Inflation, singular instantons, and eleven dimensional cosmology

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We investigate cosmological solutions of eleven dimensional supergravity compactified on a squashed seven manifold. The effective action for the four dimensional theory contains scalar fields describing the size and squashing of the compactifying space. The potential for these fields consists of a sum of exponential terms. At early times only one such term is expected to dominate. The condition for an exponential potential to admit inflationary solutions is derived and it is shown that inflation is not possible in our model. The criterion for an exponential potential to admit a Hawking-Turok instanton is also derived. It is shown that the instanton remains singular in eleven dimensions. [S0556-2821(98)01024-8]

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I. INTRODUCTION

Until recently it was thought that slow-roll inflation always gives rise to a flat universe. This assumption was proved incorrect in [1,2] (building on the earlier work of [3] and [4] on bubble nucleation and ''old'' inflation) where it was demonstrated that an open universe can arise after quantum tunneling of a scalar field initially trapped in a false vacuum. (One can also obtain open inflation in two field models [5,6].) However, such models of open inflation appear rather contrived owing to the special form that the scalar potential must be assumed to take. They also do not address the problem of the initial conditions for the universe, i.e., no explanation is given of how the scalar field became trapped in the false vacuum. These two objections were confronted in [7] within the framework of the ''no boundary proposal'' [8]. It was described there how an open universe could be created without assuming any special form for the potential. The approach was to construct an instanton (i.e., a solution to the Euclidean field equations) and analytically continue to Lorentzian signature. The novel feature of the instanton is that it is singular although the singularity is sufficiently mild for the instanton to possess a finite action. Several objections have been raised against the use of such instantons, the most serious of which is Vilenkin's argument [9] that if such instantons are allowed then flat space should be unstable to the nucleation of singular bubbles. Another objection is that the singularity can be viewed as a boundary of the instanton (there is a finite contribution to the action from the boundary [9]) which is unacceptable according to the no boundary proposal.

There have been three different approaches to dealing with the problems raised by a singular instanton. The first is to regularize the singularity with matter in the form of a membrane [10,11]. An alternative approach [12] is to analytically continue the instanton across a deformed surface that does not include the singularity. The problem with this is that the surface does not have vanishing second fundamental form which means that one obtains a region of spacetime which does not have purely Lorentzian signature. It was pointed out that this region is not in the open universe and so it may not have observable consequences. The third approach, due to Garriga [13], is to construct a four dimensional singular instanton from a higher dimensional non-singular one. This approach is particularly appealing because *M*-theory is eleven dimensional. Garriga gives a non-singular five dimensional instanton that reduces to Vilenkin's in four dimensions but with a cutoff to the scale of bubble nucleation that makes the decay rate of flat space unobservably small. He also gives a five dimensional solution with cosmological constant that reduces to a four dimensional instanton of Hawking-Turok type. (Garriga's five dimensional instantons are just Euclidean Schwarzschild and the five sphere respectively.) One purpose of this paper is to examine whether it is possible to obtain Hawking-Turok instantons in four dimensions from non-singular instantons of eleven dimensional supergravity, the low energy limit of *M*-theory.

Our second aim is to investigate whether solutions of eleven dimensional supergravity corresponding to four dimensional inflating universes exist. Since inflation is now widely accepted as the standard explanation of several cosmological problems (see e.g. [14]), one would expect the existence of inflationary solutions of *M*-theory if it is indeed the correct theory of everything. However, compactifications of $D=11$ supergravity usually give a *negative* cosmological constant (see [15]) which is precisely the opposite of what we need for inflation. The reason for this is that if the compactifying space has positive curvature then the field equations imply that our space has negative curvature. This suggests that a way around the problem may be to look for solutions with the seven dimensional compactifying space M_7 negatively curved at early times but positively curved at late times. We do this by taking M_7 to be a coset space and squashing it (the meaning of squashing is explained below), treating the squashing parameters as dynamical scalar fields.

Upon reduction to four dimensions we obtain a model with scalar fields evolving according to a potential consisting of a sum of exponential terms. At early times only one term in the sum is expected to be significant. Cosmological solutions involving scalar fields with exponential potentials have been investigated by several workers. Lucchin and Matarrese [16] showed that power-law inflation can result from such potentials. This was further investigated by Barrow [17] who gave an exact scaling solution to the equations of motion

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which was subsequently generalised by Liddle [18]. Halliwell [19] has conducted a phase-plane analysis of the equations of motion resulting from an exponential potential. Wetterich has derived scaling solutions for cosmologies with the scalar field coupled to other matter [20]. For the single scalar case we have found a first integral of the equations of motion and give an exact expression for the number of inflationary *e*-foldings. It is found that a significant inflationary period only results from solutions that approach the scaling solutions at late times. The results are generalized to the multi-scalar case. We have analyzed the behavior of scalars with an exponential potential near the singularity of the instanton and give a criterion for the singularity to be integrable. (This was discussed in [21] but the analysis was incomplete.)

Applying the results on exponential potentials to our model from eleven dimensions yields the disappointing result that the potential is too steep for inflation to occur. We find that unlike in Garriga's models the instanton is singular in eleven dimensions. The reason for this is that Garriga's potential comes from a five dimensional cosmological constant whereas ours comes from the Ricci scalar of the compactifying space and has too steep a dependence on the scalar field that measures the size of the internal space (i.e. its ''breathing'' mode). It is this same dependence that rules out inflationary behavior which leads us to speculate that if one could fix the size of the internal space then a solution with more appealing properties might be found.

As this paper was nearing completion we received a paper by Bremer *et al.* [22] which has some overlap with our work. They also consider cosmological solutions with dynamical squashing in various dimensions. Their S^7 example is not the same as ours: they obtain squashed metrics on S^7 by viewing it as a $U(1)$ bundle over CP^3 and squashing corresponds to varying the length of the $U(1)$ fibers whereas we treat S^7 as a S^3 bundle over S^4 and squashing corresponds to varying the size of the S^3 fibers. Our methods are applicable to any squashed coset space (although we always use the Freund-Rubin ansatz [23]). Integrability of the instanton singularity is not discussed in [22] (indeed the examples discussed there all appear to be non-integrable) and neither is the condition for inflation. (In the conclusions section of [22] it is stated that the instanton solutions can be continued to give open inflationary universes. This is not the case: the potentials are too steep to yield a significant inflationary period.)

Note added in proof. Cosmological equations resulting from compactifying an S^7 with dynamical squashing and a non-zero four form on S^7 were derived in [28] but solutions of these equations were not discussed.

II. ELEVEN DIMENSIONAL SUPERGRAVITY

The action for the bosonic sector of $D=11$ supergravity is [15]

$$\hat{S}=\int d^{11}x\sqrt{-\hat{g}}\left(\frac{1}{2\hat{\kappa}^2}\hat{R}-\frac{1}{48}\hat{F}_{MNPQ}\hat{F}^{MNPQ}+\frac{\sqrt{2}\hat{\kappa}}{(12)^4}\frac{1}{\sqrt{-\hat{g}}}\epsilon^{M_1\cdots M_{11}}\hat{F}_{M_1\cdots M_4}\hat{F}_{M_5\cdots M_8}\hat{A}_{M_9\cdots M_{11}}\right)+S_{\text{boundary}}. \tag{2.1}$$

Carets will be used to distinguish eleven dimensional quantities from four dimensional ones. Uppercase Roman letters will be used for eleven dimensional indices and lowercase Greek letters for four dimensional ones. $\hat{\kappa}^2=8\pi\hat{G}$ is the eleven dimensional Planck scale. We will use a positive signature metric and a curvature convention such that a sphere has positive Ricci scalar. $\epsilon^{M_1\cdots M_{11}}$ is the alternating tensor density. The four form $\hat{F}_{MNPQ}$ is related to its three form potential $\hat{A}_{MNP}$ by

$$\hat{F}_{MNPQ}=4\partial_{[M}\hat{A}_{NPQ]} \tag{2.2}$$

where square brackets denote antisymmetrization.

S_{boundary} is a sum of boundary terms which are essential in quantum cosmology:

$$S_{\text{boundary}}=B_1+B_2, \tag{2.3}$$

where B_1 is the Gibbons-Hawking boundary term [24] and B_2 is needed because we want to consider the Hartle-Hawking wave function [8] as a function of the four-form on the boundary, hence it is the variation of the four form that should vanish on the boundary, not that of the three form. See [12] for a discussion of this point. We shall only consider solutions with a vanishing Chern-Simons term, for which

$$B_2=\frac{1}{6}\int d^{11}x\,\partial_M(\sqrt{-\hat{g}}\hat{F}^{MNPQ}\hat{A}_{NPQ}). \tag{2.4}$$

The equations of motion following from the action (2.1) are

$$\hat{R}_{MN}=\frac{\hat{\kappa}^2}{6}\left(\hat{F}_{MPQR}\hat{F}_N^{PQR}-\frac{1}{12}\hat{F}_{PQRS}\hat{F}^{PQRS}\hat{g}_{MN}\right), \tag{2.5}$$

$$\partial_M(\sqrt{-\hat{g}}\hat{F}^{MNPQ})=-\frac{\hat{\kappa}}{576\sqrt{2}}\epsilon^{NPQM_1\cdots M_8}\hat{F}_{M_1\cdots M_4}\hat{F}_{M_5\cdots M_8}. \tag{2.6}$$

If $F\wedge F\wedge n$ vanishes on the boundary (where n is the 1-form normal to the boundary), then the action is gauge invariant and the second boundary term is

$$B_2=\frac{1}{24}\int d^{11}x\sqrt{-\hat{g}}\hat{F}^{MNPQ}\hat{F}_{MNPQ}. \tag{2.7}$$

III. SQUASHED MANIFOLDS

Given a Lie group G, the manifolds admitting a transitive action of G can be viewed as coset spaces G/H where H is the isotropy subgroup. We are interested in the most general G-invariant metric on such a manifold (i.e. the most general metric for which the left action of G yields a group of isometries). If G/H is isotropy irreducible (see [15]), then there is a unique (up to scale) such metric which is actually an Einstein metric. For example, if $G=SO(8)$ and $H=SO(7)$, then the unique G-invariant metric on G/H is the round metric on S^7. If the coset space is not isotropy irreducible, then the general G-invariant metric contains arbitrary parameters. This is what is meant by squashing. An example is $G=SO(5)$ and H the $SO(3)$ subgroup such that G/H has S^7 topology. The most general G-invariant metric contains seven arbitrary parameters and there are *two* G-invariant Einstein metrics.

It is discussed in [25] how one can squash a coset space by rescaling the vielbein i.e. $e^a \to \lambda_a e^a$ (no summation). A criterion is given for deciding if a particular rescaling will preserve the isometry group of the metric. This is the most general kind of deformation that preserves the G-invariant metric on G/H (see [26] for a review).

IV. DIMENSIONAL REDUCTION WITH DYNAMICAL SQUASHING

Our metric ansatz is

$$d\hat{s}^2 = e^{2B(x)} g_{\mu\nu}(x) dx^\mu dx^\nu + g_{mn}(x,y) dy^m dy^n, \quad (4.1)$$

where x^μ and y^m are coordinates on the four and seven dimensional manifolds with metrics $g_{\mu\nu}$ and g_{mn} respectively. We shall choose the field $B(x)$ so that the reduced action is in the Einstein frame. The siebenbein on the internal manifold is assumed to be

$$e_m^a(x,y) = e^{A_a(x)} \bar{e}_m^a(y) \quad \text{(no summation)}, \quad (4.2)$$

where $\bar{g}_{mn}(y) = \Sigma_a \bar{e}_m^a(y) \bar{e}_n^a(y)$ is the unsquashed metric. The squashing is described by the seven scalar fields $A_a(x)$. Note that for squashing in the sense described above (i.e. preserving the isometry group) these scalar fields will not be independent.

In the following discussion we shall not specify a particular squashed coset for M_7. The choice is not arbitrary: the eleven dimensional field equations have to be satisfied. We shall assume that a suitable coset has been found but our conclusions will be independent of the details of the internal space. As an example we shall consider S^7 as a $SO(5)/SO(3)$ coset with a two parameter family of metrics (i.e. only two of the A_a are independent), one parameter being the size and the other the squashing. This choice does satisfy the $D=11$ field equations.

$B(x)$ is calculated by observing

$$\sqrt{-\hat{g}}\hat{R} = \sqrt{-g}\sqrt{\bar{g}_7} e^{\Sigma A_a(x)} e^{2B(x)} (R + \cdots), \quad (4.3)$$

so after integrating over y^m the reduced action will be in the Einstein frame provided that

$$B(x) = -\frac{1}{2} \sum_a A_a(x). \quad (4.4)$$

In the Einstein frame the components of the eleven dimensional Ricci tensor are (see Appendix A)

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} \nabla^2 \left(\sum_a A_a \right) g_{\mu\nu} - \frac{3}{2} \sum_a A^a_{,\mu} A^a_{,\nu} - \frac{1}{2} \sum_{a \neq b} A^a_{,\mu} A^b_{,\nu}, \quad (4.5)$$

$$\hat{R}_{ab} = R_{ab}[M_7] - e^{\Sigma A_a} \operatorname{diag}(\nabla^2 A_a). \quad (4.6)$$

The four dimensional part has been written with curved indices and the seven dimensional part with tangent space indices for notational clarity.

We shall use the Freund-Rubin ansatz [23] for the four form i.e.

$$\hat{F}_{\mu\nu\rho\sigma}(x,y) = F_{\mu\nu\rho\sigma}(x) \quad \text{other components vanish.} \quad (4.7)$$

(Some other ansätze for the four form were considered in [22].)

One can substitute these ansätze into the field equations to obtain equations of motion for the effective four dimensional theory. Alternatively one can obtain the same equations by varying the reduced action obtained by substituting into Eq. (2.1):

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{4\kappa^2} \sum_{a,b} (\partial A_a) M_{ab} (\partial A_b) + \frac{1}{2\kappa^2} \exp\left(-\sum A_a \right) R[M_7] - \frac{1}{48} \exp\left(3 \sum A_a \right) F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \right] + B, \quad (4.8)$$

where B is a boundary term and indices are raised with $g^{\mu\nu}$. The integration over y^m gives a volume factor $V_7 = \int d^7y \sqrt{\bar{g}_7}$ which is absorbed into the definition of the four dimensional Planck scale: $\hat{\kappa}^2 = \kappa^2 V_7$. A factor of $\sqrt{V_7}$ has also been absorbed into $F_{\mu\nu\rho\sigma}$. The matrix M_{ab} has 3's on its diagonal and 1's everywhere else. $R[M_7]$ is the (seven dimensional) Ricci scalar of the internal space computed treating $A_a(x)$ as constant parameters.

Note that there is no guarantee that solutions of the four dimensional equations of motion obtained from this action are solutions of the eleven dimensional field equations. This is because we have not considered the field equation associated with the seven internal dimensions. However, if one chooses a coset such that this field equation can be satisfied, then the resulting equations of motion will be the same as those obtained from the reduced action. In our S^7 example the Ricci tensor of the internal space (see Appendix B) splits into two independent diagonal parts. This will give two independent field equations. In order to satisfy them (at least for non-constant scalar fields), we must include at least two degrees of freedom in the metric on S^7. So in addition to squashing S^7 we allow its size to vary. This is achieved by multiplying its metric by an overall conformal factor $e^{2C(x)}$. Then the scalars A_a are given by

$$A_1=A_2=A_3=A_4=C, \quad A_5=A_6=A_7=A+C, \quad (4.9)$$

where e^A is the squashing parameter defined in Appendix B. The two field equations coming from the internal space give the equations of motion for A and C. The same equations of motion can be obtained from the four dimensional reduced action.

Returning to the general case, the kinetic term can be diagonalized by defining

$$\phi_k=\frac{1}{\kappa\sqrt{k(k+1)}}\left(\sum_{j=1}^{k} A_j-kA_{k+1}\right), \quad k=1,\ldots,6, \quad (4.10)$$

$$\psi=\frac{3}{\kappa\sqrt{14}}\sum_{j=1}^{7} A_j . \quad (4.11)$$

If the scalar fields A_a are not linearly independent, then the fields ϕ_k will not be independent and the kinetic terms will still not be correctly normalized. This occurs in our squashed S^7 example:

$$\phi_1=\phi_2=\phi_3=0,$$

$$\sqrt{4.5}\phi_4=\sqrt{5.6}\phi_5=\sqrt{6.7}\phi_6=-\frac{4A}{\kappa},$$

$$\psi=\frac{3}{\kappa\sqrt{14}}(3A+7C). \quad (4.12)$$

Since ϕ_4, ϕ_5 and ϕ_6 are not independent, we define $\phi=\sqrt{(12/7)}(A/\kappa)$ so that

$$\frac{1}{2}(\partial\phi_4)^2+\frac{1}{2}(\partial\phi_5)^2+\frac{1}{2}(\partial\phi_6)^2=\frac{1}{2}(\partial\phi)^2; \quad (4.13)$$

so now the scalar fields A and C have been replaced by ϕ and ψ with diagonal kinetic terms.

Note that a scaling of the internal manifold $A_a(x)\to A_a(x)+C(x)$ only affects ψ, which measures its size. In general one must allow the size to vary in order to satisfy the $D=11$ field equations (i.e. one could not impose $\psi=$const except in special cases corresponding to static solutions); hence ψ and ϕ_k will be independent. Thus the kinetic term for ψ is correctly normalized and ψ will not need rescaling. It is this that will allow us to draw general conclusions later on about the possibility of inflation or higher dimensional non-singular instantons in our model.

The inverse transformation relating A_j to ϕ_k and ψ is

$$A_j=\kappa\left(-\sqrt{\frac{j-1}{j}}\,\phi_{j-1}+\sum_{k=j}^{6}\frac{1}{\sqrt{k(k+1)}}\,\phi_k+\frac{\sqrt{14}}{21}\,\psi\right). \quad (4.14)$$

Substituting into the reduced action gives

$$S=\int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R-\sum_{k=1}^{6}\frac{1}{2}(\partial\phi_k)^2-\frac{1}{2}(\partial\psi)^2 - W(\phi_k,\psi)-\frac{1}{48}e^{\kappa\sqrt{14}\psi}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}\right)+B \quad (4.15)$$

where the scalar potential is

$$W(\phi_k,\psi)=-\frac{1}{2\kappa^2}e^{(-\sqrt{14}/3)\kappa\psi}R[M_7]. \quad (4.16)$$

The equations of motion following from this action are

$$R_{\mu\nu}=2\kappa^2\left[\sum_k\frac{1}{2}\partial_\mu\phi_k\partial_\nu\phi_k+\frac{1}{2}\partial_\mu\psi\partial_\nu\psi+\frac{1}{2}Wg_{\mu\nu} +\frac{1}{12}e^{\kappa\sqrt{14}\psi}\left(F_{\mu\rho\sigma\tau}F_\nu^{\rho\sigma\tau}-\frac{3}{8}F_{\lambda\rho\sigma\tau}F^{\lambda\rho\sigma\tau}g_{\mu\nu}\right)\right], \quad (4.17)$$

$$\nabla^2\phi_k=\frac{\partial W}{\partial\phi_k}, \quad (4.18)$$

$$\nabla^2\psi=\frac{\partial W}{\partial\psi}+\frac{\kappa\sqrt{14}}{48}e^{\kappa\sqrt{14}\psi}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}, \quad (4.19)$$

$$\partial_\mu(\sqrt{-g}e^{\kappa\sqrt{14}\psi}F^{\mu\nu\rho\sigma})=0. \quad (4.20)$$

Note that the final equation is obtained by varying $A_{\mu\nu\rho}$. This equation has the unique solution

$$F_{\mu\nu\rho\sigma}=\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}e^{-\kappa\sqrt{14}\psi}F, \quad (4.21)$$

for some constant F. If we substitute this solution into Eq. (4.19), then we get

$$\nabla^2\psi=\frac{\partial V}{\partial\psi}, \quad (4.22)$$

where

$$V(\phi_k,\psi)=W(\phi_k,\psi)+\frac{1}{2}F^2e^{-\kappa\sqrt{14}\psi} \quad (4.23)$$

is the effective potential that determines the evolution of the field ψ. Note that we can replace W by V in the field equation for ϕ_k and substitute the solution for $F_{\mu\nu\rho\sigma}$ into Eq. (4.17) to yield

$$R_{\mu\nu}=2\kappa^2\left(\sum_k\frac{1}{2}(\partial_\mu\phi_k)(\partial_\nu\phi_k) +\frac{1}{2}(\partial_\mu\psi)(\partial_\nu\psi)+\frac{1}{2}Vg_{\mu\nu}\right); \quad (4.24)$$

so now V occurs in all of the equations of motion and one can forget about W.

For our squashed S^7 example, the potential is

$$V(\phi,\psi)=-\frac{1}{2\kappa^2}\,e^{-3\sqrt{14}\kappa\psi/7}\left(\frac{3}{2}\,e^{-4\sqrt{21}\kappa\phi/21}+12e^{\sqrt{21}\kappa\phi/7}-3e^{10\sqrt{21}\kappa\phi/21}\right)+\frac{1}{2}\,F^2e^{-\sqrt{14}\kappa\psi}. \tag{4.25}$$

Plotted as a function of ϕ this tends to $\pm\infty$ as $\phi\rightarrow\pm\infty$. There is a local minimum at $\phi=0$ corresponding to the round metric on the S^7. There is also a local maximum at a negative value of ϕ corresponding to the squashed Einstein metric on S^7. The qualitative behavior as ψ varies depends on the value of ϕ. There is a positive constant ϕ_0 such that at $\phi=\phi_0$ the F-independent part of the potential vanishes. For $\phi>\phi_0$, V is a monotonically decreasing function of ψ tending to $+\infty$ as $\psi\rightarrow-\infty$ and to 0 as $\psi\rightarrow+\infty$. For $\phi<\phi_0$ (which includes the two Einstein metrics), the asymptotic behavior is similar but there is a local minimum at some value of ψ corresponding to a negative value of V. Hence there exist static solutions of $D=11$ supergravity with ψ sitting at this minimum and ϕ corresponding to either the round or the squashed Einstein metric. These have been extensively discussed from the point of view of Kaluza-Klein theory [15]. We are interested in solutions with a positive potential at early times in the hope that these may exhibit inflationary behavior. Such solutions start with ϕ large and positive, corresponding to a negatively curved metric on S^7. One would expect solutions to exist in which ϕ rolls down to the local minimum so that the solution settles into the Freund-Rubin solution [23] $\mathrm{AdS}_4\times S^7$ (with a round metric) at late times. Note that this solution appears unstable because ϕ can tunnel past the local maximum and roll off to $-\infty$. However, Breitenlohner and Freedman [27] have shown how boundary conditions at infinity can stabilize AdS space, at least against small perturbations; so one would expect a similar argument to be valid here.

A second example that we have considered involves taking the compactifying space to be $S^1\times S^3\times S^3$. Here S^3 is a group manifold, and so one can squash all three directions independently [15]. Thus this M_7 can be squashed with all seven A_a independent. The $D=11$ field equations can be satisfied with this M_7. The Ricci scalar of S^3 with squashing described by A_5, A_6 and A_7 is

$$R=e^{-2A_5}+e^{-2A_6}+e^{-2A_7}-\frac{1}{2}\left(e^{2(A_5-A_6-A_7)}+e^{2(A_6-A_7-A_5)}+e^{2(A_7-A_5-A_6)}\right). \tag{4.26}$$

Thus in order to get a negatively curved S^3 the second group of terms must dominate the first: Note that there is no static Freund-Rubin solution in this case because $S^1\times S^3\times S^3$ cannot be given an Einstein metric. Thus the potential V does not have any extrema.

V. CONDITION FOR INFLATION

We shall now seek a solution of the field equations derived above that describes a four dimensional universe. Spatial homogeneity and isotropy imply that the metric must take the form

$$ds^2=-dt^2+a(t)^2ds_3^2 \tag{5.1}$$

where ds_3^2 is the line element of a three-space of constant curvature. Substituting this into Eq. (4.17) yields the equations

$$\left(\frac{\dot a}{a}\right)^2=\frac{\kappa^2}{3}\left(\sum_k\ \frac{1}{2}\,\dot\phi_k^2+\frac{1}{2}\,\dot\psi^2+V\right)-\frac{k}{a^2}, \tag{5.2}$$

$$\ddot a=-\frac{\kappa^2}{3}\,a\left(\sum_k\ \dot\phi_k^2+\dot\psi^2-V\right). \tag{5.3}$$

k is the sign of the curvature of the spatial sections.

Inflation is defined by $\ddot a>0$; so Eq. (5.3) implies that for inflation we need

$$V>\sum_k\ \dot\phi_k^2+\dot\psi^2. \tag{5.4}$$

Since the potentials we have obtained consist of a sum of exponential terms with no extrema of ψ at positive values of the potential, one would expect that if inflation occurs then it would do so at a large value of the potential where typically only one exponential is significant. For simplicity we shall consider a single scalar model

$$V(\phi)=V_0e^{\alpha\kappa\phi}. \tag{5.5}$$

The equation of motion for the scalar field is

$$-\ddot\phi-3H\dot\phi=\frac{dV}{d\phi} \tag{5.6}$$

where the Hubble parameter is $H(t)=\dot a/a$. We shall assume that $\alpha>0$, which can always be achieved by reversing the signs of α and ϕ.

For inflation we need V to be larger than the scalar kinetic term and curvature term (if $k\neq 0$); so one would expect the Hubble parameter to behave like $e^{\alpha\phi/2}$. (We have set $\kappa=1$.) After substituting this into the equation of motion for the scalar field it is natural to seek a solution of the form $\dot\phi\propto e^{\alpha\phi/2}$. With this in mind, define a new variable by

$$\Phi=-\dot\phi e^{-\alpha\phi/2}. \tag{5.7}$$

If one replaces $\ddot\phi$ by $(d/d\phi)[(1/2)\dot\phi^2]$, eliminates $\dot\phi$ in favor of Φ and neglects the curvature term (this is the only approximation that we shall make), then one obtains

$$\frac{1}{2}\,\frac{d}{d\phi}\,\Phi^2+\frac{\alpha}{2}\,\Phi^2-\sqrt{3\left(\frac{1}{2}\,\Phi^2+V_0\right)}\,\Phi+\alpha V_0=0. \tag{5.8}$$

Now define x by $\Phi=\sqrt{2V_0}\,\sinh x$ to give

$$\frac{dx}{d\phi}=\frac{1}{2}\,(\sqrt{6}-\alpha\,\coth x). \tag{5.9}$$

It is obvious that there is a solution with $x=\text{const}$ when $\alpha<\sqrt{6}$. This is the solution obtained previously by Barrow [17] and Liddle [18]. However, we can investigate the general solution by using the change of variable $y=e^{2x}$ to give

$$e^{\alpha(\phi-\phi_*)/2}=F(y)\equiv y^{\alpha/2(\sqrt{6}+\alpha)}\left|y-\frac{\sqrt{6}+\alpha}{\sqrt{6}-\alpha}\right|^{\alpha^2/(6-\alpha^2)}. \tag{5.10}$$

ϕ_* is a constant of integration. If we are interested in real solutions obtained by analytical continuation from a Euclidean instanton, then we must impose the initial conditions

$$\phi=\phi_0,\quad \dot{\phi}=0\Rightarrow y=1. \tag{5.11}$$

In the model of open inflation described in [7], one analytically continues the instanton to an open universe at a point where the scale factor vanishes. This means that initially it is not a good approximation to neglect the curvature term in the Einstein constraint equation as we have done here. However, if there is a significant inflationary period, then the curvature term will rapidly become negligible. So, strictly speaking, our analysis is only applicable after this term has become negligible, by which time the above boundary conditions will not hold (since then $\dot{\phi}<0$ and so $y>1$). However, as we shall show, the condition for a significant period of inflation is not sensitive to the initial value of $\dot{\phi}$, and so we shall take the above value for simplicity. Of course our results are exact for flat $(k=0)$ universes.

With these boundary conditions, Eq. (5.10) becomes

$$e^{\alpha(\phi-\phi_0)/2}=\frac{F(y)}{F(1)}. \tag{5.12}$$

The condition for inflation is $\dot{\phi}^2<V$ or, equivalently, $\sinh^2 x<1/2$. This is satisfied if, and only if,

$$2-\sqrt{3}<y<2+\sqrt{3}. \tag{5.13}$$

Since we are starting with $\dot{\phi}=0$, we will always get *some* inflation. How much we get depends on $F(y)$, the qualitative behavior of which depends on the magnitude of α. There are two cases to consider: (i) $\alpha^2<6$ and (ii) $\alpha^2>6$. In the first case $F(y)$ has zeros at $y=0$ and $y=y_0\equiv(\sqrt{6}+\alpha)/(\sqrt{6}-\alpha)$ and a local maximum at $y=1$. For large y, $F(y)$ tends to infinity as a power of y. The solution $x=\text{const}$ corresponds to the second zero of $F(y)$ (but this solution is incompatible with the boundary condition $\dot{\phi}=0$ since it corresponds to an eternally inflating universe).

If $\alpha^2<2$, then the second zero of $F(y)$ lies within the range of values of y corresponding to inflation. This implies that the solution will inflate all the way to $F(y)=0$ i.e., to $\phi=-\infty$. For larger α inflation will stop before $F(y)=0$ is reached. In case (ii) the only zero of $F(y)$ is at $y=0$. Once again $F(y)$ has a maximum at $y=1$, beyond which it decreases monotonically to zero as $y\to\infty$.

We have succeeded in finding a first integral for the scalar field equation of motion. This relates ϕ and $\dot{\phi}$ implicitly. It does not seem possible to integrate this again to find an explicit solution for $\phi(t)$ but this is not necessary in order to calculate the number of inflationary efoldings N, defined by

$$N=\int_0^{t_{\max}}H(t)dt, \tag{5.14}$$

where $t_{\max}$ is the (comoving) time at which inflation stops. One can now substitute the expression for $H(t)$ in terms of Φ and ϕ and then substitute for Φ and ϕ in terms of y using the definition of y and Eq. (5.10). To transform the integral over t into an integral over y one needs to know dy/dt, which is obtained by differentiating Eq. (5.10) with respect to t. Here y runs from 1 (when $\dot{\phi}=0$) to $2+\sqrt{3}$ (end of inflation). This gives

$$N=\frac{2}{\alpha\sqrt{3}}\int_1^{2+\sqrt{3}}\left(\frac{y+1}{y-1}\right)\left(\frac{-F'(y)}{F(y)}\right)dy, \tag{5.15}$$

which is infinite if $\alpha<\sqrt{2}$ and otherwise evaluates to

$$N=\frac{\sqrt{6}}{3(\sqrt{6}+\alpha)}\left[\frac{1}{2}\log(2+\sqrt{3})+\frac{\sqrt{6}}{\sqrt{6}-\alpha}\log\left(1+\frac{\sqrt{6}-\alpha}{\sqrt{3}(\alpha-\sqrt{2})}\right)\right]. \tag{5.16}$$

This is small unless α is exponentially close to $\sqrt{2}$. We can conclude that an exponential potential can only give a significant inflationary period if $\alpha\leqslant\sqrt{2}$. Note that the result is independent of ϕ_0 in contrast with the result for power law potentials. As mentioned above, the initial value of $\dot{\phi}$ does not significantly affect the amount of inflation as can be verified by changing the lower limit of integration in Eq. (5.15).

It is easy to calculate the asymptotic behavior of the solutions found above. $\phi\to-\infty$ at late times, and so $F(y)\to 0$. Hence $y\to y_0$ in case (i) and $y\to\infty$ in case (ii). In the first case one has $\Phi\to\Phi_0=\text{const}$; so using the definition of Φ one obtains the solution for $\phi(t)$. The kinetic and potential energy densities and the scale factor have the following asymptotic behavior:

$$\frac{1}{2}\,\dot{\phi}^2=\frac{2}{\alpha^2(t-t_0)^2}, \tag{5.17}$$

$$V=V_0e^{\alpha\phi}=\frac{6-\alpha^2}{\alpha^4}\,\frac{2}{(t-t_0)^2}, \tag{5.18}$$

$$a=a_0(t-t_0)^{2/\alpha^2}. \tag{5.19}$$

It is clear from these expression that it is only consistent to neglect the curvature term in the Einstein constraint equation at large times if $\alpha^2<2$; otherwise these results are restricted to flat $(k=0)$ cosmologies.

In case (ii), $y\rightarrow\infty$ implies $x\rightarrow\infty$. Substituting this into Eq. (5.9) and solving gives the following:

$$\frac{1}{2}\dot{\phi}^2=\frac{1}{3(t-t_0)^2}, \tag{5.20}$$

$$V=V_0e^{\alpha\phi}=\frac{1}{(t-t_0)^{2\alpha/\sqrt{6}}}, \tag{5.21}$$

$$a=a_0(t-t_0)^{1/3}. \tag{5.22}$$

Note that the potential energy density is negligible compared with the kinetic energy density as $t\rightarrow\infty$ (indeed this asymptotic solution may be obtained by simply neglecting V in the field equations). The curvature term in not negligible in this case, and so these results are only valid for $k=0$.

If we include extra scalar fields but still assume that the potential is dominated by a single exponential term $V_0e^{\alpha\phi}e^{\beta\psi}$, then the situation just gets worse because this potential must now dominate two kinetic terms to yield inflation. One can make progress analytically by defining $\Theta=\beta\phi-\alpha\psi$. Then the equations of motion for ϕ and ψ imply that θ obeys

$$\ddot{\Theta}+3H\dot{\Theta}=0, \tag{5.23}$$

where the Hubble parameter is given by the Einstein constraint equation with two scalar fields. This equation can be integrated to give

$$\dot{\Theta}=A\,\exp\left(-3\int^t H(t')dt'\right), \tag{5.24}$$

where A is a constant. If the scale factor grows sufficiently fast, then this term will be asymptotically negligible and $\Theta\approx$const will be a good approximation. Then we can write $\psi=(\beta/\alpha)\phi+$const. Now define

$$\theta=\frac{\sqrt{\alpha^2+\beta^2}}{\alpha}\,\phi \tag{5.25}$$

and the equations of motion reduce to those for a single scalar field θ moving in an exponential potential with parameter $\sqrt{\alpha^2+\beta^2}$. We can now apply the results derived above to give the asymptotic behavior of $\theta(t)$ and the scale factor. The asymptotic behavior of $\dot{\Theta}$ can then be found: when $\alpha^2+\beta^2<6$, $\dot{\Theta}\propto(t-t_0)^{-6/(\alpha^2+\beta^2)}$ and otherwise $\dot{\Theta}\propto(t-t_0)^{-1}$. In both cases, $\dot{\phi}$ and $\dot{\psi}$ are proportional to $(t-t_0)^{-1}$, and so only when $\alpha^2+\beta^2<6$ is it consistent to neglect $\dot{\Theta}$. In this case one simply applies the single scalar result with parameter $\sqrt{\alpha^2+\beta^2}$ to concluded that the solution is still inflating at large times only when $\alpha^2+\beta^2<2$. By analogy with the single scalar results one would only expect a significant inflationary period from such solutions.

If $\alpha^2+\beta^2>6$, then an asymptotic solution (for $k=0$) can be found in analogy with the single scalar case by neglecting V. The scalar equations of motion can be immediately integrated and the result plugged into the Einstein constraint equation to give the scale factor. The results are similar to the single scalar case. Our results can obviously be generalized when there are more than two scalar fields present.

We can now return to our model obtained from eleven dimensional supergravity. The final (F) term in V is too steep to drive inflation, and so we turn to the term coming from the (seven dimensional) Ricci scalar. This depends upon the specific internal manifold that we choose to squash but it is possible to extract the ψ dependence in the general case. To see this, note that the scalars A_j all have the same dependence on ψ Eq. (4.14); hence the metric on the internal space depends on ψ only through the conformal factor $e^{2\sqrt{14}\psi/2l}$. It follows that the dependence of the first term in V on ψ is given by a factor $e^{(-3\sqrt{14})\psi/7}$. This multiplies a ϕ_k dependent piece $\tilde{V}$. Since $(3\sqrt{14}/7)^2>2$, the above work shows that it is not possible to get inflationary behavior driven by a single exponential term in the potential. If inflationary solutions are possible, then they must arise from a combination of several such terms leading to a less steep region of the potential, for example near a local maximum. However, the potential cannot possess a local maximum in ψ and only possesses a local minimum when $\tilde{V}<0$ and this occurs at a negative value for V, which is obviously not suitable for inflation.

VI. SINGULAR INSTANTONS

The behavior of the Hawking-Turok instanton [7] corresponding to an exponential potential can be analyzed in a similar manner. The instanton is assumed to possess an $O(4)$ symmetry; so its metric can be written

$$ds^2=d\sigma^2+b(\sigma)^2d\Omega^2. \tag{6.1}$$

The Euclidean field equations are

$$\left(\frac{b'}{b}\right)^2=\frac{1}{3}\left(\frac{1}{2}\,\phi'^2-V\right)+\frac{1}{b^2}, \tag{6.2}$$

$$b''=-\frac{1}{3}\,(\phi'^2+V)b, \tag{6.3}$$

$$\phi''+3\frac{b'}{b}\,\phi'=\frac{dV}{d\phi}. \tag{6.4}$$

Hawking and Turok consider solutions to these equations that look like deformed four-spheres, regular at the North pole and singular at the South pole. As the singularity is approached they assume that the scalar kinetic term dominates its potential. We shall investigate this assumption for $V=V_0e^{\alpha\phi}$. If the curvature term $1/b^2$ is negligible in the Einstein constraint equation, then near the singularity we can write

$$\frac{b'}{b} = -\sqrt{\frac{1}{3}\left(\frac{1}{2}\phi'^2 - V\right)}. \tag{6.5}$$

Substituting this into the scalar equation of motion and defining $\Phi = \phi' e^{-\alpha\phi/2}$ gives

$$\frac{d}{d\phi}\left(\frac{1}{2}\Phi^2\right) + \frac{\alpha}{2}\Phi^2 - \sqrt{3\left(\frac{1}{2}\Phi^2 - V_0\right)}\Phi - \alpha V_0 = 0. \tag{6.6}$$

Now let $\Phi = \sqrt{2V_0}\cosh x$; so we are assuming that the kinetic term is larger than the potential term (otherwise the above expressions do not make sense). For definiteness, take $x \geqslant 0$. This gives the equation

$$\frac{dx}{d\phi} = \frac{1}{2}(\sqrt{6} - \alpha \tanh x), \tag{6.7}$$

which can be integrated by defining $y = e^{2x}$ to give

$$e^{\alpha(\phi-\phi_*)/2} = G(y) \equiv y^{\alpha/2(\sqrt{6}+\alpha)}\left| y + \frac{\sqrt{6}+\alpha}{\sqrt{6}-\alpha}\right|^{\alpha^2/(6-\alpha^2)}. \tag{6.8}$$

There are two cases to consider: (i) $\alpha < \sqrt{6}$ and (ii) $\alpha > \sqrt{6}$. (Once again we can restrict $\alpha \geqslant 0$ through reversing the signs of α and ϕ.)

Case (i). $G(y)$ is a monotonically increasing function; so y becomes large as ϕ becomes large. Asymptotically we have

$$e^{(\phi-\phi_*)/2} \approx y^{\alpha/2(\sqrt{6}-\alpha)} = e^{[\alpha/(\sqrt{6}-\alpha)]x} \Rightarrow x \approx \frac{1}{2}(\sqrt{6}-\alpha)(\phi-\phi_*). \tag{6.9}$$

This gives

$$\phi' e^{\alpha\phi/2} = \Phi \approx \sqrt{2V_0}\cosh\frac{1}{2}(\sqrt{6}-\alpha)(\phi-\phi_*) \approx \sqrt{\frac{V_0}{2}}\, e^{(\sqrt{6}-\alpha)(\phi-\phi_*)/2}, \tag{6.10}$$

which is a differential equation that we can solve for ϕ to give

$$e^{-\sqrt{6}\phi/2} \approx \sqrt{\frac{3V_0}{4}}\, e^{-(\sqrt{6}-\alpha)\phi_*/2}(\sigma_f - \sigma), \tag{6.11}$$

where σ_f is a constant of integration corresponding to the coordinate of the singularity. The behavior of the potential and kinetic terms near the singularity is

$$V \propto (\sigma_f - \sigma)^{-2\alpha/\sqrt{6}}, \tag{6.12}$$

$$\frac{1}{2}\phi'^2 \approx \frac{1}{3}(\sigma_f - \sigma)^{-2}; \tag{6.13}$$

so near the singularity the kinetic term will dominate. The behavior of the scale factor is easily obtained from the Einstein constraint equation

$$b \approx b_0(\sigma_f - \sigma)^{1/3}. \tag{6.14}$$

b_0 is a constant that is determined by matching the solution near the South pole to the solution at the North pole. Note that it is consistent to neglect the curvature term near the singularity. (These results agree with those obtained in [21] but it was not pointed out there that they are only valid for $\alpha < \sqrt{6}$.)

For the singularity to be integrable, $\int d\sigma b^3 V$ must converge [7] (one must include the boundary term B_2 to derive this result). It is easy to see that it does in this case.

Case (ii). For $\alpha > \sqrt{6}$,

$$G(y) = y^{\alpha/2(\alpha+\sqrt{6})}\left| y - \frac{\alpha+\sqrt{6}}{\alpha-\sqrt{6}}\right|^{-\alpha^2/(\alpha^2-6)}. \tag{6.15}$$

Now $G(y)$ has a singularity at $y = y_0 \equiv (\alpha+\sqrt{6})/(\alpha-\sqrt{6})$ and tends to zero at large y. Thus near the singularity the behavior is $\phi \to \infty$, $y \to y_0$ which implies $x \to x_0$ and $\Phi \to \Phi_0$. This gives a differential equation with solution

$$e^{\alpha\phi/2} \approx \frac{2}{\alpha\Phi_0}(\sigma_f - \sigma)^{-1}. \tag{6.16}$$

Hence near the singularity the potential and kinetic terms behave as follows:

$$V \approx \frac{2}{\alpha^2\cosh^2 x_0}(\sigma_f - \sigma)^{-2} \tag{6.17}$$

$$\frac{1}{2}\phi'^2 \approx \frac{2}{\alpha^2}(\sigma_f - \sigma)^{-2}; \tag{6.18}$$

so now they only differ by a constant factor, which must be taken account of in order to determine the solution for b. Substituting these asymptotic solutions into the Einstein constraint equation and eliminating x_0 in favor of α yields

$$b \approx b_0(\sigma_f - \sigma)^{2/\alpha^2}. \tag{6.19}$$

(Hence it is consistent to neglect the curvature term.) So now we have

$$b^3 V \propto (\sigma_f - \sigma)^{6/\alpha^2 - 2}, \tag{6.20}$$

but $6/\alpha^2 - 2 < -1$, and so the singularity is not integrable in this case.

In the two scalar case, arguments similar to those presented in the previous section show that the singularity is only integrable for $\alpha^2 + \beta^2 < 6$, with obvious generalization to more than two fields.

In our model the potential V contains a term coming from the 4-form. We shall assume that the correct analytic continuation of the 4-form to Euclidean signature is the one that leaves V unchanged. This means that F must be unchanged;

so $F_{\mu\nu\rho\sigma}$ must be imaginary in the Euclidean theory (because $\sqrt{-g}\to i\sqrt{g}$) in agreement with the discussion in [22].

If the F-term is the dominant term in the potential near the singularity, then the above work shows that the singularity is not integrable. Hence for a Hawking-Turok instanton to exist the dominant part of the potential must come from the Ricci scalar of the internal space. If one exponential term $V_0\exp(-3\sqrt{14}\psi/7)\ \exp(\Sigma\lambda_i\phi_i)$ is dominant, then the condition for an integrable singularity is $\Sigma\lambda_i^2<24/7$. This is not satisfied in the case of the squashed S^7 considered above since $(10\sqrt{21}/21)^2>24/7$. Hence the squashed S^7 does not give an integrable singularity. For the $S^1\times S^3\times S^3$ example we need at least one of the three spheres to have negative curvature; so the dominant term must be one of those in the second set of brackets in Eq. (4.26). Without loss of generality we may assume it is the first one. If one writes this in terms of the fields ϕ_k and ψ, then one finds that the exponent is $2\kappa[\sqrt{4/5}\phi_4+\sqrt{6/5}\phi_5+\sqrt{6/7}\phi_6+(\sqrt{14}/21)\psi]$; so the sum of squares of ϕ_k coefficients is $80/7>24/7$ and hence the singularity is not integrable in this case either.

VII. SINGULARITY IN ELEVEN DIMENSIONS

The examples provided by Garriga [13] are encouraging evidence in favor of being able to obtain Hawking-Turok instantons in four dimensions by dimensional reduction of higher dimensional non-singular instantons. However, Garriga's cosmological (as opposed to flat) example is special because it requires the presence of a cosmological constant in the higher dimensional theory. This always gives rise to a potential in the dimensionally reduced action that gives both inflation and an integrable singularity on the instanton. Eleven dimensional supergravity does not have a cosmological constant which is why we have been considering squashing as an alternative mechanism of generating a positive potential in the four dimensional effective action. Unfortunately it is easy to see that the instantons of the type that we have considered remain singular even when viewed from within the higher dimensional framework. If one takes the trace of the eleven dimensional field equations, then one obtains

$$\hat{R}=\frac{\hat{\kappa}^2}{72}\hat{F}_{PQRS}\hat{F}^{PQRS}, \tag{7.1}$$

which, upon substituting the solution for $\hat{F}_{PQRS}$ (and remembering that a factor of $\sqrt{V_7}$ was absorbed into F), becomes

$$\hat{R}=-\frac{\kappa^2}{3}F^2e^{-2\sqrt{14}\kappa\psi/3}. \tag{7.2}$$

Since $\psi\to-\infty$ at the (four dimensional) singularity independently of the sign of $R[M_7]$ (the exponents in V are negative, and so $\psi\to-\infty$ rather than $+\infty$ as considered above), one sees immediately that $\hat{R}\to-\infty$; so the Hawking-Turok singularity is an eleven dimensional curvature singularity.

VIII. DISCUSSION AND CONCLUSIONS

The results of the previous sections are rather disheartening: our aim was to find a non-singular instanton in eleven dimensions that gives rise to a Hawking-Turok instanton in four dimensions and an inflationary period after continuing to Lorentzian signature. Instead we have found that our model realizes neither of these objectives. However, the stumbling block appears to be the same in both cases. Inflation was ruled out because the potential depends too steeply on ψ. The singularity of the eleven dimensional instanton is also due to the dependence on ψ. Note that the eleven dimensional Ricci scalar is independent of ϕ_k, and so if some means were found of fixing the size of the compactifying space (i.e. keeping ψ constant), then the instanton may become non-singular in eleven dimensions even with $\phi_k\to\pm\infty$ (and hence singular in four dimensions). The problem with keeping ψ fixed is in satisfying the eleven dimensional field equations. One would have to introduce extra degrees of freedom in the metric of the compactifying space, which involves going beyond squashing.

It is conceivable that by modifying our ansatz for the four-form more interesting results might be obtained. The work in this paper is only applicable to cosmological solutions using the Freund-Rubin ansatz [23]. Bremer *et al.* [22] consider solutions with some more general four-form configurations. However, neither of their S^7 examples (round or squashed) appear to admit inflationary solutions or instantons with integrable singularities. The Freund-Rubin ansatz is attractive as an explanation of why there are four non-compact spacetime dimensions but leads to a large negative cosmological constant. In [12] it was suggested that this may be balanced by a contribution from supersymmetry breaking. Such symmetry breaking would also have a dynamical effect at early times and might generate corrections to the effective potential that make inflation possible.

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APPENDIX A: DERIVATION OF RICCI TENSOR

The non-zero Christoffel symbols for the metric (4.1) are given in terms of those for the $(D=11)$ metric with $B=0$ by

$$\hat{\Gamma}^{\mu}_{\nu\rho}=\Gamma^{\mu}_{\nu\rho}+\delta^{\mu}_{\nu}B_{,\rho}+\delta^{\mu}_{\rho}B_{,\nu}-B_{,}^{\mu}g_{\nu\rho},$$

$$\hat{\Gamma}^{\mu}_{mn}=-e^{-2B}\sum_a A_{a,}{}^{\mu}e^a_m e^a_n,$$

$$\hat{\Gamma}^{m}_{n\rho}=\sum_a A_{a,\rho}e^m_a e^a_n, \quad \hat{\Gamma}^m_{np}=\Gamma^m_{np}. \tag{A1}$$

Indices are raised with $g_{\mu\nu}$ and e^a_m is the rescaled siebenbein.

The Ricci tensor is most easily computed using normal coordinates in the four dimensional spacetime i.e. $\Gamma^{\mu}_{\nu\rho}=0$.

(Note that we are not free to choose normal coordinates on the whole eleven dimensional manifold because such coordinates will not preserve the product form we have assumed for the metric.) The result is

$$\hat{R}_{\mu\nu}=R_{\mu\nu}-\left[\nabla^2 B+\left(2B_{,\rho}+\sum_a A_{a,\rho}\right)B_{,}^{\rho}\right]g_{\mu\nu}$$
$$-\nabla_\mu\nabla_\nu\left(2B+\sum_a A_a\right)+2B_{,\mu}B_{,\nu}$$
$$+\sum_a (A_{a,\mu}B_{,\nu}+A_{a,\nu}B_{,\mu})-\sum_a A_{a,\mu}A_{a,\nu}, \quad \text{(A2)}$$

$$\hat{R}_{mn}=R_{mn}[M_7]-e^{-2B}$$
$$\times\sum_a e^a_m e^a_n\left[\nabla^2 A_a+A_{a,}{}^{\rho}\left(2B_{,\rho}+\sum_b A_{b,\rho}\right)\right]. \quad \text{(A3)}$$

$R_{mn}[M_7]$ is the Ricci tensor of the squashed internal manifold computed treating the scalars A_a as constants. Note that substantial simplification occurs when we use the Einstein frame, given by Eq. (4.3).

APPENDIX B: THE SQUASHED SEVEN SPHERE

As discussed in Sec. III, S^7 can be regarded as the coset $SO(5)/SO(3)$. The most general $SO(5)$ invariant metric on S^7 contains seven parameters. Here we restrict ourselves to a two parameter subset. The particular squashing we use here is described in more detail in [15].

Using letters near the start of the Greek or Roman alphabet to denote tangent space indices, the metric is given in terms of the siebenbein as $g_{mn}=\delta_{ab}e^a_m e^b_n$ where

$$e^0=d\mu, \quad e^i=\frac{1}{2}\sin\mu\omega_i, \quad e^{\hat{i}}=\frac{1}{2}e^{A(x)}(\nu_i+\cos\mu\omega_i). \quad \text{(B1)}$$

The indices i and $\hat{i}$ run from 1 to 3. Here μ is a coordinate taking values in the range $[0,\pi]$ and $A(x)$ is a scalar field that measures the amount of squashing. The round S^7 is given by $A=0$. The one forms ν_i and ω_i are given by

$$\nu_i=\sigma_i+\Sigma_i, \quad \omega_i=\sigma_i-\Sigma_i \quad \text{(B2)}$$

where σ_i and Σ_i each satisfy the $SU(2)$ algebra:

$$d\sigma_i=-\frac{1}{2}\epsilon_{ijk}\sigma_j\wedge\sigma_k, \quad d\Sigma_i=-\frac{1}{2}\epsilon_{ijk}\Sigma_j\wedge\Sigma_k. \quad \text{(B3)}$$

The tangent space components of the Ricci tensor are [15]

$$R_{ab}=\mathrm{diag}(\alpha,\alpha,\alpha,\alpha,\beta,\beta,\beta), \quad \text{(B4)}$$

where

$$\alpha=3\left(1-\frac{1}{2}e^{2A}\right), \quad \beta=e^{2A}+\frac{1}{2}e^{-2A}. \quad \text{(B5)}$$

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