Improved staggered quark actions with reduced flavor symmetry violations for lattice QCD

J.-F. Lagaë and D. K. Sinclair
HEP Division, Argonne National Laboratory, 9700 South Cass Avenue, Argonne, Illinois 60439
(Received 17 June 1998; published 8 December 1998)

We introduce a new class of actions for staggered quarks in lattice QCD which significantly reduce flavor symmetry violations in the pion mass spectrum. An action introduced by the MILC Collaboration for the same purpose is seen to be a special case. We discuss how such actions arise from a systematic attempt to reduce flavor symmetry violations in the weak coupling limit. It is shown that for quenched lattice QCD at $6/g^2 = 5.7$, representative actions of this class give a considerable reduction in flavor symmetry violation over the standard staggered action, and a significant reduction over what is achieved by the MILC action.
[S0556-2821(98)00221-5]

PACS number(s): 12.38.Gc

I. INTRODUCTION

Lattice QCD simulations have always been limited by the requirement that the lattice be large compared with the correlation lengths (in particular with the pion Compton wavelength) of the theory, while the lattice spacing be small enough for physical observables to exhibit the scaling properties dictated by asymptotic freedom. In addition the symmetries of the lattice theory should approximate the continuum Lorentz and flavor symmetries. With the standard action this requires a lattice with a large number of sites. For this reason there has been considerable effort to find improved actions which obtain the desired results with appreciably larger lattice spacings.

Two methods have emerged for producing such improved actions. (For a recent summary of such methods and a more complete set of references see [1].) In the first method, the action is improved in powers of the lattice spacing a, by adding higher dimensional operators [2]. The coefficients can be calculated perturbatively [3] or non-perturbatively [4]. The second method uses renormalization group analyses to determine the fixed point, ''perfect'' action [5] (this paper references to the earlier literature). For the pure gauge sector of lattice QCD, actions of the Lüscher-Weisz form [6] and truncated perfect actions [5] work very well. For the fermion part of the action, Sheikholeslami and Wohlert have identified the operators which contribute at low orders in a [7]. For Wilson fermions both methods based on expansions in a and on perfect actions have had considerable success with lattice spacings as large as ~ 0.5 fm [8,9].

For staggered fermions, the improvement of the action in powers of a was discussed by Naik [10]. More recently Luo has enumerated all operators which contribute at low orders in a [11]. Perfect action methods have also been applied to staggered fermions [12,13]. While simple actions of both these classes show improvements over the standard action, they do not significantly reduce the flavor symmetry violations in the pion spectrum [14,15]. These flavor symmetry violations are a major part of the reason one is forced to use small lattice spacings when using staggered fermions.

The reason why these simple improvements to the staggered quark action fail to make significant improvements to the flavor symmetry, is that they concentrate on making the fermion dispersion relations near the corners of the Brillouin zone (where the components of the fermion momentum are close to 0 or π in lattice units) more continuum-like, and the interactions of these fermions with low momentum gauge fields also more continuum-like. However, the reason for the flavor symmetry violations in the staggered fermion method is that the gauge fields can transfer momenta large enough to take the quark from the neighborhood of one corner of the Brillouin zone to the vicinity of another, which changes that quark's flavor. Such large momentum transfers are not suppressed by these simple improved actions, and so flavor symmetry violations are not significantly suppressed.

Recently, the MILC Collaboration introduced a new staggered fermion action in which the gauge link is replaced by the linear combination of a single gauge link and the sum over the six 3-link ''staples'' joining the same sites [16]. Such a replacement tends to smooth the interaction between the gauge and fermion fields, thus reducing the coupling of the high-momentum components of the gauge fields to the quarks. Not surprisingly this action significantly decreases the flavor symmetry violation in the pion spectrum.

We have examined how to systematically suppress such flavor symmetry violations. At tree level, this leads to an action which reduces the flavor symmetry breaking by an extra power of a^2. We then assume that a good choice of action beyond tree level will have the same form, but with different coefficients. The MILC action is then seen to be a special case of this more general class of actions.

We have compared the spectrum of light hadrons, with particular emphasis on the pions, obtained with the standard staggered quark action, the MILC action, and a subset of our new actions. For a preliminary search of the parameter space for these actions we calculated the spectrum of light hadrons on a set of quenched gauge configurations at $\beta = 6/g^2 = 5.7$ on an $8^3 \times 32$ lattice. A summary of these results was presented at Lattice'97 [17]. We have since calculated the spectrum of light hadrons for the standard action, a near optimal MILC action and a promising choice from our new actions on quenched gauge configurations with $\beta = 6/g^2 = 5.7$ on a $16^3 \times 32$ lattice. From these calculations we conclude that, for an appropriate choice of parameters, our new class of actions represents a significant improvement over the MILC action, and confirm that both actions represent a considerable improvement over the standard staggered action.

In Sec. II we describe how to systematically reduce flavor symmetry breaking at tree level for lattice QCD at weak coupling, and introduce our new class of improved actions based on this analysis. In Sec. III we present our measurements of the hadron spectrum on quenched configurations. Section IV gives our summary and conclusions.

II. IMPROVING THE STAGGERED QUARK ACTION

In the staggered fermion transcription of quarks to the lattice, the quark field on site n of the lattice, $\psi(n)$, is a 3 component object—a color triplet. It lacks Dirac or flavor indices. The 4 flavors and 4 Dirac components are associated with the 16 poles (per color) of the free lattice Dirac propagator. For massless quarks, these occur when each component of the momentum $p_\mu=0$ or π. Since interactions, in general, change the momenta of the quarks, they induce mixings between the degrees of freedom associated with different flavors and hence break flavor symmetry.

At tree level, one can suppress flavor mixing to higher order in a, if one suppresses the coupling of fermions to gluons whose momentum components are all either 0 or π but not all 0. To see how this might be done, let us for the moment ignore the requirements of gauge invariance. Then the quark gluon coupling term in the Lagrangian could be replaced by

$$i\psi^\dagger(n)\,\eta_\mu(n)A_\mu(n)\psi(n+\mu)-\text{H.c.}, \tag{1}$$

which again gives mixing between quark flavors. We now replace A_μ in this term by

$$A_\mu(n)\to\frac{1}{256}(2+D_1+D_{-1})(2+D_2+D_{-2})\times(2+D_3+D_{-3})(2+D_4+D_{-4})A_\mu(n) \tag{2}$$

where

$$D_{\pm\nu}A_\mu(n)=A_\mu(n\pm\nu). \tag{3}$$

In momentum space this is equivalent to the substitution

$$A_\mu(k)\to\frac{1}{16}(1+\cos k_1)(1+\cos k_2)(1+\cos k_3)\times(1+\cos k_4)A_\mu(k). \tag{4}$$

The right hand side of this equation $\to A_\mu(k)$ as $k\to 0$ and differs from $A_\mu(k)$ by a factor of only $1+\mathcal{O}(a^2)$ for $|k|=\mathcal{O}(a)$. It vanishes when any component of k equals π and is suppressed by a factor of $\mathcal{O}(a^2)$ when any component of k is within $\mathcal{O}(a)$ of π. Hence this modification of the action would suppress the tree level flavor symmetry violations by $\mathcal{O}(a^2)$, which is what we want for our improved action.

We now return to the gauge invariant theory. The quark-gluon coupling term in the Lagrangian is now

$$\psi^\dagger(n)\,\eta_\mu(n)U_\mu(n)\psi(n+\mu)-\text{H.c.} \tag{5}$$

where $U_\mu=\exp iA_\mu$. Our ansatz for a tree-level improved action is obtained by replacing A_μ by U_μ in Eq. (2), multiplying out the prefactors, and replacing the products of displacement operators by appropriately symmetrized covariant displacement operators. This leads to the replacement $U_\mu\to\mathbf{U}_\mu$ where

$$\mathbf{U}_\mu=\frac{1}{16}\Bigg\{2+\sum_\nu\left[\frac{1}{2}D_\nu+\frac{1}{4}(D_{\mu\nu}+D_{-\mu\nu})\right]+\sum_{\nu\rho}\left[\frac{1}{4}D_{\nu\rho}+\frac{1}{8}(D_{\mu\nu\rho}+D_{-\mu\nu\rho})\right]+\sum_{\nu\rho\lambda}\left[\frac{1}{8}D_{\nu\rho\lambda}+\frac{1}{16}(D_{\mu\nu\rho\lambda}+D_{-\mu\nu\rho\lambda})\right]\Bigg\}U_\mu \tag{6}$$

where ν, ρ and λ are summed over ±1, ±2, ±3, ±4 with $|\mu|$, $|\nu|$, $|\rho|$, $|\lambda|$ all different. The D's are the covariant displacement operators. For example D_ν is defined by

$$D_\nu U_\mu(n)=U_\nu(n)U_\mu(n+\nu)U_\nu^\dagger(n+\mu). \tag{7}$$

The operational definition of the D's is as follows. The link is displaced one unit in each of the subscript directions. The ends of the undisplaced link are joined to the ends of the displaced link by products of links over the shortest paths joining the two. We then symmetrize over all such paths.

We now must check that this replacement suppresses the quark-gluon interaction when at least one component of the gluon momentum is close to π and the rest are near to 0. Since we are interested primarily in what happens to leading order in a we can write $U_\mu\approx 1+iA_\mu$ and $\mathbf{U}_\mu\approx 1+i\mathbf{A}_\mu$. We then evaluate $\mathbf{A}_\mu$ when each component of the momentum k of A_μ is 0 or π. When $k_1=k_2=k_3=k_4=0$ we find $\mathbf{A}_\mu=A_\mu$. At the other corners of the Brillouin zone we find that $\mathbf{A}_\mu$ is longitudinal, and hence decouples, which is the desired result.

To go beyond tree level we assume that we can use a replacement of the same form as the tree replacement, i.e.

$$\mathbf{U}_\mu=C\Bigg\{x_0+2y_0+\sum_\nu[x_1D_\nu+y_1(D_{\mu\nu}+D_{-\mu\nu})]+\sum_{\nu\rho}[x_2D_{\nu\rho}+y_2(D_{\mu\nu\rho}+D_{-\mu\nu\rho})]+\sum_{\nu\rho\lambda}[x_3D_{\nu\rho\lambda}+y_3(D_{\mu\nu\rho\lambda}+D_{-\mu\nu\rho\lambda})]\Bigg\}U_\mu \tag{8}$$

with $C=1/(x_0+2y_0+6x_1+12y_1+12x_2+24y_2+8x_3+16y_3)$ Here we could modify the definitions of the displacement operators D which include the index $\pm\mu$ to use different weights, depending on the positions of the links in the $\pm\mu$ directions. Indeed standard tadpole improvement [3] would require such changes. We have chosen not to exercise this option in our choices of improved actions, since even without, this class of action has 6 free parameters. We note

TABLE I. Hadron masses for various choices of the staggered quark action at $\beta=5.7$ on an $8^3\times32$ lattice. Standard is the standard staggered action, MILC5, MILC1 and MILCu are the MILC action with $\omega=0.5$, 1 and ∞ respectively and naive, imp2 and impu are our improved action with $x=0.5$, 1 and ∞ respectively.

ACTION	m_q	m_π	m_{π_2}	m_ρ	m_N
Standard	0.012	0.3156(10)	0.659(37)	0.891(20)	1.217(22)
Standard	0.020	0.4001(9)	0.709(21)	0.916(12)	1.291(15)
Standard	0.040	0.5495(8)	0.858(18)	1.006(11)	1.548(31)
MILC5	0.012	0.3227(32)	0.462(17)	0.793(12)	
MILC5	0.020	0.4083(32)	0.533(11)	0.809(8)	1.181(52)
MILC5	0.040	0.5550(16)	0.674(8)	0.893(8)	1.378(16)
MILC1	0.012	0.3267(33)	0.453(15)	0.787(11)	
MILC1	0.020	0.4100(29)	0.530(11)	0.807(8)	1.177(48)
MILC1	0.040	0.5629(16)	0.664(6)	0.893(8)	1.377(15)
MILCu	0.012	0.3371(32)	0.471(13)	0.781(10)	
MILCu	0.020	0.4246(28)	0.544(10)	0.807(7)	1.198(43)
MILCu	0.040	0.5852(15)	0.688(5)	0.904(7)	1.384(20)
naive	0.012	0.3562(32)	0.453(9)	0.788(9)	1.170(15)
naive	0.020	0.4471(30)	0.525(6)	0.809(7)	1.214(11)
imp2	0.012	0.3868(33)	0.456(7)	0.786(8)	1.177(13)
imp2	0.020	0.4855(22)	0.538(5)	0.815(6)	1.264(16)
imp2	0.040	0.6707(18)	0.712(3)	0.921(5)	1.469(40)
impu	0.006	0.3660(23)	0.448(8)	0.770(10)	1.163(14)
impu	0.012	0.5018(22)	0.554(4)	0.813(6)	1.328(23)

that the MILC action belongs to this class, being the special case where $x_0=1$, $x_1=\omega$, and $x_2=x_3=y_0=y_1=y_2=y_3=0$.

III. THE HADRON SPECTRUM WITH IMPROVED GAUGE ACTIONS

One of the most visible effects of flavor symmetry violation for staggered quarks is seen in the pion mass spectrum. Only one of the pions is a true Goldstone boson whose mass vanishes as the quark mass is taken to zero. The mass differences between the non-Goldstone pions is typically somewhat less than that between them and the Goldstone pion. For our measurements, we have chosen π_2, the other local pion, as our representative non-Goldstone pion. For inverse lattice spacings ~ 1 GeV, i.e. for lattice spacings ~ 0.2 fm, this symmetry breaking is quite large. In the chiral ($m_q\to 0$) limit $m^2_{\pi_2}/m^2_\rho\approx 0.5$ rather than 0, a 50% effect. Since in the real world, $m^2_{\pi_2}/m^2_\rho\approx 0.03$, this is a major impediment to working at such lattice spacings.

For our measurements we have chosen to work with quenched gauge field configurations at $\beta=5.7$ where the inverse lattice spacing is ~ 1 GeV. To search the parameter space we performed spectrum calculations on 203 independent quenched configurations generated using the standard (Wilson) gauge action on an $8^3\times32$ lattice. Hadron spectra were calculated using a single wall source on the (*odd*, *odd*, *odd*) sites of the first time slice of each configuration, gauge fixed to Coulomb gauge. The propagators for local hadrons were measured using point sinks.

Our measurements used the standard staggered quark action, the MILC action and our improved action. The π, π_2, ρ and nucleon masses are given in Table I. For the MILC action we measured the spectrum for $\omega=0.5$, $\omega=1.0$ and $\omega=\infty$ at quark mass $m_q=0.012$, 0.02 and 0.04. From Table I, we note that the flavor symmetry breaking measured as $(m^2_{\pi_2}-m^2_\pi)/m^2_\rho$ appears smallest for the $\omega=1.0$ measurements, but that the difference between the 3 ω values is small. From this we conclude that $\omega=1.0$ is a close to optimal choice for $\beta=5.7$. For our improved action, even with our restricted parametrization, there are 6 independent parameters. To limit our choices, we chose a 1 parameter subclass parametrized by x, for which $x_n=x^n$ and $y_n=x^{n+1}$. This choice was influenced by tadpole improvement. The results we present here are for the tree level coefficients ($x=1/2$) with quark masses $m_q=0.012$ and 0.02, for $x=1$ with quark masses 0.012, 0.02 and 0.04 and for $x=\infty$ with quark masses 0.006 and 0.012. Here symmetry breaking appears to be smallest for $x=1$. We note also that symmetry breaking for our best improved action is appreciably smaller than for the best MILC action. Since the ρ masses are very close, this does not appear to be simply due to differences in the perceived lattice spacing.

Since, even at $\beta=5.7$, 8^3 is a rather small spatial lattice, we have confirmed and quantified our results using a set of 158 quenched configurations on a $16^3\times32$ lattice, also at $\beta=5.7$. The larger lattice also permitted us to go to smaller quark masses. For this larger lattice we have calculated the light hadron spectrum with the standard staggered quark action, the MILC action with $\omega=1$ and our improved action with $x=1$. For these measurements we used a single source on time-slice 1 of each configuration. This source was a con-

TABLE II. Hadron masses for various choices of the staggered quark action at $\beta=5.7$ on a $16^3\times32$ lattice. The notation is as for Table I.

ACTION	m_q	m_π	m_{π_2}	m_ρ	m_N
Standard	0.006	0.2256(5)	0.673(23)	0.881(10)	1.346(14)
Standard	0.012	0.3145(4)	0.723(14)	0.909(7)	1.392(10)
Standard	0.020	0.4000(4)	0.813(21)	0.941(5)	1.442(9)
Standard	0.040	0.5503(4)	0.909(11)	1.026(5)	1.551(7)
MILC1	0.006	0.2233(6)	0.415(12)	0.795(23)	1.152(34)
MILC1	0.012	0.3122(6)	0.458(6)	0.819(13)	1.194(16)
MILC1	0.020	0.3993(5)	0.521(4)	0.850(8)	1.255(12)
MILC1	0.040	0.5571(5)	0.656(3)	0.909(5)	1.394(8)
imp2	0.006	0.2681(6)	0.377(6)	0.763(15)	1.111(18)
imp2	0.012	0.3743(6)	0.454(4)	0.808(8)	1.184(11)
imp2	0.020	0.4785(6)	0.538(3)	0.841(6)	1.254(12)
imp2	0.040	0.6690(7)	0.713(2)	0.938(3)	1.410(9)

stant for all (*odd*, *odd*, *odd*) sites of an $8\times8\times8$ cube and zero elsewhere, making it identical to the source we used on the smaller lattice, since this seemed to produce flat effective mass plots. Again we worked in Coulomb gauge and used point sinks. The masses we obtained from fits to these propagators are presented in Table II.

We note that the Goldstone π masses show very little finite size effect in going from an $8^3\times32$ to a $16^3\times32$ lattice. The π_2 masses for the MILC and improved actions also show relatively little finite size effect while that for the standard action shows considerably more. However, we note that the errors for the standard action are large, and all the errors in these tables are purely statistical—no estimate of the systematic error associated with choice or appropriateness of fits is included—so that is not clear how significant this is. Similar comments can be made about any apparent finite size effects in the ρ and nucleon masses. We assume that for a $16^3\times32$ lattice, where the spatial box size is >3 fm at this β, the finite size effects will be relatively small. We refer the reader to recent, more extensive quenched spectrum calculations at $\beta=5.7$ for serious finite size studies and masses with which our standard action masses can be compared, and also for chiral extrapolation studies [18,19].

Just comparing $(m^2_{\pi_2}-m^2_\pi)/m^2_\rho$ indicates that both the MILC and our improved actions give a considerable reduction in flavor symmetry violation over the standard action. In addition our improved action gives improvement over that of the MILC action. To make this more quantitative, we have chirally extrapolated our masses to zero quark mass. We do this first because, since the relationship between the lattice quark mass and the physical ($\overline{\rm MS}$) mass is different for each action, and there is the ambiguity as to which observable should be used to determine which lattice quark masses correspond to one another. $m_q=0$ is the same for each lattice action. Secondly, the physical u and d quark masses are small enough that the chiral limit is a good approximation to the real world.

Since we have 4 quark masses for each action, we can use a 3-parameter fit for our chiral extrapolation. For the Goldstone pion we have chosen to fit to

$$m^2_\pi=am_q+bm_q^{3/2}+cm_q^2. \tag{9}$$

For the second pion π_2 we have used a 2-parameter fit

$$m^2_{\pi_2}=a+bm_q. \tag{10}$$

Our pion masses and these fits are plotted in Fig. 1. Since our Goldstone pion masses have been forced to zero in the $m_q=0$ limit, $m^2_{\pi_2}$ is a good measure of flavor symmetry violation. From our fits we obtain $m^2_{\pi_2}=0.367(44)$ for the standard action $m^2_{\pi_2}=0.114(8)$ for the MILC action and $m^2_{\pi_2}=0.073(5)$ for our improved action. These would be valid measures of improvement if we consider that the true lattice spacing is that obtained from some pure gluonic observable such as the string tension, or the ρ mass from some unknown ''correct'' fermion action.

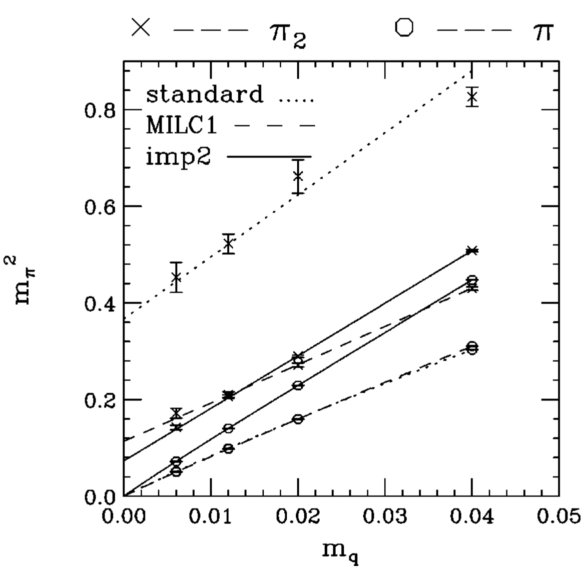

FIG. 1. Pion masses squared as functions of the quark mass on a $16^3\times32$ lattice at $\beta=5.7$. The lines are the fits described in the text.

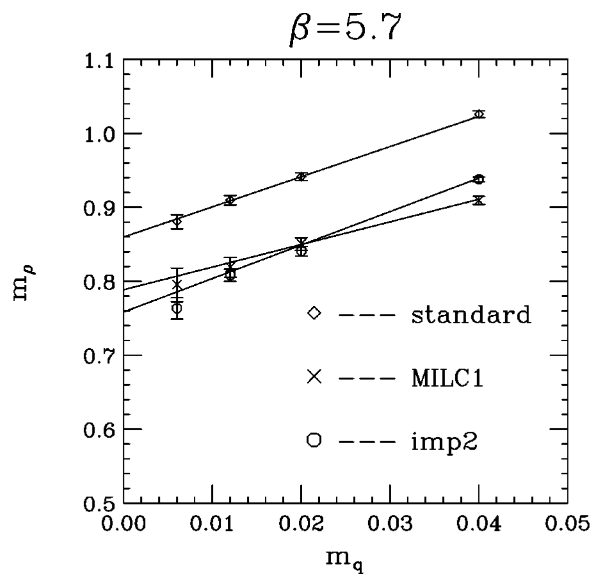

FIG. 2. ρ masses as functions of m_q for a $16^3\times32$ lattice, with linear fits.

If, however, we follow the MILC Collaboration, and consider that each fermion action determines its own lattice spacing, we need to extrapolate whatever hadron mass is to be used to determine this spacing to $m_q=0$ for each action separately. For the ρ and nucleon, we have used a simple linear extrapolation in m_q. (Even though such fits were not great, we were unable to find any 3 parameter fits which did significantly better.) These fits for our 3 actions are shown in Figs. 2 and 3. This gives $m_\rho=0.859(9)$ and $m_N=1.330(14)$ for the standard action, $m_\rho=0.788(14)$ and $m_N=1.116(20)$ for the MILC action and $m_\rho=0.758(10)$ and $m_N=1.090(17)$ for our improved action in the chiral limit. Hence if we use the ρ mass to set the scale, we get $m^2_{\pi_2}/m^2_\rho=0.497(59)$ for the standard action $m^2_{\pi_2}/m^2_\rho=0.183(12)$ for the MILC action and $m^2_{\pi_2}/m^2_\rho=0.127(9)$ for our improved action.

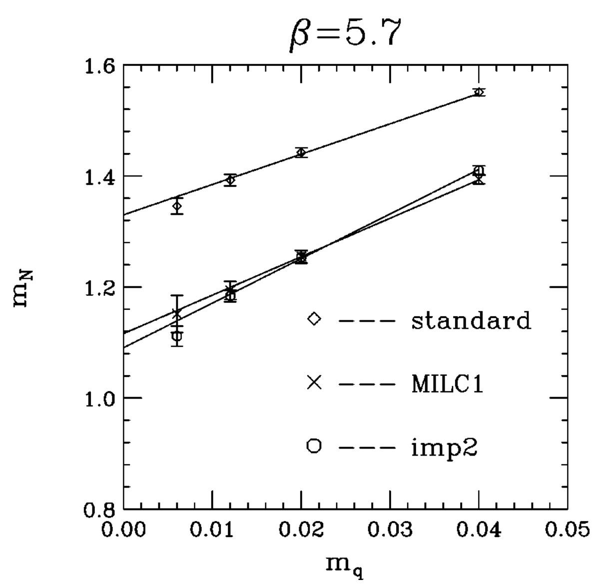

FIG. 3. Nucleon masses as functions of m_q for a $16^3\times32$ lattice, with linear fits.

As discussed by the MILC Collaboration, one of the effects of improving the quark action is to reduce the lattice spacing as determined by the ρ mass. (This is evident in our results quoted above.) Before we conclude how successful our program has been, we therefore need to know how much flavor symmetry violation would have been improved merely by making such decreases in lattice spacing, without changing our quark action. To do this one would need to know how large the standard action flavor symmetry violations would be at a lattice spacings where the ρ mass for the standard action would have the values the MILC ρ mass and our improved ρ mass have respectively at $\beta=5.7$. In addition, we would need to know the size of the MILC action flavor symmetry violations at the lattice spacing where the MILC ρ mass has the value our improved ρ mass has at $\beta=5.7$. We do not have this information. However, we know that the leading flavor symmetry violations should be $\mathcal{O}(a^2)$ and the ratios of relevant a's are given by the inverse ratios of the corresponding ρ masses. This would give an estimate of $m^2_{\pi_2}/m^2_\rho=0.418(50)$ for the standard action at a lattice spacing set by the MILC ρ mass, and $m^2_{\pi_2}/m^2_\rho=0.387(46)$ at a lattice spacing set by our improved ρ mass. Similarly we estimate the flavor symmetry violation for the MILC action to be $m^2_{\pi_2}/m^2_\rho=0.169(11)$ at a lattice spacing set by our improved ρ mass.

We now check this $\mathcal{O}(a^2)$ dependence against published results [16] at higher β values, performing the required linear extrapolations as best we can. At $\beta=5.85$ the chirally extrapolated ρ mass is 0.5676(42) from which we predict $m^2_{\pi_2}/m^2_\rho=0.217(26)$ compared with the value calculated from extrapolated π_2 and ρ masses namely $m^2_{\pi_2}/m^2_\rho=0.267(6)$. At $\beta=5.95$ the chirally extrapolated ρ mass is 0.4629(40) from which we predict $m^2_{\pi_2}/m^2_\rho=0.144(17)$ compared with the direct extrapolation $m^2_{\pi_2}/m^2_\rho=0.170(13)$. Thus we conclude that our predictions from the assumed $\mathcal{O}(a^2)$ dependence will be if anything lower than the actual values.

IV. DISCUSSION AND CONCLUSIONS

We have introduced a new class of single-link actions for staggered fermions which reduce the flavor symmetry violations from $\mathcal{O}(a^2)$ to $\mathcal{O}(a^4)$ at the tree level, where a is the lattice spacing, by suppressing the coupling of high momentum gluons to quarks which is responsible for flavor mixing. On quenched configurations at $\beta=5.7$ where $a^{-1}\approx1$ GeV, the flavor symmetry violations for local pions are reduced by $\approx$65–75%, over those of the standard action and by $\approx$25–30% over those in the MILC action. For our improved action this means that the flavor symmetry violations at $\beta=5.7$ are approximately the size of those for the standard action at $\beta=6.0$, i.e. at approximately half the lattice spacing. Since all these actions are single link, inverting the Dirac operator is no more expensive than with the standard action. In fact, it is considerably less expensive with the improved actions, since they require many less conjugate gradient iterations to reach the same level of convergence. For example, at $m_q = 0.006$, our improved action required 1000–1050 conjugate gradient iterations compared with 1700–1750 for the MILC action and 2600–2700 for the standard action.

Of course, although flavor symmetry violations are one of the most important barriers to using staggered quarks at lattice spacings of $\gtrsim 0.1$ fm, they are not the only barriers. Our improvements need to be combined with improvements to the free fermion dispersion relations and to the gauge action. In the case of the MILC action, work of this nature has recently been done by Orginos and Toussaint [20], who have also included dynamical quarks.

The Originos-Toussaint paper does, however, point out that although their actions improve the $\pi - \pi_2$, mass splitting, the point-split pions do not show as much improvement. Although we have not measured the spectrum of these point-split pions, we expect the improvement to be more uniform across the pion multiplet with our action than with the MILC action. The reason is that at tree level, our action uniformly suppresses all flavor mixings. On the other hand the MILC action, at tree level, can be adjusted to maximally suppress some flavor mixings, but it will at best only partially suppress the others. However, only explicit measurement will tell if our expectations are correct.

Another aspect of flavor symmetry violation for staggered quarks is the extent to which they fail to obey the Atiyah-Singer index theorem. We have shown that the MILC action produces only limited improvement in this area [21]. It is to be hoped that our improved action might fare better.

A more serious study is needed to determine the optimal parameters in our action. Our action is still far from ideal for treating light u and d quarks. A tadpole improved perturbative calculation of the coefficients might be helpful, although perturbative calculations for staggered fermions have proved disappointing in the past. One might hope that it might be possible to reduce the flavor symmetry violations to $\mathcal{O}(a^4)$ as suggested by the tree level calculations, within the restrictions of a single link fermion action. The analysis of Luo [11] should be helpful in reducing the operators in our action to an independent set.

Although it is obvious that we need to reduce the coupling of high momentum gluons to staggered quarks, to reduce flavor symmetry violations, it is also important to reduce such coupling for Wilson quarks. In the case of Wilson quarks this is to decrease chiral symmetry violations. This has been addressed most recently by DeGrand [9] who introduced smeared link fields into Wilson fermions calculations. Such smearing could also be useful in reducing effects of small instantons on Wilson fermions which have the potential for creating problems if they are to be used as the basis for domain-wall fermions [22].

ACKNOWLEDGMENTS

This work was supported by U.S. Department of Energy contract W-31-109-ENG-38. The computing was performed on the CRAY J-90's at NERSC.

[1] F. Niedermayer, Nucl. Phys. B (Proc. Suppl.) **53**, 56 (1997).
[2] K. Symanzik, Nucl. Phys. **B226**, 187 (1983); **B226**, 205 (1983).
[3] G. P. Lepage and P. B. Mackenzie, Phys. Rev. D **48**, 2250 (1993).
[4] S. Capitani *et al.*, Nucl. Phys. B (Proc. Suppl.) **63**, 153 (1998), and references contained therein.
[5] P. Hasenfratz, Nucl. Phys. B (Proc. Suppl.) **63**, 53 (1998).
[6] M. Lüscher and P. Weisz, Commun. Math. Phys. **97**, 19 (1985); Phys. Lett. **158B**, 250 (1985).
[7] B. Sheikholeslami and R. Wohlert, Nucl. Phys. **B259**, 572 (1985).
[8] M. Alford *et al.*, Phys. Lett. B **361**, 87 (1995); Nucl. Phys. **B496**, 377 (1997).
[9] T. DeGrand, Phys. Rev. D **58**, 094503 (1998).
[10] S. Naik, Nucl. Phys. **B316**, 238 (1989).
[11] Y. Luo, Phys. Rev. D **57**, 265 (1998).
[12] W. Bietenholz *et al.*, Nucl. Phys. **B495**, 285 (1997).
[13] W. Bietenholz and H. Dilger, e-print hep-lat/9803018 (1998), and references contained therein.
[14] C. Bernard *et al.*, Phys. Rev. D **580**, 4503 (1998).
[15] A. Peikert *et al.*, Nucl. Phys. B (Proc. Suppl.) **63**, 895 (1998).
[16] T. Blum *et al.*, Phys. Rev. D **55**, R1133 (1997).
[17] J.-F. Lagaë and D. K. Sinclair, Nucl. Phys. B (Proc. Suppl.) **63**, 892 (1998).
[18] C. Bernard *et al.*, hep-lat/9805004 (1998); Nucl. Phys. B (Proc. Suppl.) **47**, 345 (1996); **53**, 212 (1997).
[19] S. Gottlieb, Nucl. Phys. B (Proc. Suppl.) **53**, 155 (1997), and references contained therein.
[20] K. Orginos and D. Toussaint, e-print hep-lat/9805009 (1998).
[21] J. B. Kogut, J.-F. Lagaë and D. K. Sinclair, Phys. Rev. D **58**, 054504 (1998).
[22] R. G. Edwards, U. M. Heller, and R. Narayanan, Nucl. Phys. **B522**, 2851 (1998); R. G. Edwards *et al.*, *ibid.* **B518**, 319 (1998).