-\infty}^{\infty}\mathrm{d}\kappa\,\mathrm{e}^{-(4\pi^2 s/\beta^2)\kappa^2}\right). \tag{52}$$

We compute now the basic thermal form factor, Eq. (31) with Eq. (23), using this identity and separating the $k=0$ term out of the sum,

$$\gamma(\beta\sqrt{-\triangle})=\frac{2\pi^2}{\beta\sqrt{-\triangle}}+\frac{8\pi}{\beta\sqrt{-\triangle}}\left(\sum_{k=1}^{\infty}\arctan\left(\frac{\beta\sqrt{\triangle}}{4\pi k}\right)-\int_0^\infty \mathrm{d}\kappa\,\arctan\left(\frac{\beta\sqrt{\triangle}}{4\pi\kappa}\right)\right). \tag{53}$$

The $k=0$ mode of the sum gives precisely the leading infinite temperature contribution, while the rest can be calculated by employing the following sum:

$$\sum_{k=1}^{\infty}\left(\arctan\left(\frac{b}{k}\right)-\frac{b}{k}\right)=\frac{\mathrm{i}}{2}\ln\left(\frac{\Gamma(1+\mathrm{i}b)}{\Gamma(1-\mathrm{i}b)}\right)-b\mathrm{C}. \tag{54}$$

Adding up the regularized zero temperature form factor,

$$\gamma(-\triangle)=-\ln\left(\frac{-\triangle}{\mu^2}\right)+2, \tag{55}$$

we obtain an expression which is valid at any temperature,

$$\gamma(\beta\sqrt{-\triangle})=\frac{2\pi^2}{\beta\sqrt{-\triangle}}+\frac{4\pi\mathrm{i}}{\beta\sqrt{-\triangle}}\ln\left(\frac{\Gamma\left(1+\mathrm{i}\frac{\beta\sqrt{-\triangle}}{4\pi}\right)}{\Gamma\left(1-\mathrm{i}\frac{\beta\sqrt{-\triangle}}{4\pi}\right)}\right)+2\ln\left(\frac{\beta\mu}{4\pi}\right). \tag{56}$$

Besides an obvious advantage of Eq. (56), namely, that it is the formula in terms of usual elementary and special functions, the leading infinite temperature contributions are present here explicitly. Taking $\beta\to\infty$ and $\beta\to 0$ limits, one can readily find zero temperature (55) and high temperature (48) asymptotics of this basic thermal form factor. In fact, one can see that the logarithm of the gamma functions' ratio in the main result (56) is a sum of all positive powers of β in the high temperature limit (48). Hence, it gives a partial (in the given curvature order) summation formula for the $\beta\to 0$ series [19]. Eventually, one has to transform Eq. (56) into a spectral form, the procedure we can complete again only at high temperatures, Eq. (50). This is the reason why we refrain from deriving the total free energy in this new representation.

IV. FREE ENERGY OF MASSIVE SCALAR FIELDS

The use of the curvature expansion is crucial for derivation of the massless field free energy, because it allows one to avoid artificial infrared divergences. Two other advantages of perturbation theory, namely, that free energy can be found at arbitrary finite temperature and important nonlocal contributions can be obtained, work for a thermodynamic system of massive fields as well. Besides, this is the most studied field model, so let us investigate an ensemble of multicomponent scalar massive fields satisfying equation

$$\left[\left(\Box-\frac{1}{6}R-m^2\right)\hat{1}+\hat{P}(\phi)\right]\varphi=0. \tag{57}$$

Because the mass term can be factorized out of the heat kernel, one can still use massless heat kernel (21) to derive the free energy,

$$F^{\beta}=-\frac{1}{2\beta}\int_0^{\infty}\frac{\mathrm{d}s}{s}\,\mathrm{e}^{-sm^2}\mathrm{Tr}\,K^{\beta}(s). \tag{58}$$

As usual, we subtract $n=0$ mode from the image sum, Eq. (14). Let us first treat local terms. The result in terms of the modified Bessel functions reads

$$\begin{aligned}F^{\beta}_{\mathrm{ren}}-F^{\infty}_{\mathrm{ren}}=-\frac{1}{32\pi^2}\int \mathrm{d}^3x\,g^{1/2}\sum_{n=1}^{\infty}\mathrm{tr}\Bigg\{&\left(\frac{16m^2}{\beta^2n^2}K_0(m\beta n)\right.\\&\left.+\frac{16m}{\beta^3n^3}K_1(m\beta n)\right)\hat{1}\\&+\frac{8m}{\beta n}K_1(m\beta n)\hat{P}+\mathrm{O}[\mathfrak{R}^2]\Bigg\}.\end{aligned} \tag{59}$$

So far this expression is valid at any nonzero temperature. However, we are able to proceed and obtain explicit formulas only in high temperature limit. Simple expansions of the Bessel functions at $\beta\to 0$ with the subsequent n-sum evaluation produces known local contributions [28,29]. The total result for free energy of massive fields at high temperature looks like

$$\begin{aligned}F^{\beta}_{\mathrm{ren}}-F^{\infty}_{\mathrm{ren}}=-\int \mathrm{d}^3x\,g^{1/2}\,\mathrm{tr}\Bigg\{&\frac{\pi^2}{90\beta^4}\hat{1}+\frac{1}{24\beta^2}(\hat{P}-m^2\hat{1})\\&+\frac{1}{32\pi^2}[R_{ij}\gamma_1^{\beta}(-\triangle)R^{ij}\hat{1}+R\gamma_2^{\beta}(-\triangle)R\hat{1}\\&+\hat{P}\gamma_3^{\beta}(-\triangle)R+\hat{P}\gamma_4^{\beta}(-\triangle)\hat{P}]+\mathrm{O}[\mathfrak{R}^3]\Bigg\},\\&\beta\to 0.\end{aligned} \tag{60}$$

The computational procedure for second order terms is performed after Poisson resummation (51). Applying Eq. (52) to the basic form factor of nonlocal free energy for massive fields,

$$\gamma^{\beta}(-\triangle)=\int_0^{\infty}\frac{\mathrm{d}s}{s}[\theta_3(0,\mathrm{e}^{-(\beta^2/4s)})-1]\mathrm{e}^{-sm^2}f(-s\triangle), \tag{61}$$

[vacuum contribution subtracted in Eq. (61) is dealt with at the end of the present section], and using the integral

$$\int_0^1\mathrm{d}\alpha(m^2-\alpha(1-\alpha)\triangle)^{(-1/2)}=\frac{2}{\sqrt{-\triangle}}\arctan\left(\frac{\sqrt{-\triangle}}{2m}\right) \tag{62}$$

we get

$$\begin{aligned}\gamma(\beta\sqrt{-\triangle})=&\frac{4\pi}{\beta\sqrt{-\triangle}}\arctan\left(\frac{\sqrt{-\triangle}}{2m}\right)+\frac{8\pi}{\beta\sqrt{-\triangle}}\\&\times\left[\sum_{k=1}^{\infty}\arctan\left(\frac{\sqrt{-\triangle}}{\sqrt{4m^2+16\pi^2k^2/\beta^2}}\right)\right.\\&\left.-\int_0^{\infty}\mathrm{d}\kappa\,\arctan\left(\frac{\sqrt{-\triangle}}{\sqrt{4m^2+16\pi^2\kappa^2/\beta^2}}\right)\right].\end{aligned} \tag{63}$$

This equation is valid at arbitrary finite temperature, therefore, free energy of massive fields is nonlocal at any temperature. The first term of Eq. (63) came from $k=0$ mode of the sum, and it is nothing but the leading term of high temperature expansion, $\beta\to 0$. The difference of two divergent terms in the square brackets is finite, however, we are unable to give the result in a closed form. Thus, we restrict consideration to leading terms of high temperature expansion and understand the basic thermal form factor as

$$\gamma^{\beta}(-\triangle)=\frac{4\pi}{\beta\sqrt{-\triangle}}\arctan\left(\frac{\sqrt{-\triangle}}{2m}\right)+\mathrm{O}[\beta],\quad \beta\to 0. \tag{64}$$

The main nonlocality is contained in the leading term (64). Subleading terms combined with vacuum contributions are not important at high temperatures. The full table of form factors $\gamma_i(q)$, $i=1,\ldots,4$, in terms of $q=m/\sqrt{-\triangle}$ reads

$$\gamma_1(q)=\frac{\pi q}{\beta m}\left[-\frac{5}{12}q-q^3+(1+4q^2)^2\arctan\left(\frac{1}{2q}\right)\right], \tag{65}$$

$$\gamma_2(q)=\frac{\pi q}{24\beta m}\left[\frac{13}{14}q-3q^3-\left(\frac{11}{3}+28q^2+48q^4\right)\times\arctan\left(\frac{1}{2q}\right)\right], \tag{66}$$

$$\gamma_3(q)=\frac{\pi q}{2\beta m}\left[2q+(1-4q^2)\arctan\left(\frac{1}{2q}\right)\right], \tag{67}$$

$$\gamma_4(q)=\frac{2\pi q}{\beta m}\arctan\left(\frac{1}{2q}\right). \tag{68}$$

For practical purposes of physical applications we need to know spectral form representations for Eqs. (65)–(68). A spectral form for the basic form factor (64) is obvious,

$$\gamma^{\beta}(-\triangle)=\frac{8\pi}{\beta}\int_m^{\infty}d\tilde{m}\frac{1}{4\tilde{m}^2-\triangle}. \tag{69}$$

Its massless limit immediately gives Eq. (50). Spectral forms for form factors with subtractions are obtained similarly (see Appendix B). Then, all form factors (65)–(68) admit the form

$$\gamma_i^{\beta}(-\triangle)=\frac{\pi}{\beta}\int_m^{\infty}d\tilde{m}\,\rho_i(\tilde{m}^2)\frac{1}{4\tilde{m}^2-\triangle}, \tag{70}$$

where mass spectral weights are given in the table,

$$\rho_1(\tilde{m}^2)=\frac{1}{4}\left(1-\frac{2m^2}{\tilde{m}^2}+\frac{m^4}{\tilde{m}^4}\right), \tag{71}$$

$$\rho_2(\tilde{m}^2)=-\frac{1}{32}\left(\frac{25}{9}-\frac{14}{3}\frac{m^2}{\tilde{m}^2}+\frac{m^4}{\tilde{m}^4}\right), \tag{72}$$

$$\rho_3(\tilde{m}^2)=-\frac{1}{3}+\frac{m^2}{\tilde{m}^2}, \tag{73}$$

$$\rho_4(\tilde{m}^2)=4. \tag{74}$$

Now we need to complete our derivation with the regularized free energy at zero temperature F^{∞}. Nonlocal effective action for massive fields in an arbitrary spacetime dimension has been calculated first by Avramidi [42]. His approach is a direct summation of derivatives in a massive field theory, but we can make use of the massless heat kernel (21) obtained with the covariant perturbation theory and arrive at the same result. We compute zeta function regularized effective action according to the equation

$$W_{\rm ren}^{\infty}=-\frac{1}{2}\frac{\partial}{\partial\epsilon}\left[\frac{\mu^{2\epsilon}}{\Gamma(\epsilon)}\int_0^{\infty}\frac{d\,s}{s^{1-\epsilon}}e^{-sm^2}\mathrm{Tr}K(s)\right]_{\epsilon=0}. \tag{75}$$

Then, we get the following result for zero temperature free energy (the specific form of the effective action in Ref. [42] in four dimensions):

$$F_{\rm ren}^{\infty}=-\frac{1}{32\pi^2}\int d^3x\,g^{1/2}\mathrm{tr}\left\{-\frac{m^4}{2}\left(\ln\left(\frac{m^2}{\mu^2}\right)-\frac{3}{2}\right)\hat{1}+m^2\left(\ln\left(\frac{m^2}{\mu^2}\right)-1\right)\hat{P}+[R_{ij}\gamma_1(-\triangle)R^{ij}+R\gamma_2(-\triangle)R+\hat{P}\gamma_3(-\triangle)R+\hat{P}\gamma_4(-\triangle)\hat{P}]+\mathrm{O}[\Re^3]\right\}, \tag{76}$$

where form factors γ_i are given in terms of dimensionless argument $q=m/\sqrt{-\triangle}$ by the following expressions:

$$\gamma_1(q)=\frac{1}{60}\left[-\ln\left(\frac{m^2}{\mu^2}\right)+\frac{46}{15}+\frac{56}{3}q^2+32q^4-2(1+4q^2)^{5/2}\mathrm{arctanh}\left(\frac{1}{\sqrt{1+4q^2}}\right)\right], \tag{77}$$

$$\gamma_2(q)=\frac{1}{180}\left[\ln\left(\frac{m^2}{\mu^2}\right)-\frac{97}{30}-17q^2-12q^4+2\sqrt{1+4q^2}\times(1+8q^2+6q^4)\mathrm{arctanh}\left(\frac{1}{\sqrt{1+4q^2}}\right)\right], \tag{78}$$

$$\gamma_3(q)=\frac{1}{6}\left[-\frac{1}{3}-4q^2+4q^2\sqrt{1+4q^2}\mathrm{arctanh}\left(\frac{1}{\sqrt{1+4q^2}}\right)\right], \tag{79}$$

$$\gamma_4(q)=-\frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)+1-\sqrt{1+4q^2}\mathrm{arctanh}\left(\frac{1}{\sqrt{1+4q^2}}\right). \tag{80}$$

This effective action may look more similar to Eq. (43) if the inverse hyperbolic tangents in functions γ_i are expressed in terms of logarithms. We have to remark here that form factor γ_3 is different from the others. Similarly to that of massless fields it does not depend on the regularization parameter μ. However, Eq. (79) is nonlocal in contrast to local (46). Of course, in the zero mass limit Eqs. (76)–(80) turn to Eqs. (43)–(47).

Finally, it should be noted that in order to use Eq. (76) for specific physical models, the convenient way to work with form factors (77)–(80) is to treat them in a spectral form representation [39]. Then, the following mass spectrum integral is to be used:

$$\frac{4}{\sqrt{-\triangle}}\,\frac{1}{\sqrt{4m^2-\triangle}}\mathrm{arctanh}\left(\frac{\sqrt{-\triangle}}{\sqrt{4m^2-\triangle}}\right)=\int_m^\infty \mathrm{d}\tilde{m}\frac{1}{\sqrt{\tilde{m}^2-m^2}}\,\frac{1}{4\tilde{m}^2-\triangle}. \tag{81}$$

V. FREE ENERGY OF MASSLESS SPINOR FIELDS

In this section we consider the massless Dirac spinors ψ in a Euclidean ultrastatic spacetime at finite temperature. The massless covariant Dirac equation is taken as

$$\nabla\!\!\!\!/\,\psi=0, \tag{82}$$

where the standard notation $\nabla\!\!\!\!/=\gamma^\mu\nabla_\mu$ is used (see [21] for general definitions). The method of calculation of the effective action for spin-1/2 fields, $W_{(1/2)}$, is similar to the one for spin-0 fields. The main difference is that fermions are antiperiodic in the Euclidean time and, therefore, they satisfy boundary conditions (2) with the minus sign. The local form of the one-loop effective action $W_{(1/2)}=-\mathrm{Tr}\ \ln \nabla\!\!\!\!/$ was studied first in Ref. [9]. It is defined in terms of the heat kernel (or zeta function) of operator (82), however, following DeWitt's idea we consider the squared operator $\nabla\!\!\!\!/^2$, thus,

$$W^\beta_{(1/2)}=\beta F^\beta_{(1/2)}=\frac{1}{2}\int_0^\infty\frac{\mathrm{d}s}{s}\mathrm{Tr}\ K^\beta_{(1/2)}(s), \tag{83}$$

with the heat kernel $K_{(1/2)}(s)$ corresponding to the squared Dirac operator. It can be shown [9] that the heat kernel (Green function) of the operator $\nabla\!\!\!\!/^2$ is equivalent to the spinor heat kernel which is a solution of the equation,

$$\left\{\frac{\mathrm{d}}{\mathrm{d}s}-\hat{1}\left[\Box-\frac{1}{4}R(x)\right]\right\}\hat{K}_{(1/2)}(s|x;x')=\hat{1}\,\delta(s)\,\delta(x;x'). \tag{84}$$

One can represent the heat kernel (84) at finite temperature in a form of the image sum [43,29],

$$\hat{K}^\beta_{(1/2)}(s|\tau,\boldsymbol{x};\tau',\boldsymbol{x}')=\sum_{n=-\infty}^{\infty}(-1)^n\hat{K}_{(1/2)}(s|\tau,\boldsymbol{x};\tau'+\beta n,\boldsymbol{x}') \tag{85}$$

[cf. Eq. (12)]. Because time dependence of the heat kernel in ultrastatic spacetimes factorizes out Eq. (13), the trace of the heat kernel can be written in the form

$$\mathrm{Tr}K^\beta_{(1/2)}(s)=\theta_2(0,\mathrm{e}^{-(\beta^2/4s)})\,\frac{\beta}{(4\pi s)^{1/2}}\int \mathrm{d}^3x\ \mathrm{tr}\,K^{(3)}_{(1/2)}(s|\boldsymbol{x},\boldsymbol{x}), \tag{86}$$

where θ_2 is the Jacobi theta function. It is convenient to use the θ functions identity

$$\theta_2(0,\mathrm{e}^{-z})=\theta_3(0,\mathrm{e}^{-z/4})-\theta_3(0,\mathrm{e}^{-z}) \tag{87}$$

and express the heat kernel trace at finite temperature (86) in terms of the Jacobi functions θ_3. Thanks to this fact [29,43] we do not have to repeat all calculations and can get the free energy of Dirac spinors at finite temperature using mathematical derivations of Sec. III. Spinor thermal form factors $\gamma_{(1/2)}$ are obtained then by a simple combination of scalar form factors γ

$$\gamma_{(1/2)}(\beta\sqrt{-\triangle})=-[2\gamma(2\beta\sqrt{-\triangle})-\gamma(\beta\sqrt{-\triangle})]. \tag{88}$$

To form the operator of Eq. (84) the potential term should be taken $\hat{P}=-\frac{1}{12}R\hat{1}$. The commutator curvature is not zero when covariant derivatives act on spinors, but we need to know only that $\mathrm{tr}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu}=-\frac{1}{8}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\mathrm{tr}\hat{1}$, where the squared Riemann tensor must be expressed via Ricci tensor and scalar curvature with help of the Gauss-Bonnet identity [44]. All other matrix structures are reduced to $\mathrm{tr}\hat{1}=4$. Taking into account these properties the free energy of massless spinors reads (from now on we omit spinor indices $_{(1/2)}$)

$$F^\beta_{\mathrm{ren}}-F^\infty_{\mathrm{ren}}=-\int \mathrm{d}^3xg^{1/2}\left\{\frac{7\pi^2}{180\beta^4}-\frac{1}{144\beta^2}R+\frac{1}{8\pi^2}\times[R_{ij}\gamma^\beta_1(-\triangle)R^{ij}+R\gamma^\beta_2(-\triangle)R]+\mathrm{O}[R^3]\right\}, \tag{89}$$

where the thermal spinor form factors

$$\gamma_i(\beta\sqrt{-\triangle})=\int_0^\infty \mathrm{d}t\ g_i(t)\left[\frac{2\pi}{\beta\sqrt{-\triangle}\,t}\ \frac{1}{\mathrm{sh}(2\pi t/(\beta\sqrt{-\triangle}))}-\frac{1}{t^2}\right], \tag{90}$$

with the trigonometric polynomials

$$g_1(t)=-\frac{1}{4}\left(\frac{\cos(t)}{t}-3\frac{\sin(t)}{t^2}\right)+\frac{3}{2}\left(\frac{\cos(t)}{t^3}-\frac{\sin(t)}{t^4}\right), \tag{91}$$

$$g_2(t)=-\frac{1}{16}\left[2\frac{\sin(t)}{t^2}-\frac{\cos(t)}{t}+3\left[\frac{\cos(t)}{t^3}-\frac{\sin(t)}{t^4}\right]\right]. \tag{92}$$

Note, that the only difference of Eq. (90) from Eq. (38) is the hyperbolic sinus instead of the hyperbolic tangent.

This result is to be combined with the regularized contribution F^∞_{ren} of the image sum (85). The zeta regularized effective action (83) at zero temperature takes the form

$$F^{\infty}_{\rm ren}=-\frac{1}{8\pi^2}\int d^3x\, g^{1/2}{\rm tr}\{R_{ij}\gamma_1(-\triangle)R^{ij}+R\gamma_2(-\triangle)R+{\rm O}[\mathfrak{R}^3]\}, \tag{93}$$

where zero temperature spinor form factors $\gamma_i(-\triangle)$, $i=1,2$, are

$$\gamma_1(-\triangle)=-\frac{1}{40}\left[\ln\left(-\frac{\triangle}{\mu^2}\right)-\frac{12}{5}\right], \tag{94}$$

$$\gamma_2(-\triangle)=\frac{1}{120}\left[\ln\left(-\frac{\triangle}{\mu^2}\right)-\frac{77}{30}\right]. \tag{95}$$

The main result for the finite temperature free energy $F^{\beta}_{\rm ren}$ is the sum of Eqs. (89) and (93). It is an essentially nonlocal functional. On the other hand, it is well known that only local terms survive in high temperature limit [43]. To check this property we consider the high temperature limit of our result.

The calculation of $\beta\to 0$ asymptotic of the thermal form factors (90) are analogous to similar derivations in Sec. III. A different hyperbolic function appearing in thermal form factors, namely, the hyperbolic sinus, results in the absence of linear in temperature nonlocal terms in high temperature expansion, for example, the basic form factor $[g_i(t)=-2\sin(t)$ in Eq. (90)] reads

$$\gamma(\beta\sqrt{-\triangle})=-2\left[{\rm C}+\ln\left(\frac{\beta\sqrt{-\triangle}}{\pi}\right)-1\right]+\frac{7\zeta(3)}{24\pi^2}\beta^2(-\triangle)-\frac{31\zeta(5)}{640\pi^4}\beta^4(-\triangle)^2+{\rm O}[\beta^6],\quad \beta\to 0. \tag{96}$$

Uniting the vacuum free energy (93) and the high temperature expansion of Eq. (89) into one expression, we observe that the logarithmic nonlocality also cancels in the sum. The resulting $T=1/\beta\to\infty$ expansion looks like

$$\begin{aligned}F^{\beta}_{\rm ren}=-\int d^3x\, g^{1/2}\Bigg\{&\frac{7\pi^2}{180\beta^4}-\frac{1}{144\beta^2}R+\frac{1}{16\pi^2}\\&\times\left(\ln\left(\frac{\beta\mu}{\pi}\right)+{\rm C}\right)\left[\frac{1}{30}RR-\frac{1}{10}R_{ij}R^{ij}\right]\\&+\beta^2\frac{7}{128}\frac{\zeta(3)}{\pi^4}\left[\frac{1}{280}R\triangle R-\frac{1}{84}R_{ij}\triangle R^{ij}\right]\\&+\beta^4\frac{93}{1024}\frac{\zeta(5)}{\pi^6}\left[\frac{1}{3780}R\triangle^2R\right.\\&\left.-\frac{1}{1080}R_{ij}\triangle^2R^{ij}\right]+{\rm O}[R^3]+{\rm O}[\beta^6]\Bigg\},\quad\beta\to 0.\end{aligned} \tag{97}$$

Combinations of quadratic in curvatures terms in the square brackets are the functional traces of Schwinger-DeWitt coefficient a_2, a_3, and a_4 after substituting the Riemann curvature with the Ricci tensor and scalar curvature [9,10,38]. Expression (97) coincides with the massless limit of a similar expression in Ref. [29], which corrects apparent misprints in higher β-order terms of Ref. [43].

Massive spinor fields can be treated in a similar way. At finite temperature the corresponding effective action is nonlocal. We do not present the result here because of its complexity. In the physically interesting limit of high temperatures the effective action of Fermi fields becomes local and can be found in Ref. [29]. A small correction should be made to the result of [29] because the effective actions corresponding to the first order operator $(\nabla\!\!\!/+m)$ and the squared one differ by local terms [see Eq. (4.2) of Ref. [45]].

VI. CONCLUSIONS

Free energy of quantum fields in ultrastatic (optical) spaces has been a subject of study in a large number of papers [19,30,43,46,47]. When effects of gravitational fields are negligible, the metric under consideration is flat, and the spacetime is automatically ultrastatic. But even this simple case has not been studied sufficiently, as most works are concerned with finding the effective action either on constant background fields or (and) at very high temperature [4,5,40]. However, rapidly oscillating background fields generate nonlocal terms in the effective action which contribute to vacuum polarization effects [22,14,48,49]. Needless to say, that an interesting and important high temperature limit of the effective action is still only an asymptotic, and knowledge of the effective action behavior at arbitrary finite temperature is necessary.

We have obtained nonlocal structures of the one-loop Euclidean effective action and free energy for thermal fields in asymptotically flat ultrastatic curved spacetimes. For nonconformal massless scalar fields this expression has been found at arbitrary finite temperatures. Explicit formulas for the high temperature limit of this general expression have been obtained. For massive scalar fields the free energy is derived in the high temperature limit. With help of these results we calculated also the free energy of massless spinor fields.

The calculated one-loop Euclidean effective action is known to be a generating functional of one-particle irreducible Green functions [9,50]. Therefore, variations of the nonlocal effective action over background fields generate the Green functions [51] and the energy-momentum tensor [48], while variations of the free energy over thermodynamical variables, such as temperature, provides one with thermodynamical potentials, entropy, etc. [19,21].

We would like to emphasize again that the technique we adopted, namely, the nonlocal covariant perturbation theory [16], is a limit opposite to the effective potential method [5,8] and the derivative expansions [6,47,52] suitable for constant or slowly changing background fields. Therefore, to make comparisons one has to expand our nonlocal results in powers of the derivatives and compare them with the corresponding terms in the known derivative expansions expanded, in turn, in powers of field strengths. Our results are in agreement with all known expansions of this kind.

It is important to stress the value of the nonlocal effective action approach. As has been shown in Sec. III the effective action for massless fields is naturally infrared finite, which demonstrates that infrared infinities are absent both at zero and finite temperatures. As for the applications of the covariant perturbation theory to other models, one can refer to Refs. [53,54], where spinor quantum electrodynamics in flat spacetime and scalar electrodynamics with gravity in the context of conformal anomaly have been studied.

We used in this paper the imaginary time formalism, which means that the notion of the global periodic time is introduced. This greatly reduces the class of physical systems and spacetimes that can be considered. Alternatively, there exists real time formalism [55] which can give a covariant form of the partition function [56] and free energy. To work out nonlocal structures of the effective action using the perturbation theory, which is applicable to nonequilibrium systems and covariant from the outset, one should derive it from the scratch. It would be an extremely interesting project to do.

Obvious next step is to apply conformal transformation technique [28,29] to the obtained results and find the nonlocal free energy on more general static gravitational backgrounds. This problem is facilitated by the fact that conformal transformations of the nonlocal effective action are studied in detail in Refs. [57]. In order to study particle creation by external gravitational fields the third curvature order is required [23,49]. Our intention here was to develop a general method for calculations of nonlocal free energy, which is the most transparent while working in the second order in curvatures. Derivation of the next perturbative order, if needed, will not pose serious technical problems.

Effects of space boundaries are also interesting and important [19,46] and they deserve investigation, however, to do so by means of curvature expansion a major revision of covariant perturbation theory is required.

The results of this paper can be generalized to the very important case of background gauge fields. This requires more caution in dealing with temporal components of gauge fields and leads to the appearance of other noncovariant terms in the effective action [58]. It is not surprising, since at finite temperatures Lorenz invariance is broken. But the general structure of nonlocalities in the effective action is still the same.

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APPENDIX A: MASSLESS FORM FACTORS WITH SUBTRACTIONS

Complex form factors entering the trace of the heat kernel (21) are to be treated similarly to the basic one, as is described in Sec. III. The heat kernel form factor with one subtraction generates a thermal form factor in the effective action by the following computational procedure:

$$\eta_1(\beta\sqrt{-\triangle})=\int_0^\infty\frac{ds}{s}\left[\theta_3(0,\mathrm{e}^{-(\beta^2/4s)})-1\right]\frac{f(-s\triangle)-1}{s\triangle}. \tag{A1}$$

Using integral representation,

$$\frac{f(-s\triangle)-1}{s\triangle}=\int_0^1 d\alpha_1\,\alpha_1(1-\alpha_1)\int_0^1 d\alpha_2\,\mathrm{e}^{\alpha_1(1-\alpha_1)\alpha_2 s\triangle}, \tag{A2}$$

and taking the integral over proper time, we reduce it to the form

$$\eta_1(z)=4\int_0^1 d\alpha_1\,\alpha_1(1-\alpha_1)\int_0^1 d\alpha_2 \times K_0(nz\sqrt{\alpha_2}\sqrt{\alpha_1(1-\alpha_1)}), \tag{A3}$$

where $z=\beta\sqrt{-\triangle}$ is a new dimensionless variable. This is a table integral [32] with respect to α_2:

$$\int_0^1 dy\,y^2\,K_0(by)=-\frac{1}{b}K_1(b)+\frac{2}{b^2}, \tag{A4}$$

which, after introducing a new variable, $y=2\sqrt{\alpha(1-\alpha)}$, gives us

$$\eta_1(z)=-4\sum_{n=1}^\infty\left[\frac{1}{nz}\int_0^1 dy\,\frac{y^2}{\sqrt{1-y^2}}K_1\left(\frac{nzy}{2}\right)-2\frac{1}{n^2z^2}\right]. \tag{A5}$$

Applying the Bessel function relationship,

$$K_1(y)=-\frac{d}{dy}K_0(y),$$

to K_1 and using Eq. (35) for the resulting integral, we get

$$\eta_1(z)=-4\sum_{n=1}^\infty\left[\int_0^\infty dt\,\frac{\sin(t)}{(t^2+n^2z^2/4)^2}-2\frac{1}{n^2z^2}\right]. \tag{A6}$$

After integration by parts, to reduce the power of the denominator, η_1 admits the form similar to Eq. (35),

$$\eta_1(z)=2\sum_{n=1}^\infty\int_0^\infty dt\left(\frac{\sin(t)}{t^2}-\frac{\cos(t)}{t}\right)\frac{1}{(t^2+n^2z^2/4)}. \tag{A7}$$

With use of Eq. (36) it gives the final answer,

$$\eta_1(z)=\int_0^\infty dt\left(\frac{\sin(t)}{t^2}-\frac{\cos(t)}{t}\right)\left[\frac{2\pi}{zt}\frac{1}{\mathrm{th}(2\pi t/z)}-\frac{1}{t^2}\right]. \tag{A8}$$

Let us treat now the form factor with two subtractions,

$$\eta_2(\beta\sqrt{-\triangle})=\int_0^\infty \frac{ds}{s}[\theta_3(0,e^{-(\beta^2/4s)})-1]\times\frac{f(-s\triangle)-1-\frac{1}{6}s\triangle}{(s\triangle)^2}. \qquad (A9)$$

In the representation,

$$\frac{f(-s\triangle)-1-\frac{1}{6}s\triangle}{s\triangle}=\int_0^1 d\alpha_1\,\alpha_1^2(1-\alpha_1)^2\int_0^1 d\alpha_2\,\alpha_2\times\int_0^1 d\alpha_3\,e^{\alpha_1(1-\alpha_1)\alpha_2\alpha_3 s\triangle}, \qquad (A10)$$

it can be integrated first over the proper time. Integration over α_3 is exactly Eq. (A4), and integral over α_2 parameter is

$$\int_0^1 dy\,y^2K_1(by)=-\frac{1}{b}K_2(b)+\frac{2}{b^3}. \qquad (A11)$$

Then, η_2 looks in variables y and z as

$$\eta_2(z)=4\sum_{n=1}^\infty\left[\frac{1}{n^2z^2}\int_0^1 dy\,\frac{y^3}{\sqrt{1-y^2}}K_2\left(\frac{nzy}{2}\right)+\frac{1}{3}\,\frac{1}{n^2z^2}-8\frac{1}{n^4z^4}\right]. \qquad (A12)$$

It is possible to reduce Eq. (A12) to γ and η_1 types of integrals employing the relation,

$$K_2(y)=\frac{2}{y}K_1(y)+K_0(y), \qquad (A13)$$

$$\eta_2(z)=4\sum_{n=1}^\infty\left[\frac{1}{n^2z^2}\int_0^\infty dt\,\sin(t)\left(\frac{1}{t^2+n^2z^2/4}+6\frac{1}{(t^2+n^2z^2/4)^2}\right)+\frac{1}{3}\,\frac{1}{n^2z^2}-12\frac{1}{n^4z^4}\right]. \qquad (A14)$$

We already know how to deal with t integral; what is new here is a sum,

$$\sum_{n=1}^\infty\frac{1}{n^2z^2}\,\frac{1}{(t^2+n^2z^2/4)}=\frac{\pi^2}{6z^2t^2}+\frac{1}{8t^4}-\frac{\pi}{4zt^3}\,\frac{1}{\mathrm{th}(2\pi t/z)}, \qquad (A15)$$

which brings η_2 to the form,

$$\eta_2(z)=\int_0^\infty dt\left(\sin(t)+3\left[\frac{\cos(t)}{t}-\frac{\sin(t)}{t^2}\right]\right)\times\left[\frac{2\pi^2}{3z^2t^2}-\frac{\pi}{zt^3\mathrm{th}(2\pi t/z)}+\frac{1}{2t^4}\right]+\frac{2\pi^2}{9z^2}. \qquad (A16)$$

And the final result reads,

$$\eta_2(z)=-\frac{1}{2}\int_0^\infty dt\left(\frac{\sin(t)}{t^2}+3\left[\frac{\cos(t)}{t^3}-\frac{\sin(t)}{t^4}\right]\right)\times\left[\frac{2\pi}{zt}\,\frac{1}{\mathrm{th}(2\pi t/z)}-\frac{1}{t^2}\right]. \qquad (A17)$$

APPENDIX B: MASSIVE FORM FACTORS WITH SUBTRACTIONS

During the computation of remaining two form factors, with one subtraction

$$\eta_1(\beta\sqrt{-\triangle})=\frac{2\sqrt{\pi}}{\beta}\int_0^\infty\frac{ds}{s^{3/2}}e^{-sm^2}\frac{f(-s\triangle)-1}{s\triangle}, \quad \beta\to 0, \qquad (B1)$$

and with two subtractions

$$\eta_2(\beta\sqrt{-\triangle})=\frac{2\sqrt{\pi}}{\beta}\int_0^\infty\frac{ds}{s^{3/2}}e^{-sm^2}\frac{f(-s\triangle)-1-\frac{1}{6}s\triangle}{(s\triangle)^2}, \quad \beta\to 0, \qquad (B2)$$

we get rid of the image sum from the outset and keep only $k=0$ term.

To perform the proper time integration we need two singular integrals that are regularized by cutoff at the lower limit,

$$A_1(b)\equiv\int_\epsilon^\infty\frac{ds}{s^{3/2}}e^{-sb}=-2\sqrt{\pi b}+\frac{2}{\sqrt{\epsilon}}, \qquad (B3)$$

$$A_2(b)\equiv\int_\epsilon^\infty\frac{ds}{s^{5/2}}e^{-sb}=\frac{4}{3}b^{3/2}\sqrt{\pi}+\frac{2}{3\epsilon^{3/2}}-\frac{2b}{\sqrt{\epsilon}}, \qquad (B4)$$

where $b>0$. The form factors (B1) and (B2) take then the form

$$\eta_1(\beta\sqrt{-\triangle})=\frac{\sqrt{4\pi}}{\beta\triangle}\left(\int_0^1 \mathrm{d}\alpha\, A_1(b(\alpha))-A_1(m^2)\right), \tag{B5}$$

$$\eta_2(\beta\sqrt{-\triangle}) =\frac{2\sqrt{\pi}}{\beta\triangle^2}\left(\int_0^1 \mathrm{d}\alpha\, A_2(b(\alpha))-A_2(m^2)-\frac{1}{6\triangle}A_1(m^2)\right), \tag{B6}$$

where $b(\alpha)=m^2-\triangle\alpha(1-\alpha)$. Now we make use of regularized integrals A_1 and A_2. Functions η_1 and η_2 are regular, therefore, all intermediate singularities in these equations mutually cancel. Integrations over α parameter do not pose a problem and they are similar to the integral (62). The only difference is that higher powers of integrand $b(\alpha)$ bring extra factors at the arctangent function:

$$\eta_1(\beta\sqrt{-\triangle})=\frac{\pi}{\beta\sqrt{-\triangle}}\left[-\frac{2m}{\sqrt{-\triangle}} +\left(1-\frac{4m^2}{\triangle}\right)\arctan\left(\frac{\sqrt{-\triangle}}{2m}\right)\right], \tag{B7}$$

$$\eta_2(\beta\sqrt{-\triangle})=\frac{\pi}{\beta\sqrt{-\triangle}}\left[-\frac{5m}{12\sqrt{-\triangle}}-\frac{m^3}{(-\triangle)^{3/2}} +\left(1-\frac{4m^2}{\triangle}\right)^2\arctan\left(\frac{\sqrt{-\triangle}}{2m}\right)\right]. \tag{B8}$$

Spectral forms for η_1 and η_2 are readily obtained with help of basic spectral integral (69) applied to arctangent functions in Eqs. (B7),(B8). To transform operator factors to spectral weights we only need to know the identity,

$$\frac{1}{\triangle}\,\frac{1}{4\tilde{m}^2-\triangle}=\frac{1}{4\tilde{m}^2\triangle}-\frac{1}{4\tilde{m}^2}\,\frac{1}{4\tilde{m}^2-\triangle}. \tag{B9}$$

The spectral representations read,

$$\eta_1(\beta\sqrt{-\triangle})=\frac{2\pi}{\beta}\int_m^\infty \mathrm{d}\tilde{m}\left(1-\frac{m^2}{\tilde{m}^2}\right)\frac{1}{4\tilde{m}^2-\triangle}, \tag{B10}$$

$$\eta_2(\beta\sqrt{-\triangle})=\frac{\pi}{4\beta}\int_m^\infty \mathrm{d}\tilde{m}\left(1-\frac{2m^2}{\tilde{m}^2}+\frac{m^4}{\tilde{m}^4}\right)\frac{1}{4\tilde{m}^2-\triangle}. \tag{B11}$$

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