Relation between quark masses and weak mixings

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Simple transformation formulas between fermion matrices and observables, and numerical values of quark matrices, are obtained on a particular weak basis with one quark matrix diagonal and the other with vanishing elements 1-1, 1-3 and 3-1, and with only the element 2-2 complex. When we choose M_u diagonal, then M_d shows intriguing numerical properties which suggest a four parameter description of it, which implies $V_{us} \simeq \sqrt{m_d/m_s}$, $V_{cb} \simeq (3/\sqrt{5})(m_s/m_b)$ and $V_{ub} \simeq (1/\sqrt{5})(\sqrt{m_d m_s}/m_b)$. A few comments on mass-mixing relations are added. [S0556-2821(98)05923-2]

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In the standard model Lagrangian [1], written in a general weak basis, quark mass matrix elements are not explicitly related to physical observables, that is, quark masses and weak mixings. The problem of finding such a relation, without extra symmetries, has been addressed in [2–5]. In particular, in [2] it was shown that it is always possible to find a weak basis where the quark mass matrices have the nearest neighbor interaction form and depend on twelve real parameters. Two of these twelve parameters are arbitrary [3] and related to the phase convention of the weak mixing matrix [4]. Then, in [5], it was shown that it is always possible to set one quark matrix in the diagonal form and the other in a form with zero entries in positions 1-1, 2-2 and 3-1, and with only the element 1-2 complex. In such a way mass matrices contain ten real parameters, exactly the same number of physical observables: six quark masses and three mixing angles and one phase. This corresponds to the choice of a *minimal parameter basis* [6]. As one mass matrix is chosen to be diagonal, it is relatively easy to obtain exact transformation formulas between mass matrices and observables. Other minimal parameter bases are considered in [7,8]. Here we describe a further minimal parameter basis, which shows interesting properties and on which transformation formulas are simple.

In fact it is also always possible [9] to choose a weak basis for which

$$M_d = \mathrm{diag}(m_d, m_s, m_b) \tag{1}$$

and

$$M_u = \begin{pmatrix} 0 & M_{12} & 0 \\ M_{21} & M_{22} & M_{23} \\ 0 & M_{32} & M_{33} \end{pmatrix} \tag{2}$$

[or M_u is diagonal and M_d has the form (2)].

On this basis the relation between mass matrices and observables is given by

$$M_u M_u^+ = K^+ \mathrm{diag}(m_u^2, m_c^2, m_t^2) K \equiv X^u \tag{3}$$

where K is the Cabibbo-Kobayashi-Maskawa (CKM) matrix [10]. In the case of M_u diagonal we have, instead,

$$M_d M_d^+ = K\ \mathrm{diag}(m_d^2, m_s^2, m_b^2) K^+ \equiv X^d. \tag{4}$$

By writing $M_{ij} = m_{ij} e^{i r_{ij}}$ we can reconstruct the usual representations of K [11] by means of three nonvanishing phases r_{12}, r_{22} and r_{23}. The product $M_u M_u^+$ is then given by

$$\begin{pmatrix} m_{12}^2 & m_{12} m_{22} e^{i(r_{12}-r_{22})} & m_{12} m_{32} e^{i r_{12}} \\ m_{12} m_{22} e^{-i(r_{12}-r_{22})} & m_{21}^2 + m_{22}^2 + m_{23}^2 & m_{22} m_{32} e^{i r_{22}} + m_{23} m_{33} e^{i r_{23}} \\ m_{12} m_{32} e^{-i r_{12}} & m_{22} m_{32} e^{-i r_{22}} + m_{23} m_{33} e^{-i r_{23}} & m_{32}^2 + m_{33}^2 \end{pmatrix}. \tag{5}$$

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and the transformation formulas between masses and mixings in X and mass matrix elements in M are written in a very simple form

$$m_{12}=\sqrt{X^u_{11}} \tag{6}$$

$$m_{22}=|X^u_{12}|/m_{12} \tag{7}$$

$$m_{32}=|X^u_{13}|/m_{12} \tag{8}$$

$$m_{33}=\sqrt{X^u_{33}-m^2_{32}} \tag{9}$$

$$r_{12}=\text{phase}(X^u_{13}) \tag{10}$$

$$r_{22}=r_{12}-\text{phase}(X^u_{12}) \tag{11}$$

$$M_{23}=(X^u_{23}-m_{22}m_{32}e^{ir_{22}})/m_{33}$$

$$m_{23}=|M_{23}| \tag{12}$$

$$r_{23}=\text{phase}(M_{23}) \tag{13}$$

$$m_{21}=\sqrt{X^u_{22}-m^2_{22}-m^2_{23}}. \tag{14}$$

In the case of M_u diagonal the same formulas hold with $X^u \to X^d$. With a phase transformation of quark fields,

$$\begin{aligned} M_{u,d}\to&\text{diag}(e^{-ir_{12}},e^{-ir_{23}},1)M_{u,d}\\ &\times\text{diag}(e^{ir_{23}},1,1), \end{aligned} \tag{15}$$

only a phase $r'_{22}=r_{22}-r_{23}$ remains in the element 2-2, and we obtain, using numerical values of quark masses at $\mu=M_Z$ as in [7] ($m_u=0.00233$, $m_c=0.677$, $m_t=181$, $m_d=0.00469$, $m_s=0.0934$, $m_b=3.00$ GeV) and mixings as in [11] (with $\delta=1.35$),

$$M_u=\begin{pmatrix} 0 & 1.591 & 0 \\ 0.011 & 7.118\ e^{1.334i} & 0.269 \\ 0 & 180.1 & 17.02 \end{pmatrix}\ \text{GeV}, \tag{16}$$

and if instead we choose M_u to be diagonal,

$$M_d=\begin{pmatrix} 0 & 0.024 & 0 \\ 0.021 & 0.105\ e^{-1.205i} & 0.106 \\ 0 & 1.333 & 2.685 \end{pmatrix}\ \text{GeV}. \tag{17}$$

We can see that in Eq. (16), due to the large value of the top quark mass, the biggest matrix element is not 3-3, as in Eq. (17), but the element 3-2. This feature is different from the basis in [5] where the biggest element is the element 3-3 either if M_u or M_d is diagonal. Moreover, the numerical values in Eq. (17) suggest to take M_u diagonal and

$$M_d=\begin{pmatrix} 0 & a & 0 \\ a & be^{i\varphi} & b \\ 0 & c & 2c \end{pmatrix}, \tag{18}$$

where a, b and c are of order 10^{-2}, 10^{-1} and 1 GeV, respectively. From Eq. (18) we obtain the approximate expression

$$M_d\simeq\begin{pmatrix} 0 & \sqrt{m_dm_s} & 0 \\ \sqrt{m_dm_s} & m_se^{i\varphi} & m_s \\ 0 & m_b/\sqrt{5} & 2m_b/\sqrt{5} \end{pmatrix}. \tag{19}$$

In the heavy quark limit $m_b \gg m_s, m_d$ we have the effective matrix for the two lightest down quarks

$$M_d\simeq\begin{pmatrix} 0 & \sqrt{m_dm_s} \\ \sqrt{m_dm_s} & m_s \end{pmatrix} \tag{20}$$

which gives the famous relation [12,13]

$$V_{us}\simeq\sqrt{\frac{m_d}{m_s}}. \tag{21}$$

In the chiral limit $m_d \ll m_s, m_b$ we have instead, for the two heaviest down quarks,

$$M_d\simeq\begin{pmatrix} m_s & m_s \\ m_b/\sqrt{5} & 2m_b/\sqrt{5} \end{pmatrix} \tag{22}$$

and when we diagonalize the Hermitian matrix

$$M_dM_d^+\simeq\begin{pmatrix} m_s^2 & 3m_sm_b/\sqrt{5} \\ 3m_sm_b/\sqrt{5} & m_b^2 \end{pmatrix} \tag{23}$$

we obtain the relation

$$V_{cb}\simeq\frac{3}{\sqrt{5}}\frac{m_s}{m_b}, \tag{24}$$

which gives $V_{cb}=0.042$ to be compared with the experimental value 0.041 ± 0.005 [11]. Finally, taking the full matrix

$$M_dM_d^+\simeq\begin{pmatrix} m_dm_s & m_s\sqrt{m_dm_s}e^{-i\varphi} & m_b\sqrt{m_dm_s/5} \\ m_s\sqrt{m_dm_s}e^{i\varphi} & m_s(m_d+2m_s) & m_sm_b(e^{i\varphi}+2)/\sqrt{5} \\ m_b\sqrt{m_dm_s/5} & m_sm_b(e^{-i\varphi}+2)/\sqrt{5} & m_b^2 \end{pmatrix} \tag{25}$$

we have the relation

$$V_{ub} \simeq \frac{1}{\sqrt{5}} \frac{\sqrt{m_d m_s}}{m_b}, \tag{26}$$

which gives $V_{ub}=0.003$ to be compared with the experimental range 0.002–0.005 [11]. From Eqs. (21), (24) and (26) we yield also

$$\frac{V_{ub}}{V_{cb}} \simeq \frac{1}{3} V_{us}. \tag{27}$$

Setting $x=\sqrt{m_d/m_s}$ and $y=\sqrt{m_s/m_b}$, we have $V_{us}\simeq x$, $V_{cb}\simeq(3/\sqrt{5})y^2$ and $V_{ub}\simeq(1/\sqrt{5})xy^2$ which means that, on this basis, weak mixings, apart from numerical coefficients not so different from 1, are generated by square roots of quark mass ratios x and y. Of course $x\sim y\sim\lambda$ leads to the Wolfenstein parametrization [14] of the CKM matrix. On the basis with M_d diagonal and M_u given by Eq. (2) such simple features are lost. Nevertheless weak mixings appear related to up quark ratios (for example $V_{cb}\simeq 11\ m_c/m_t$). Then, from the paper [15], where the relations $V_{us}\simeq\sqrt{m_d/m_s}$, $V_{cb}\simeq m_s/m_b$, but $V_{ub}\simeq\sqrt{m_u/m_t}$ were inferred, and our work, we argue that choosing different weak bases we can accordingly obtain different relations between mixings and masses, each of them in agreement with experimental data. Hence, each weak basis may be useful to describe some features of fermion masses and mixings. As a last remark on the basis considered here, we observe that, as written in footnote 6 of [2], in left-right symmetric models [16] both M_u and M_d can always take the form in Eq. (2).

In conclusion, we have obtained very simple formulas for relating fermion matrices to observables, and numerical values of quark matrices on a basis with one quark mass matrix diagonal and the other with three zeros in positions 1-1, 1-3 and 3-1. Such numerical values suggest a simple form for M_d which does imply relations (21), (24), (26), and (27). Moreover, on this basis weak mixings have a simple expression.

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