Towards $N=2$ SUSY homogeneous quantum cosmology: Einstein-Yang-Mills systems

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(Received 21 July 1998; published 24 December 1998)

The application of $N=2$ supersymmetric quantum mechanics for the quantization of homogeneous systems coupled with gravity is discussed. Starting with the superfield formulation of an $N=2$ SUSY sigma model, Hermitian self-adjoint expressions for quantum Hamiltonians and Lagrangians for any signature of a sigma-model metric are obtained. This approach is then applied to coupled SU(2) Einstein-Yang-Mills (EYM) systems in axially symmetric *Bianchi*-type I, II, VIII, IX, *Kantowski-Sachs*, and closed *Friedmann-Robertson-Walker* cosmological models. It is shown that all these models admit the embedding into the $N=2$ SUSY sigma model with the explicit expressions for superpotentials being direct sums of gravitational and Yang-Mills (YM) parts. In addition, the YM parts of superpotentials exactly coincide with the corresponding Chern-Simons terms. The spontaneous SUSY breaking caused by YM instantons in EYM systems is discussed in a number of examples. [S0556-2821(98)06424-8]

PACS number(s): 98.80.Hw, 04.65.+e

I. INTRODUCTION

In order to quantize a pure bosonic system one can apply supersymmetry as a mighty tool for dealing with the problems of a quantum theory [1–5]. The quantization can be done in two ways. The first one is to embed the system in a four-dimensional supersymmetric field theory and then reduce it to one dimension [2,3,6] or, alternatively, to consider the desired Lagrangian as a bosonic part of a supersymmetric sigma model after dimensional reduction [7,8]. These two approaches are not equivalent in general and the results can be different. The second method, i.e., the method of supersymmetric quantum mechanics, seems more convenient for our purposes and we shall follow it hereafter.

In spatially homogeneous cosmological models the only dynamical variable is time t; other (spatial) coordinates can be integrated out from the action. Therefore, one can simply consider the corresponding mechanical system and then try to make a supersymmetric sigma-model extension. The case of pure gravity and gravity with scalar fields was investigated recently by Graham and Bene in the framework of $N=2$ supersymmetric (SUSY) quantum mechanics. However, construction of the quantum Hamiltonian, proposed there, turned out to be Hermitian not self-dual for the case of indefinite signature of the metric in minisuperspace. In this paper we use another construction of the corresponding Hamiltonian, which, in accordance with general lines of quantization, is Hermitian self-adjoint for any type of signature of the metric in minisuperspace. The obtained quantum states coincide with those found in [7,8] only in null fermion and filled fermion sectors, while in other fermion sectors they exist only if the manifold, determined by the minisuperspace metric, has corresponding nontrivial cohomologies.

We apply developed $N=2$ SUSY sigma-model technique for the quantization of SU(2) Einstein-Yang-Mills (EYM) system in homogeneous axially symmetric *Bianchi* type I, II, VIII, IX, *Kantowski-Sachs* (KS), and closed *Friedman-Robertson-Walker* (FRW) cosmological models. Since the work by Bartnik and McKinnon [9] where an infinite set of regular particle-like SU(2) non-Abelian EYM configurations was obtained, further interest in the EYM system has been caused by the unexpected properties of their classical solutions. In particular, it has been shown that non-Abelian EYM black holes violate the naive ''no-hair'' conjecture in an external region [10], as well as demonstrating rather unusual internal structure [11,12] with the generic space-time singularity being an infinitely oscillating, but not of a mixmaster, type. The metric in the space-time region under an event horizon of a spherically symmetric black hole is equivalent to the homogeneous cosmological Kantowski-Sachs metric and this correspondence allows us to apply the methods developed in quantum cosmology for the study of black hole singularities. Classical EYM solutions in different (Bianchi) cosmologies have still not been investigated so far, except the axially symmetric Bianchi type I model, where chaotic behavior of the metric, inspired by chaos in YM equations of motion, has been observed [13,14]. In all the classical EYM systems mentioned above, the nonlinear nature of the source YM field produces nontrivial space-time configurations mainly in strong field regions, i.e., near black hole or cosmological space-time singularities, where a pure classical description of space-time should be replaced by a quantum field theory and our present work is one step towards this goal.

We show that all considered EYM models, containing initially purely bosonic (gravitational and YM) degrees of freedom, admit $N=2$ supersymmetrization in the framework of the $N=2$ SUSY sigma model. The inclusion of non-

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Abelian gauge fields to pure gravitational systems produces additional parts in superpotentials, which, as we shall see below, are equal to Yang-Mills Chern-Simons terms. The connection between the superpotential and the ''winding number'' in some supersymmetric Yang-Mills field theories and sigma models was discussed earlier [2,3]. However, direct generalization to the EYM supersymmetric sigma models is not straightforward, since the expression for the space-time metric, which in turn determines the form of the *Ansatz* for the Yang-Mills field, can be arbitrary. Therefore the fact that the $N=2$ supersymmetric sigma model based on axially symmetric homogeneous EYM systems respects this result is quite nontrivial.

The paper is organized as follows. In Sec. II we discuss formal aspects of $N=2$ SUSY sigma models, starting with the superfield approach. In Sec. III the desired embedding of EYM systems into the $N=2$ SUSY sigma model is described and explicit expressions for superpotentials are given. The quantization and SUSY breaking by YM instantons are discussed in Sec. IV.

II. $N=2$ SUSY QUANTUM MECHANICS

Let us first recall some main features of $N=2$ supersymmetric quantum mechanics, developed mainly in [1–5]. We shall follow the superfield approach, since it is more geometrical, rather than the component one, and the component form of the corresponding Lagrangian obtained is obviously invariant under the desired SUSY transformations. Consider superspace, spanned by the coordinates $(t,\theta,\overline{\theta})$, where t is time, while θ and its conjugate $\overline{\theta}$ are nilpotent Grassman variables. The $N=2$ supersymmetry transformations in superspace with the complex odd parameter ϵ have the following form:

$$\delta t = i\epsilon\overline{\theta} + i\overline{\epsilon}\theta,$$
$$\delta\theta = \epsilon \quad \delta\overline{\theta} = \overline{\epsilon}, \tag{1}$$

which are generated by the linear differential operators

$$\Omega = \frac{\partial}{\partial\overline{\theta}} + i\theta\frac{\partial}{\partial t} \quad \text{and} \quad \overline{\Omega} = \frac{\partial}{\partial\theta} + i\overline{\theta}\frac{\partial}{\partial t}. \tag{2}$$

Now one can introduce the main object of the theory — the real vector superfield Φ^i:

$$\Phi^i = q^i + \overline{\theta}\xi^i - \theta\overline{\xi^i} + \overline{\theta}\theta F^i, \tag{3}$$

where q^i stands for all bosonic degrees of freedom of the system, ξ^i and $\overline{\xi^i}$ are their fermionic superpartners, and F^i is an auxiliary bosonic field. Since the superfield Φ^i transforms under the supersymmetry transformations as

$$\delta\Phi^i = (\overline{\epsilon}\Omega + \epsilon\overline{\Omega})\Phi^i, \tag{4}$$

the most general supersymmetric Lagrangian can be obtained in terms of the supercovariant derivatives

$$D = \frac{\partial}{\partial\theta} - i\overline{\theta}\frac{\partial}{\partial t} \quad \text{and} \quad \overline{D} = \frac{\partial}{\partial\overline{\theta}} - i\theta\frac{\partial}{\partial t}, \tag{5}$$

which anticommute with Ω and $\overline{\Omega}$; the resulting Lagrangian

$$L = \int d\theta d\overline{\theta}\left(-\frac{1}{2}g_{ij}(D\Phi^i)(\overline{D}\Phi^i) + W\right) \tag{6}$$

is invariant under supersymmetry transformations (4) by construction and it corresponds to the one-dimensional $N=2$ supersymmetric sigma model, characterized by the metric $g_{ij}(i,j=1,\ldots,n)$ of the ''target'' manifold $M(g_{ij})$ and the superpotential W, both being a function of the superfield Φ^i.

Note that the Lagrangian (6) is *self-adjoint* for any signature of the metric g_{ij}. This fact is especially important for considering homogeneous systems coupled with gravity, since in these cases the manifold M described by the metric g_{ij} is not Riemannian.

After integration over the Grassman variables and elimination of an auxiliary field F^i, one gets a more familiar component form of the Lagrangian:

$$L = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j + ig_{ij}(q)\overline{\xi^i}(\dot{\xi}^j + \Gamma^j_{kl}\dot{q}^k\xi^l) + \frac{1}{2}R_{ijkl}\overline{\xi^i}\xi^j\overline{\xi^k}\xi^l - \frac{1}{2}g^{ij}(q)\partial_i W\partial_j W - \partial_i\partial_j W\overline{\xi^i}\xi^j, \tag{7}$$

where R_{ijkl} and Γ^i_{jk} are the Riemann curvature and Christoffel connection, corresponding to the metric g_{ij}. The supersymmetry transformations can be also written in the component form

$$\delta q^i = \overline{\epsilon}\xi^i - \epsilon\overline{\xi^i},$$
$$\delta\xi^i = \epsilon(-i\dot{q}^i + \Gamma^i_{jk}\overline{\xi^j}\xi^k - \partial^i W),$$
$$\delta\overline{\xi^i} = \overline{\epsilon}(i\dot{q}^i + \Gamma^i_{jk}\overline{\xi^j}\xi^k - \partial^i W), \tag{8}$$

which allow us to find the conserved supercharges using the standard Noether theorem technique:

$$Q = \xi^i(g_{ij}\dot{q}^j + i\partial_i W),$$
$$\overline{Q} = \overline{\xi^i}(g_{ij}\dot{q}^j - i\partial_i W). \tag{9}$$

Following the general lines of quantization of the system with bosonic and fermionic degrees of freedom [15], we introduce the canonical Poisson brackets

$$\{q^i,P_{q^j}\} = \delta^i_j, \quad \{\xi^i,P_{\xi^j}\} = -\delta^i_j, \quad \{\overline{\xi^i},P_{\overline{\xi^j}}\} = -\delta^i_j, \tag{10}$$

where P_{q^j}, P_{ξ^j}, and $P_{\overline{\xi^j}}$ are momenta, conjugate to q^i, ξ^i, and $\overline{\xi^i}$. After finding their explicit form

$$P_{q^i} = g_{ij}\dot{q}^i + i\Gamma_{j;ik}\overline{\xi^j}\xi^k, \tag{11}$$

$$P_{\xi^i}=-ig_{ij}\overline{\xi^j},\quad P_{\overline{\xi^i}}=0, \tag{12}$$

one can conclude from Eqs. (12) that the system possesses the second class fermionic constraints

$$\chi_{\xi^i}=P_{\xi^i}+ig_{ij}\overline{\xi^j}\quad\text{and}\quad \chi_{\overline{\xi^i}}=P_{\overline{\xi^i}}, \tag{13}$$

since

$$\{\chi_{\xi^i},\chi_{\overline{\xi^j}}\}=-ig_{ij}. \tag{14}$$

Therefore, the quantization has to be done using the Dirac brackets, defined for any two functions V_a and V_b as

$$\{V_a,V_b\}_D=\{V_a,V_b\}-\{V_a,\chi_c\}\frac{1}{\{\chi_c,\chi_d\}}\{\chi_d,V_b\}. \tag{15}$$

Using Eq. (15), one can easily find nonvanishing Dirac brackets between bosonic and fermionic degrees of freedom:

$$\{q^i,P_{q^j}\}_D=\delta^i_j,\quad \{\xi^i,\overline{\xi^j}\}_D=-ig^{ij}. \tag{16}$$

Then, after replacing the Dirac brackets with a graded commutator

$$\{,\}_D\rightarrow i[,]_\pm, \tag{17}$$

one obtains the following (anti)commutation relations:

$$[q^i,P_{q^j}]_-=i\delta^i_j,\quad [\xi^i,\overline{\xi^j}]_+=g^{ij}. \tag{18}$$

To make a quantum expression for supercharges (9) it is convenient to introduce the projected fermionic operators $\overline{\xi^a}=e^a_\mu\overline{\xi^\mu}$ and $\xi^a=e^a_\mu\xi^\mu$ where e^i_a is inverse to the tetrad $e^a_i(e^a_ie^i_b=\delta^a_b)$, related to the metric g_{ij} of the "target" manifold M and to the metric of its tangent space η_{ab} in the usual way, $e^a_ie^b_j\eta_{ab}=g_{ij}$.

However, the explicit form of the supercharges depends on the choice of operator ordering and therefore is ambiguous. We take it as in [3]:

$$Q=\xi^ae^i_a(P_i+i\omega_{iab}\overline{\xi^a}\xi^b+i\partial_iW),$$
$$\overline{Q}=\overline{\xi^a}e^i_a(P_i+i\omega_{iab}\overline{\xi^a}\xi^b-i\partial_iW), \tag{19}$$

where ω_{iab} is the corresponding spin connection.

In what follows, we shall consider systems subject to the classical Hamiltonian constraint

$$H_0=\frac{1}{2}g^{ij}P_iP_j+\frac{1}{2}g^{ij}(q)\partial_iW\partial_jW=0, \tag{20}$$

which in the quantum case should be replaced by the condition on the quantum state $|\rho\rangle$,

$$H|\rho\rangle=0, \tag{21}$$

with the Hamiltonian

$$H=\frac{1}{2}[Q,\overline{Q}]_+, \tag{22}$$

giving H_0 in the classical limit, i.e., when all fermionic fields are set equal to zero.

The important point is that the operators (9) are nilpotent and mutually Hermitian adjoint with respect to the measure $\sqrt{|-g|}d^nq$ and, therefore, the energy operator H is self-adjoint for any signature of the metric g_{ij}. Now the Lagrangian (7) is self-adjoint after the fashion of construction, since we use real superfields and hence the complex Noether charges and their quantum mechanical expressions are Hermitian adjoint to each other.

Obviously, now one can consider two first order differential equations on the wave function,

$$\overline{Q}|\rho\rangle=0\quad\text{and}\quad Q|\rho\rangle=0, \tag{23}$$

and therefore linearize the operator equation (21); the existence of normalizable solutions of the system (23) means, in turn, that supersymmetry is unbroken quantum mechanically.

In order to solve the system consider the Fock space spanned by the fermionic creation and annihilation operators $\overline{\xi^a}$ and ξ^a, respectively, with $[\xi^a,\overline{\xi^b}]_+=\eta^{ab}$. The general state in this Fock space is obtained in terms of the series expansion

$$|\rho\rangle=F(q)|0\rangle+\cdots+\frac{1}{n!}\overline{\xi^{a_1}}\cdots\overline{\xi^{a_n}}F_{a_1\cdots a_n}(q)|0\rangle$$
$$=F(q)|0\rangle+\cdots+\frac{1}{n!}\overline{\xi^{i_1}}\cdots\overline{\xi^{i_n}}F_{i_1\cdots i_n}(q)|0\rangle, \tag{24}$$

where the coefficients in expansions of this series are p-forms defined on the manifold $M(g_{ij})$, and their number due to the nilpotency of fermionic creation operators is finite. Since the fermion number operator $N=\overline{\xi^a}\xi_a$ commutes with the Hamiltonian H and

$$[N,Q]_-=-Q,\quad [N,\overline{Q}]_-=\overline{Q}, \tag{25}$$

one can consider states characterized by the different fermion numbers separately. Now the solution in empty and filled fermion sectors is simply expressed in terms of the superpotential W as follows:

$$|\rho_0\rangle=\text{const}\times e^{-W}|0\rangle, \tag{26}$$

$$|\rho_n\rangle=\text{const}\times\frac{1}{n!}\overline{\xi^{a_1}}\cdots\overline{\xi^{a_n}}\epsilon_{a_1\cdots a_n}e^{+W}|0\rangle. \tag{27}$$

In order to investigate the solutions in other fermion sectors, let us first recall [2] that in the case of vanishing superpotential operators $\overline{Q_0}$ and Q_0 (supercharges with $W=0$) act on the p-forms F as exterior and co-exterior derivatives, respectively. So solution of the equation $\overline{Q_0}|\rho\rangle=0$ cannot be written as

$$|\rho_p\rangle=\overline{Q_0}|\sigma_{p-1}\rangle \tag{28}$$

only if the corresponding pth cohomology group $H^p(M)$ of the manifold $M(g_{ij})$ is nontrivial. Before generalizing this result to systems with nonzero superpotential W, first note that

$$\overline{Q}=e^{-W}\overline{Q}_0e^{W} \quad \text{and} \quad Q=e^{W}Q_0e^{-W}. \tag{29}$$

Now, using Eqs. (28) and (29) one can prove that the general solution in p-fermion sectors $(p=1,\ldots,n-1)$ of the first equation in Eqs. (23) for the case of trivial cohomology group $H^p(M)$ is

$$|\rho_p\rangle=\overline{Q}|\sigma_{p-1}\rangle. \tag{30}$$

However, because Q and $\overline{Q}$ are Hermitian adjoint to each other, the second equation in Eqs. (23) indicates that this state has zero norm and consequently is unphysical. Therefore the possible existence of supersymmetric ground states, i.e., solutions of the zero-energy Schrödinger-type equation (21), is directly related to the topology of the considered manifold $M(g_{ij})$, since all states except those in purely bosonic and filled fermion sectors can be excluded even without solving the system (23), if the topology of the manifold $M(g_{ij})$ is trivial.

For purely bosonic systems with nonvanishing potential energy the described $N=2$ supersymmetrization turns out to be the simplest possible one and it can be applied for canonical quantization of any appropriate homogeneous cosmological model coupled with matter. After the choice of operator ordering in the supercharges, Eq. (21) in the null fermion sector corresponds to the Wheeler-DeWitt equation for the considered Einstein-matter system and its solution (26) is then easily obtained in terms of superpotential W, since SUSY allows us to linearize the quantum Hamiltonian equation.

III. $N=2$ SUPERSYMMETRIZATION OF SU(2) EINSTEIN-YANG-MILLS COSMOLOGICAL MODELS

Now we are in a position to make the $N=2$ supersymmetric extension of homogeneous axially symmetric SU(2) Einstein-Yang-Mills systems given by the action

$$S=\int d^4x\sqrt{-G}\left(R-\frac{1}{2}F^A_{\mu\nu}F^{A\mu\nu}\right). \tag{31}$$

We restrict ourselves to a subclass of homogeneous space-times which admit a representation in the form of an unconstrained Hamiltonian system for a corresponding classical coupled system of equations; i.e., we consider axially symmetric Bianchi type I, II, VIII, IX (axially symmetric Bianchi type VII is equivalent to Bianchi type I), Kantowski-Sachs, and closed Friedmann-Robertson-Walker cosmological models.

The general diagonal Bianchi-type axially symmetric space-times are parametrized by two independent functions of a cosmological time $b_1(t)$ and $b_3(t)$,

$$ds^2=-dt^2+b_1^2(t)[(\omega^1)^2+(\omega^2)^2]+b_3^2(t)(\omega^3)^2, \tag{32}$$

where ω^i are basis left-invariant one-forms $(d\omega^i=\frac{1}{2}C^i_{jk}\omega^j\wedge\omega^k)$ for the spatially homogeneous three-metrics, depending on three spatial (not necessarily Cartesian) coordinates x,y,z: Bianchi type I:

$$\omega^1=dx,\quad \omega^2=dy,\quad \omega^3=dz. \tag{33}$$

Bianchi type II:

$$\omega^1=dz,\quad \omega^2=dx,\quad \omega^3=dy-xdz. \tag{34}$$

Bianchi type VIII:

$$\begin{aligned}\omega^1&=dx+(1+x^2)dy+(x-y-x^2y)dz,\\ \omega^2&=dx+(-1+x^2)dy+(x+y-x^2y)dz,\\ \omega^3&=2xdy+(1-2xy)dz,\end{aligned} \tag{35}$$

Bianchi type IX:

$$\begin{aligned}\omega^1&=\sin z dx-\cos z\sin x dy,\\ \omega^2&=\cos z dx+\sin z\sin x dy,\\ \omega^3&=\cos x dy+dz.\end{aligned} \tag{36}$$

As was shown by Darian and Kunzle [13], the general ansatz for an SU(2) Yang-Mills field, compatible with the symmetries of axially symmetric Bianchi-type cosmological models, is also expressed in terms of two independent real-valued functions $\alpha(t)$ and $\gamma(t)$ of a cosmological time only and has the form

$$A=\alpha(t)(\omega^1\tau_1+\omega^2\tau_2)+\gamma(t)\omega^3\tau_3, \tag{37}$$

where τ_i are SU(2) group generators, normalized as $[\tau_i,\tau_j]=\epsilon_{ijk}\tau_k$.

Kantowski-Sachs space-time

$$ds^2=-dt^2+b_3^2(t)dr^2+b_1^2(t)d\theta^2+b_1^2(t)(\sin\theta)^2d\phi^2 \tag{38}$$

does not belong to Bianchi classification and admits an additional spherical symmetry; so the SU(2) YM *Ansatz* has a different form, originating from the Witten *Ansatz* for the static spherically symmetric case after the mutual replacement $r\to t, t\to r$:

$$\begin{aligned}A_0&=0,\quad A_r=\gamma(t)L_1,\\ A_\theta&=-L_3+\alpha(t)L_2,\quad A_\phi=\sin\theta[L_2+\alpha(t)L_3],\end{aligned} \tag{39}$$

where

$$\begin{aligned}L_1&=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta),\\ L_2&=(\cos\theta\cos\phi,\cos\theta\sin\phi,-\sin\theta),\\ L_3&=(-\sin\phi,\cos\phi,0)\end{aligned}$$

are spherical projections of SU(2) generators.

We also consider the closed Friedmann-Robertson-Walker model separately, because its general YM *Ansatz*

TABLE I. Potentials and superpotentials.

	Lagrangian L_0	Superpotential $W=W_{gr}+\mathbf{W_{YM}}$
Bianchi type I	$K-\left[\frac{\mathbf{1}}{\mathbf{b_3}}\alpha^2\gamma^2+\frac{\mathbf{b_3}}{\mathbf{2b_1^2}}\alpha^4\right];$	$0+\alpha^2\gamma;$
Bianchi type II	$K-\left[\frac{1}{4}\ \frac{b_3^3}{b_1^2}+\frac{\mathbf{1}}{\mathbf{b_3}}\alpha^2\gamma^2+\frac{1}{2}\ \frac{b_3}{b_1^2}(\alpha^2+\gamma)^2\right];$	$\frac{1}{2}b_3^2+(\alpha^2\gamma+\frac{1}{2}\gamma^2);$
Bianchi type VIII	$K-\left[\frac{1}{4}\ \frac{b_3^3}{b_1^2}+\mathbf{b_3}+\frac{\mathbf{1}}{\mathbf{b_3}}\alpha^2\gamma^2+\frac{1}{2}\ \frac{b_3}{b_1^2}(\alpha^2-\gamma)^2\right];$	$\frac{1}{2}(2b_1^2-b_3^2)+(\alpha^2\gamma-\frac{1}{2}\gamma^2);$
Bianchi type IX	$K-\left[\frac{1}{4}\ \frac{b_3^3}{b_1^2}-\mathbf{b_3}+\frac{\mathbf{1}}{\mathbf{b_3}}\alpha^2(\gamma-\mathbf{1})^2+\frac{1}{2}\ \frac{b_3}{b_1^2}(\alpha^2-\gamma)^2\right];$	$\frac{1}{2}(2b_1^2+b_3^2)+(\alpha^2(\gamma-\mathbf{1})-\frac{1}{2}\gamma^2);$ *or* $\frac{1}{2}(b_3^2-4b_1b_3)+[\alpha^2(\gamma-\mathbf{1})-\frac{1}{2}\gamma^2];$
KS	$K-\left[\frac{1}{b_3}\alpha^2\gamma^2-\mathbf{b_3}+\frac{1}{2}\ \frac{b_3}{b_1^2}(\alpha^2-\mathbf{1})^2\right];$	$2b_1b_3+\gamma(\alpha^2-\mathbf{1});$
FRW	$-\frac{3}{2}b\dot{b}^2+\frac{1}{2}b\dot{\alpha}^2+\frac{3}{2}b-\frac{1}{2}\ \frac{(1-\alpha^2)^2}{\mathbf{b}};$	$\frac{3}{2}b^2+(\frac{1}{3}\alpha^3-\alpha);$

[16] [SU(2) YM field on S^3] is not obtained from Bianchi-type IX after setting $\alpha(t)=\gamma(t)$ in Eq. (37). The closed FRW model with the interval

$$ds^2=-dt^2+b^2(t)[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\phi^2)] \tag{40}$$

(χ, θ, and ϕ are angles on S^3) admits the following representation for the SU(2) YM field, expressed in terms of a single real-valued function $\alpha(t)$:

$$A_0=0,\ A_j=\frac{1}{2}[\alpha(t)+1]U\partial_j U^{-1}, \tag{41}$$

$$U=\exp\{i\chi[\sin\theta(\sigma_1\cos\phi+\sigma_2\sin\phi)+\sigma_3\cos\theta]\},$$
$$j=1,2,3, \tag{42}$$

where σ_i are Pauli matrices.

Inserting these *Ansätze* into the action and integrating over all variables except t one obtains the one-dimensional Lagrangian

$$L_0=\frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j-V(q)=K-V; \tag{43}$$

here, $g_{ij}(q)$ is the metric in the extended minisuperspace, i.e., in the configuration space of spatially homogeneous axially symmetric three-metrics coupled with the corresponding SU(2) Yang-Mills fields.

Let us consider the functions $q^i=(b_1,b_3,\alpha,\gamma)$ as a bosonic components of the superfield (3). One can introduce the same number of fermionic fields ($\overline{\xi^i}$ and ξ^i) and therefore make $N=2$ supersymmetrization of the Lagrangian L_0 if, and only if, the potential $V(q)$ admits the expression via a function $W(q)$, called a superpotential:

$$V(q)=\frac{1}{2}g^{ij}(q)\frac{\partial W(q)}{\partial q^i}\frac{\partial W(q)}{\partial q^j}. \tag{44}$$

In this case $N=2$ SUSY Lagrangian (7) and the corresponding Hamiltonian, obtained after usual Legendre transformation, are self-adjoint for any signature of the metric g_{ij} in the extended minisuperspace.

The kinetic terms for all Bianchi and Kantowksi-Sachs models are the same,

$$K=-\dot{b}_1{}^2b_3-2\dot{b}_1\dot{b}_3b_1+\dot{\alpha}^2b_3+\dot{\gamma}^2\ \frac{b_1^2}{2b_3}, \tag{45}$$

and the only difference between them is due to the potential terms. Using the expression for the metric on the extended "minisuperspace,"

$$g_{b_1b_1}=-2b_3,\quad g_{b_1b_3}=-2b_1,\quad g_{\alpha\alpha}=2b_3,\quad g_{\gamma\gamma}=\frac{b_1^2}{b_3}, \tag{46}$$

and the explicit form of the potentials, we have found some superpotentials as a solution of Eq. (44), hence making an $N=2$ SUSY extension of the given Einstein-Yang-Mills systems. The results are collected in Table I.

One should note that the obtained superpotentials W in all these cases turn out to be direct sums of pure gravitational W_{gr} (first listed in [8] in terms of Misner variables) and Yang-Mills parts W_{YM}. This fact is quite interesting and does not follow *a priori* from general expectations, since in the sigma-model approach considered above gravitational and Yang-Mills variables in the Lagrangian L_0 are not separated. Moreover, it seems that the YM field is a unique one, which, being coupled with gravity, can allow the corresponding superpotential to be in the form of a direct sum. It follows from the following statement that superpotential is also the least Euclidean action—solution of the Euclidean Hamilton-Jacobi equation of the considered system. One can reconstruct from the superpotential the corresponding Euclidean solutions, those which give the main contribution to the wave function in a quasiclassical approach. So the gravitational part of the superpotential W_{gr} determines the Euclidean gravitational background configurations which should not be changed if a matter field is added. It is possible only if matter configurations do not contribute to the energy-momentum tensor. The Yang-Mills part of the superpotential W_{YM} just provides such a possibility since it produces self-dual YM instantons with the energy-momentum tensor identically vanished. We discuss this point in more detail in the next section.

Note that the full superpotential $W = W_{gr} + W_{YM}$ does not exist as a solution of Eq. (44) if we cancel one of the relevant YM function α or γ; there are no nontrivial self-dual solutions of YM equations of motion with one of YM functions canceled and W_{YM} ceases to exist in this case. The question about other solutions of Eq. (44) which are not direct sums of gravitational and YM parts is still open; however, it seems unlikely that such solutions can be obtained in a closed analytical form.

On the other hand, one more crucial observation can be done, that for all considered models the Yang-Mills part of the superpotential coincides with the corresponding Chern-Simons functional, calculated on a three-dimensional slice $t = \mathrm{const}$. Indeed, it can be checked that the YM Chern-Simons terms

$$W_{YM} = \frac{1}{2}\int d^3x\sqrt{|-G|}\,\epsilon^{0\lambda\mu\nu}\left(A^a_\lambda\partial_\mu A^a_\nu + \frac{1}{3}f^{abc}A^a_\lambda A^b_\mu A^c_\nu\right) \tag{47}$$

turn out to be solutions of the Euclidean Hamilton-Jacobi equation and therefore play the role of the Yang-Mills part of the superpotential. Such a coincidence of YM Chern-Simons terms (47) with YM superpotentials (44) in the framework of the one-dimensional sigma model describing a YM field coupled with gravity seems to be very surprising. Definitely, this statement is not true in the general case of an arbitrary space-time and takes place for the suggested models as a consequence of the symmetries of the space-time metrics and corresponding YM *Ansätze*. Note, that there exist no similar expressions for the W_{gr} part of the superpotential in terms of a functional of gravitational variables except the Bianchi-type IX model with a nonzero cosmological constant, where the Chern-Simons functional in terms of Ashtekar's variables [17] is also an exact solution of the Ashtekar-Hamilton-Jacobi equation [18].

So we have shown that the considered homogeneous axially symmetric EYM systems admit an $N=2$ supersymmetric sigma model extension with the superpotentials given explicitly in Table I and this gives us a suitable background for the quantization.

IV. QUANTIZATION AND SUSY BREAKING BY YM INSTANTONS

A. Supersymmetry at classical and quantum levels

As can be seen from the supersymmetry transformations (8), in order to prevent $N=2$ SUSY breaking at the classical level, the classical pure bosonic configurations must satisfy the properties

$$\dot{q}^i(t) = 0 \quad \text{and} \quad \partial^i W(q^i(t)) = 0, \tag{48}$$

along with the classical Hamiltonian constraint (20). Such classical configurations really exist in an usual field theory in a flat space-time, and the simplest well-known example is a scalar rest particle $(\dot{q}^i = 0)$ on a bottom of a potential with $V(q^i) = 0$.

In contrast with such examples, dealing with unconstrained homogeneous systems with gravity included, any nontrivial classical solution of Einstein (or Einstein coupled with a matter) equations never has all momenta vanished, $\dot{q}^i(t) \neq 0$. These systems satisfy Eq. (20) due to the dynamical balance between the kinetic and potential terms with both positive and negative signs.

Hence, any homogeneous Einstein (or Einstein-matter) system, being embedded into the $N=2$ supersymmetric sigma model, never has solutions of equations of motion with unbroken supersymmetry; i.e., supersymmetry is always spontaneously broken at the ''tree level.''

Let us see what happens in the quantum mechanical approach. In the Einstein-Yang-Mills systems considered above the number of bosonic functions q^i is 4, which is also the fermion number of the filled fermion sector. Therefore we shall consider solutions of the zero-energy Schrödinger-type equation (21) in these empty and filled fermion sectors.

The superpotential $W(q)$ is always defined up to the sign, since it is the ''square root'' of the bosonic potential $V(q)$. Both signs are physically acceptable and correspond to the solutions in empty Eq. (26), and filled, Eq. (27), fermion sectors when finding the supersymmetric wave functions. The normalizability of bosonic wave function for ''positive'' superpotential means in turn the normalizability of filled fermionic wave functions for the ''negative'' superpotential and vice versa. We define the norm of the physical state as $\pm\int\sqrt{|-g|}\langle\rho||\rho\rangle d^4q$ in order to avoid the problem of the negative norm in the four-fermion sector, caused by the timelike component of the fermionic field. The plus sign in the definition of the norm corresponds to $+W(q)$ while the minus sign has to be taken as $-W(q)$.

Let us accept for definiteness the positive sign of the superpotential. First consider pure gravitational systems, when α and γ functions along with their fermionic partners are set equal to zero. As was stated above, supersymmetry is spontaneously broken for any nontrivial solutions of Einstein equations. Quantum mechanically the supersymmetry is restored for Bianchi type I, II, and $\mathrm{IX}_{(1)}$, Kantowski-Sachs, and FRW models since the solution of Eq. (21), $|\rho_0^{gr}\rangle = \mathrm{const}\times e^{-W_{gr}}|0\rangle$, in the null fermion sector is normalizable:

$$\int_0^{+\infty} db_1 \int_0^{+\infty} db_3 \sqrt{|-g|}\, e^{-2W_{gr}} < \infty. \tag{49}$$

Therefore we are facing an interesting situation, where unlike ordinary supersymmetric quantum mechanics, the supersymmetry being spontaneously broken at the ''tree level'' is then restored quantum mechanically.

The only exceptions are the second (in Table I) superpotential for Bianchi type $\mathrm{IX}_{(2)}$ and Bianchi type VIII where the supersymmetry remains broken at the quantum level as well, since their norm (49) diverges at the upper limit.

Further inclusion of the Yang-Mills field spontaneously breaks the supersymmetry again, because, as one can see from Table I, the Yang-Mills part of the superpotential W_{YM} for all considered models, being the corresponding Chern-Simons term, is an odd function of α and γ; consequently, the YM parts of the wave function $|\rho_0^{YM}\rangle = \mathrm{const} \times e^{\pm W_{YM}}|0\rangle$ both in null and filled fermion sectors are not normalizable:

$$\int_{-\infty}^{+\infty} d\alpha \int_{-\infty}^{+\infty} d\gamma \sqrt{|-g|}\, e^{\pm 2W_{YM}} \rightarrow \infty. \tag{50}$$

In order to find possible supersymmetric wave functions in one-, two-, and three-fermion sectors, one has to investigate the topology of the extended minisuperspace. The simplest way of doing that is going to the Misner parametrization [19] of the space-time metric (32):

$$ds^2 = -N^2(t)dt^2 + \frac{1}{6}e^{2A(t)+2B(t)}[(\omega^1)^2+(\omega^2)^2] + \frac{1}{6}e^{2A(t)-4B(t)}(\omega^3)^2. \tag{51}$$

In terms of Misner variables the metric in the extended minisuperspace (46) has the simple diagonal form

$$g_{AA} = -1, \quad g_{BB} = 1, \quad g_{\alpha\alpha} = 2e^{-2A-2B}, \quad g_{\gamma\gamma} = e^{-2A+4B}, \tag{52}$$

which shows that the topology of the extended minisuperspace is equivalent to the Minkowski one with all cohomologies trivial, $H^p(M(g_{ij}))=0$, $p=1,2,3$, and in accordance with the discussion of Sec. II, no physical states in one-, two-, and three-fermion sectors exist since they have zero norm. Similarly, there are no physical states except the ones in null and filled fermion sectors in the considered pure gravitational systems.

B. A role of instantons

Let us discuss in more detail the mechanism of spontaneous supersymmetry breaking in the null fermion sector when the YM field is added to a pure gravitational system (such as Bianchi type I, II, $\mathrm{IX}_{(1)}$, KS, and FRW) which is quantum mechanically supersymmetric, since it admits a normalizable zero-energy solution of the Wheeler-DeWitt equation (21). This mechanism turns out to be quite similar to the one considered in [1,20–22] where the SUSY breaking by instanton configurations has been discussed.

Indeed, as was already mentioned, the superpotential $W(q)$ (if exists) is one of the solutions of the Euclidean Hamilton-Jacobi equation and represents a ''least'' Euclidean action of field configurations, giving the main quasiclassical contribution into the wave function and providing the SUSY breaking after inclusion of the Yang-Mills field. The explicit form of the superpotential allows us to reconstruct such classical configurations by solving the first order system:

$$g_{ij}\dot{q}^j = -\frac{\partial(W_{gr}+W_{YM})}{\partial q^i}. \tag{53}$$

For pure gravitational degrees of freedom these equations are equivalent to the (anti-)self-duality gravitational equations $R_{\mu\nu\lambda\sigma} = \pm\tilde{R}_{\mu\nu\lambda\sigma}$ while $W_{YM}(q)$ part of the superpotential in Eq. (53) gives rise to the (anti-)self-dual Yang-Mills equations $F^a_{\mu\nu} = \pm\tilde{F}^a_{\mu\nu}$ on a given gravitational background determined by the W_{gr}.

Then, (anti-)self-dual Yang-Mills instantons in our systems can be interpreted as a tunneling solution (with the nonvanishing Euclidean action) between topologically distinct vacua. In this case the YM instanton contribution provides the SUSY breakdown due to the energy shift from the initial zero to some positive level and this fact is expressed in the nonnormalizability of the YM part of the zero energy wave function $|\rho_0^{YM}\rangle = \mathrm{const}\times e^{-W_{YM}}|0\rangle$.

As an illustration of these statements, let us consider Euclidean configurations in Bianchi type IX and Kantowski-Sachs EYM systems.

Bianchi type IX system. The solutions of Hamilton-Jacobi equation (53), which correspond to the gravitational part of both possible superpotentials $W_{gr(BIX_{(1)})} = \frac{1}{2}(2b_1^2 + b_3^2)$ and $W_{gr(BIX_{(2)})} = \frac{1}{2}(b_3^2 - 4b_1 b_3)$, have been discussed by Gibbons and Pope [23]. For our purposes we would like to mention some of them using a slightly different notation.

One of the solutions of Eq. (53) with the normalizable superpotential $W_{gr(BIX_{(1)})}$ turns out to be the (anti-)self-dual Eguchi-Hanson [24] metric which has the form

$$ds^2 = f^2 dr^2 + \frac{r^2}{4}[(\omega^1)^2+(\omega^2)^2] + \frac{r^2}{4}f^{-2}(\omega^3)^2, \tag{54}$$

with

$$f^2 = \left[1 - \left(\frac{a}{r}\right)^4\right]^{-1}, \tag{55}$$

and ω^i is determined by Eqs. (36). In order to bring this metric to the form (32), one should introduce the ''Euclidean time'' r as

$$dt=\left[1-\left(\frac{a}{r}\right)^4\right]^{-1/2}dr. \tag{56}$$

The Eguchi-Hanson metric has vanishing Euclidean action $S_{EH}^{gr}=0$, which is completely determined by its surface contribution [25], since the volume contribution is canceled identically ($R=0$ ''on shell'') for EYM systems.

Inserting the expression for the metric functions into the Hamilton-Jacobi equations for the Yang-Mills part of the superpotential $W_{YM(BIX)}=-\alpha^2(\gamma-1)+\frac{1}{2}\gamma^2$ and differentiating with respect to the introduced variable r one obtains the system

$$\dot{\alpha}=\frac{2}{r}f^2(\alpha\gamma-\alpha), \tag{57}$$

$$\dot{\gamma}=\frac{2}{r}(\alpha^2-\gamma), \tag{58}$$

which are the self-duality YM equations on an Eguchi-Hanson background solved by the family of instanton solutions [26]

$$\alpha=\frac{a_1\sinh(\rho)}{\sinh[a_1(\rho+a_2)]},$$

$$\gamma=a_1\tanh(\rho)\coth[a_1(\rho+a_2)],\quad \frac{r^2}{a^2}=\coth(\rho), \tag{59}$$

with the action $S_{EH}^{YM}=8\pi^2(a_1^2-1)/2$ for $a_1>1$, $a_2=0$, and $S_{EH}^{YM}=8\pi^2a_1^2/2$ for $a_1>1, 0<a_2<\infty$, where a_1 and a_2 are the constants of integration.

The extremal Euclidean configurations, produced by the non-normalizable superpotential $W_{gr(BIX_{(2)})}$, are self-dual Taub-NUT (Newman-Unti-Tamburino) gravitational instantons with nonvanishing action [27]; similarly, the YM part of the superpotential gives rise to the self-dual YM instantons [28] on a Taub-NUT background.

The KS system. For the EYM system in Kantowski-Sachs space-time with $W_{gr(KS)}=2b_1b_3$ the gravitational degrees of freedom b_1 and b_3 obey the following self-duality equations:

$$\dot{b}_1b_3+\dot{b}_3b_1=b_3, \tag{60}$$

$$\dot{b}_1=1, \tag{61}$$

satisfied by

$$b_1=t \quad \text{and} \quad b_3=1, \tag{62}$$

which is nothing more than the flat Euclidean R^4 space-time metric with r and t interchanged. From the Yang-Mills part of Eq. (53) with $W_{YM(KS)}=-\gamma(\alpha^2-1)$ one obtains the usual YM (anti-)self-duality equations in R^4, written in the ''polar'' coordinates

$$\dot{\alpha}=\alpha\gamma, \tag{63}$$

$$\dot{\gamma}t^2=\alpha^2-1, \tag{64}$$

with the well-known family of YM instanton solutions, having the topological charge $k=1$ [29]:

$$\gamma=\dot{\psi} \quad \text{and} \quad \alpha=e^{\psi}\dot{g}, \tag{65}$$

where

$$\psi=-\ln\left(\frac{1-g^2}{2t}\right),\quad g=\left(\frac{a_1-t}{a_1+t}\right)\left(\frac{a_2-t}{a_2+t}\right). \tag{66}$$

Note that the dimension of moduli space $\mathcal{M}$ of SU(2) Yang-Mills instantons with a topological charge k on a given Riemannian $4-D$ manifold $\overline{\mathbf{M}}$ [which has first Betti number c_1 and the dimension c_2^- of the maximal submanifold in cohomologies $H^2(\overline{\mathbf{M}},R)$ where the corresponding intersection form is negatively defined] is [30]

$$\dim(\mathcal{M}_{\mathrm{SU}(2)})=8k-3(1-c_1+c_2^-), \tag{67}$$

and in the simplest Kantowski-Sachs case with $\overline{\mathbf{M}}=R^4(k=1, c_1=c_2^-=0)$ is equal to 5. In the framework of our approach, since we quantize the system reduced to one dimension, only some of these instantons are taken into account. In fact, we deal with the subclass of all possible YM instantons, originating from the chosen *Ansätze*, which share the space-time symmetries in the Lorentzian sector. However, their contribution breaks the supersymmetry fatally in conformity with the general expectations, as should take place in a full $4-D$ quantum theory.

To summarize, it is shown that the spontaneous supersymmetry breaking which takes place if the Yang-Mills field is added to pure gravity is caused in a quasiclassical approach by a YM instanton contribution to the wave function. This contribution, in accordance with general expectations, provides the energy shift ΔE from a zero level. To estimate this energy shift for EYM systems an instanton calculation technique can be used, which also should give the possibility to find the lowest level normalizable wave function $|\rho^1\rangle, H|\rho_1^{EYM}\rangle=\Delta E|\rho_1^{EYM}\rangle$ for the considered models. This work is in a progress now.

V. CONCLUSIONS

We would like to conclude with the following remarks. The $N=2$ SUSY quantum mechanical sigma-model approach allows us to obtain conserved supercharges as being Hermitian adjoint to each other, along with the self-adjoint expressions for the Hamiltonian and Lagrangian for any signature of a sigma-model metric. This gives the possibility to use the supersymmetry as a tool for the quantization of various homogeneous systems coupled with gravity if they can be embedded into the considered $N=2$ SUSY sigma model. The desired embedding has been done for coupled SU(2) EYM systems in some cosmological models which admit explicit expressions for the superpotentials as being direct sum of gravitational and Yang-Mills parts. After the quantization the only nontrivial zero-energy wave functions in null and filled fermion sectors turns out to have a diverging norm and this fact indicates spontaneous breaking of supersymmetry, caused by YM instantons.

ACKNOWLEDGMENTS

We would like to thank E. A. Ivanov, G. Jackeli, S. O. Krivonos, A. P. Nersessian, M. S. Volkov, and especially A. I. Pashnev for helpful discussions and comments. This work was supported in part by Russian Foundation for Basic Research Grant 96-02-18126. The work of M.M.T. was also supported in part by Grant INTAS-96-0308.

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