Equivariant gauge fixing of SU(2) lattice gauge theory

Martin Schaden
Physics Department, New York University, 4 Washington Place, New York, New York 10003
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I construct a lattice gauge theory (LGT) with a discrete Z_2 structure group and an equivariant BRST symmetry that is physically equivalent to the standard SU(2) LGT. The measure of this Z_2 LGT is invariant under *all* the discrete symmetries of the lattice and its partition function does not vanish. The topological lattice theories (TLT) that localize on the moduli spaces are explicitly constructed and their BRST symmetry is exhibited. The ghosts of the Z_2-invariant local LGT are integrated in favor of a nonlocal bosonic measure. In addition to the SU(2) link variables and the coupling g^2, this effective bosonic measure also depends on an auxiliary gauge invariant site variable of canonical dimension two and on a gauge parameter α. The relation between the expectation value of the auxiliary field, the gauge parameter α and the lattice spacing a is obtained to lowest order in the loop expansion. In four dimensions and the critical limit this expectation value is a physical scale proportional to Λ_L in the gauge $\alpha = g^2(11-n_f)/24 + O(g^4)$. Implications for the loop expansion of observables in such a critical gauge are discussed. [S0556-2821(99)00401-4]

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I. INTRODUCTION

Euclidean lattice gauge theory (LGT) is the only known rigorous non-perturbative definition of a non-Abelian gauge theory. In the vicinity of a second order phase transition for a critical value of the couplings, the LGT can be interpreted as a regularization of a continuum quantum field theory in Euclidean space-time. Apart from numerical simulations, such models also provide a mathematically rigorous foundation for various non-perturbative field theoretic ideas. These statistical models however also have peculiarities of their own that have no analog in other regularizations of a quantum field theory.

The discrete lattice by construction is devoid of any notion of ''smoothness'' and it is difficult to study effects related to topological characteristics of the continuum gauge group. The ''gauge-group'' of a LGT is simply

$$\mathcal{G} = \otimes_{sites} G_i, \qquad (1)$$

where the group G_i at the i-th site is isomorphic to the compact structure group G. Only the vanishing fraction of lattice gauge transformations that satisfy a Sobolev norm apparently correspond to continuum gauge transformations [1] in the critical limit. For lattice perturbation theory and a continuum interpretation of the lattice model it is thus desirable to reduce the rather large symmetry of the LGT to a more manageable level. This however has to be done without altering physical observables of the model. The procedure is (as we will see somewhat misleadingly) known as gauge fixing. One hopes that gauge fixing the lattice model would help disentangle lattice gauge artifacts from the physically relevant continuum dynamics. The wild ''gauge'' group of the lattice preferably should be tamed in a fashion that assures a smooth thermodynamic and critical limit of the physically equivalent gauge fixed lattice model. In analogy with covariant gauges for the continuum theory that preserve all the isometries of a space-time manifold, a gauge fixing procedure that preserves *all* the (discrete) symmetries of a periodic lattice will also be called ''covariant'' in the following. While it is relatively simple to reduce the gauge group of a LGT (by say ''fixing'' a maximal tree), it is apparently not entirely trivial to obtain a *covariantly* gauge fixed lattice measure that is normalizable [2,3].

In continuum perturbation theory, the method of choice for covariant gauge fixing is Becchi-Rouet-Stora-Tyutin (BRST) quantization. Such gauges necessarily [4] have a Gribov-ambiguity [5], i.e. an orbit generally crosses the (covariant) gauge fixing surface more than once (and some orbits approach this surface tangentially). Although apparently of little relevance for an asymptotic perturbative expansion this ambiguity does concern the non-perturbative validity of the gauge-fixed model. In the context of Chern-Simons theory it was even recently shown that a correct treatment of the generic gauge zero modes of degenerate background connections is essential for obtaining the (non-trivial) asymptotic expansion of the model [6].

A valid non-perturbative definition of the gauge fixed model is also of importance for the lattice. It has been pointed out [7] that conventional BRST-invariant Landau-gauge in fact counts the intersections of the orbit with a sign that depends on the direction in which the oriented gauge fixing surface is crossed—the ''Gribov-ambiguity'' in this case would not pose an obstruction to covariant gauge fixing as long as the degree of this map does not vanish. Quite generally the degree of this map however is zero for a covariantly gauge fixed LGT [2].

For continuum gauge theories the gauge fixing procedure was recently seen to be equivalent to the construction of a topological quantum field theory (TQFT) on the gauge group [8]. It turns out that the partition function of this TQFT is usually proportional to the generalized Euler characteristic of the gauge group manifold and thus proportional to the ''degree of the map'' of Sharpe. The TQFT construction shows that it is a topological characteristic of the *gauge group* that determines whether or not the gauge-fixed theory makes sense non-perturbatively. It allows one to continuously deform the orbit and thus enables one to handle orbits that are *on* a Gribov horizon. One can also show that the partition function of conventional covariantly gauge fixed continuum models on compact space-time indeed vanishes non-perturbatively. The very construction of a TQFT however allows one to address and solve these problems [8,9]. We will see that the method is also a powerful tool in the construction of a physically equivalent and covariantly gauge-fixed LGT.

The quest for a lattice analog of the elegant BRST-formalism of continuum gauge theories has been elusive. Neuberger [10] first formulated the analog of the conventional continuum BRST-algebra for the lattice but subsequently proved that the partition function of a gauge-fixed lattice theory with this BRST symmetry is not normalizable [3]. His proof is based on particular properties of the BRST-algebra that do not hold for the equivariant BRST construction we will consider below. For the special case of certain covariant gauge fixings on the lattice, Sharpe [2] had shown that the degree of the map is zero—and that the partition function of the gauge-fixed lattice theory therefore vanishes due to the mutual cancellation of contributions from different Gribov copies. His proof however appeared to depend on the details of the gauge fixing and raises the question whether some other covariantly gauge-fixed lattice action can be found. Sharpe proposed several models whose partition functions do not vanish. In the naive continuum limit some of them correspond to covariantly gauge fixed actions. These local lattice actions however break some of the symmetries of a periodic lattice. Determining the corresponding continuum model in this case requires a somewhat naive extrapolation.

I will translate the recent developments in continuum BRST-quantization to the mathematically more rigorous setting of LGT's on finite lattices. I use an equivariant BRST construction to reduce the gauge group of an SU(2) LGT to a physically equivalent Abelian U(1) LGT in Sec. II. [The generalization of the procedure to other lattice gauge groups $\mathcal{G}$ and subgroups $\mathcal{H}\subset\mathcal{G}$ is relatively straightforward. The essential point is to use a subgroup $\mathcal{H}$ for which the Euler characteristic of the coset manifold $\chi(\mathcal{G}/\mathcal{H})\neq 0$.] In Sec. III I examine the corresponding topological lattice theory (TLT) and show that it is a constant on the orbit space. The value of this constant is explicitly computed in Sec. IV at the trivial link configuration $U=1$. I verify that the partition function of the TLT is indeed proportional to the Euler character $\chi(\mathcal{G}/\mathcal{H}=[\mathrm{SU}(2)/\mathrm{U}(1)]^N)=2^N\neq 0$ and therefore normalizable.

This first step reduces the problem of constructing a covariant and BRST-invariant gauge-fixed LGT to that of BRST-invariant gauge fixing of an U(1) LGT. In Sec. V the presence of local fields that are *charged* under the Abelian group is utilized to build a TLT that also fixes the residual Abelian invariance. The partition function of this TLT is shown to be proportional to the number of connected components of the U(1) gauge-group and is thus normalizable. One thus obtains a local and ''lattice-covariant'' Z_2 LGT that is physically equivalent to the original SU(2) LGT. The loop expansion of this Z_2 LGT is examined in Sec. VI. I show that the measure is maximal at certain discrete pure gauge configurations *and* a non-vanishing constant configuration $\rho_i=\overline{\rho(\alpha)}$ of an auxiliary (gauge invariant) bosonic field. The unique maximum of this bosonic measure is determined in the thermodynamic limit of a four dimensional lattice.

II. EQUIVARIANT BRST: GAUGE-FIXING A SU(2) LGT TO A U(1) LGT

Consider a D-dimensional LGT with an SU(2) gauge group and for simplicity assume that the SU(2) LGT is described by a local action $S_{\mathrm{inv.}}[U]$ which depends only on the link variables $U_{ji}^\dagger=U_{ij}\in\mathrm{SU}(2)$. The generalization to the case with matter fields is straightforward. The invariance of the measure with respect to the lattice gauge group (1) implies that

$$S_{\mathrm{inv.}}[U]=S_{\mathrm{inv.}}[U^{g\in\mathcal{G}}],\quad \text{with}\quad U_{ij}^g=g_iU_{ij}g_j^\dagger,\quad g_i\in\mathrm{SU}(2). \tag{2}$$

In this section we reduce the gauge invariance of the LGT to the Abelian subgroup

$$\mathcal{H}=\otimes_{sites}\mathrm{U}(1), \tag{3}$$

while preserving the locality of the measure and its invariance with respect to the isometries of the lattice. The resulting model will exhibit an equivariant BRST-symmetry and we will prove in Secs. III and IV that it is equivalent to the original SU(2) LGT with regard to physical observables.

The construction of the equivariant BRST symmetry is analogous to the one in the continuum case [8]. Note that an infinitesimal gauge transformation with $g_i=1+\epsilon\theta_i+O(\epsilon^2)$ to order ϵ changes the links by

$$\Delta U_{ij}=\theta_iU_{ij}-U_{ij}\theta_j+O(\epsilon). \tag{4}$$

We accordingly define [10] the BRST-variation of U_{ij} as

$$sU_{ij}=(c_i+\omega_i)U_{ij}-U_{ij}(c_j+\omega_j),\quad \omega_i\in\mathrm{u}(1), \tag{5}$$

where c_i and ω_i are Lie-algebra valued Grassmannian site variables. The reason for the apparently redundant introduction of *two* ghosts c_i and ω_i instead of one for their sum is that we can thus *specify* the action of one of these ghosts and eventually decompose the Lie-algebra. For the case at hand, we take ω to be the ghost associated with the generator of the U(1) subgroup. Since our gauge fixing condition will be U(1)-invariant, it is possible to arrange matters so that the BRST-invariant action of the physically equivalent U(1) LGT does not depend on the ω ghost. Requiring that the BRST-variation be nilpotent, $s^2=0$, Eq. (5) implies

$$sc_i+s\omega_i=(c_i+\omega_i)^2=c_i^2+[\omega_i,c_i]+\omega_i^2. \tag{6}$$

Here $[\cdot,\cdot]$ is the commutator graded by the ghost number. One satisfies Eq. (6) by

$$
\begin{aligned}
sc_i &= c_i^2 + [\omega_i, c_i] + \phi_i, \quad \phi_i \in \mathrm{u}(1) \\
s\omega_i &= \omega_i^2 - \phi_i = -\phi_i \\
s\phi_i &= [\omega, \phi_i] = 0,
\end{aligned}
\tag{7}
$$

where the ghost number 2 field ϕ is introduced for the following reasons of consistency. Since the ω_i are in the Cartan sub-algebra u(1) of su(2), we can without loss of generality demand that the c_i span the remaining two generators of the Lie-algebra. The necessary Lagrange multiplier fields that implement this constraint will be introduced below. Consistency however then requires that the component in the Cartan sub-algebra of sc_i also vanish. Since c_i^2 generally will (only) have a component in the Cartan sub-algebra, we can satisfy this requirement only by introducing an additional field $\phi \in \mathrm{u}(1)$. Note that it is sufficient that ϕ take values in the Cartan sub-algebra and that ω_i generates U(1) transformations of c_i and ϕ_i. Since the subgroup generated by ω in our case is Abelian, the BRST-variation of ω and ϕ simplify in Eq. (7). In general, the equivariant BRST-construction above can be employed to reduce any group $\mathcal{G}$ to a subgroup $\mathcal{H} \subset \mathcal{G}$ also for non-Abelian $\mathcal{H}$. In [8] a similar construction was for instance used to factor the global gauge transformations of the continuum gauge theory.

To complete the equivariant BRST construction one introduces Lagrange multiplier fields as BRST-doublets that enforce the constraints. For the gauge condition we require a Nakanishi-Lautrup field b_i of vanishing ghost number. It is part of the doublet

$$
s\overline{c_i} = [\omega_i, \overline{c_i}] + b_i, \quad sb_i = [\omega_i, b_i] - [\phi_i, \overline{c_i}]. \tag{8}
$$

Note that the anti-ghost $\overline{c_i}$ here transforms under the U(1). This is a natural consequence of Eq. (5)—we cannot take $\overline{c_i}$ to be neutral under U(1), because the BRST-invariant action we intend to construct would otherwise be ω-dependent. The BRST transformation of the b-field is then given by the nilpotency of s. Note that the non-trivial transformation of the Nakanishi-Lautrup field b in Eq. (8) in the present context invalidates Neuberger's proof [3] that the partition function of a BRST-invariant lattice model is not normalizable. To impose that the components in the Cartan sub-algebra of c, sc and $\overline{c}, s\overline{c}$ vanish, we need two more doublets. The fields of these doublets take values in the Cartan sub-algebra only and therefore have the simple transformations

$$
\begin{aligned}
s\overline{\sigma} &= \sigma, \quad s\sigma = 0, \quad \overline{\sigma}, \sigma \in \mathrm{u}(1) \\
s\overline{\gamma} &= \gamma, \quad s\gamma = 0, \quad \overline{\gamma}, \gamma \in \mathrm{u}(1).
\end{aligned}
\tag{9}
$$

The construction of the partially gauge fixed action is completed by specifying a local gauge fixing function $F_i[U]$ on the lattice configuration. A sensible gauge fixing of the SU(2) LGT to a U(1) structure group has to satisfy some non-trivial conditions. For any link configuration U of the lattice there should at least be one solution $g \in \mathcal{G}$ of

$$
F_i[U^g] = 0. \tag{10}
$$

A U(1)-invariant gauge fixing furthermore requires that Eq. (10) be U(1) invariant, that is

$$
F_i[U] = 0 \Rightarrow F_i[U^h] = 0, \quad \forall h \in \mathcal{H} \subset \mathcal{G}. \tag{11}
$$

It is easy to see that Eq. (10) always has a solution if the gauge fixing function $F_i[U]$ is the Lie-derivative of a bounded Morse potential $V[U]$

$$
\sum_i \mathrm{Tr}\ \theta_i F_i[U] = \Delta V[U], \tag{12}
$$

because Eq. (10) then is the statement that $V[U^g]$, considered as a function of $g \in \mathcal{G}$ for fixed link configuration U, has at least one extremum. This is certainly the case for bounded $V[U]$. Equation (11) is furthermore automatically satisfied if the Morse potential is U(1) invariant, i.e.

$$
V[U^h] = V[U], \quad \forall h \in \mathcal{H}. \tag{13}
$$

To have a ''lattice-covariant'' gauge fixing we pick a local Morse potential $V[U]$ that is a scalar under the action of the lattice group. The simplest non-trivial Morse potential satisfying all these requirements for the problem at hand is

$$
V[U] = \sum_{\text{links}} |\mathrm{Tr}\ \tau_+ U_{ij}|^2. \tag{14}
$$

Here $\tau_+ = \tau_-^\dagger$ and τ_0 are the su(2) matrices of the fundamental representation

$$
\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tau_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{15}
$$

with the commutation relations

$$
[\tau_+, \tau_-] = 2\tau_0, \quad [\tau_0, \tau_\pm] = \pm \tau_\pm. \tag{16}
$$

The potential (14) is bounded below and on any *finite* lattice is also bounded above. From Eq. (4) and the definition (12) of the corresponding gauge fixing function $F_i[U]$ one obtains

$$
\begin{aligned}
F_i[U] = \sum_{j \sim i} & U_{ij}\tau_+ (\mathrm{Tr}\ U_{ij}^\dagger \tau_-) - \tau_+ U_{ij}^\dagger (\mathrm{Tr}\ U_{ij}\tau_-) \\
& + U_{ij}\tau_- (\mathrm{Tr}\ U_{ij}^\dagger \tau_+) - \tau_- U_{ij}^\dagger (\mathrm{Tr}\ U_{ij}\tau_+).
\end{aligned}
\tag{17}
$$

Note that the gauge fixing function (17) is anti-Hermitian and for a particular site i involves only the links to the 2D adjacent sites. With the su(2) Lie-algebra (16) one verifies that

$$
\mathrm{Tr}\ \tau_0 F_i[U] = 0, \quad \forall U, \tag{18}
$$

on any site i. This is a consequence of the U(1)-invariance of the Morse potential (14). To construct the action we also need the BRST-variation of $F_i[U]$. Because ω_i only has a component in τ_0-direction it is of the form

$$
sF_i[U] = [\omega_i, F_i[U]] + M_i[U, c], \tag{19}
$$

with

$$M_i[U,c]=\sum_{j\sim i}(c_iU_{ij}\tau_+-U_{ij}c_j\tau_+)(\mathrm{Tr}\ U_{ij}^\dagger\tau_-)$$
$$+(\tau_+U_{ij}^\dagger c_i-\tau_+c_jU_{ij}^\dagger)(\mathrm{Tr}\ U_{ij}\tau_-)$$
$$+(U_{ij}\tau_+)(\mathrm{Tr}\ U_{ij}^\dagger(\tau_-c_j-c_i\tau_-))$$
$$+(\tau_+U_{ij}^\dagger)(\mathrm{Tr}\ U_{ij}(c_j\tau_--\tau_-c_i))$$
$$+(\tau_+\leftrightarrow\tau_-). \qquad (20)$$

Using a particular parametrization for the SU(2) link variables, the intimidating expressions (17) and (20) are simplified in Appendix A. For most of the following it suffices to note that M_i only involves links attached to the site i and is linear in the ghost field c.

The action of the partially gauge fixed LGT is a local functional in the equivariant cohomology of the BRST-symmetry we have defined. It is thus of the form

$$S=S_{\mathrm{inv.}}+S_{GF}, \quad \text{with} \quad S_{GF}=sW_{GF}, \qquad (21)$$

where W_{GF} is a local lattice action of ghost number 1 that is U(1) invariant and does not involve the ω-ghost. The restriction to operators that are relevant in the critical limit imposes additional constraints on W_{GF}. The most general relevant W_{GF} for the SU(2) model is

$$W_{GF}=\sum_i \mathrm{Tr}\left[\bar{c}_iF_i[U]+\frac{\alpha}{2}\bar{c}_ib_i+\beta\bar{c}_i^2c_i\right]$$
$$+\bar{\gamma}_i\ \mathrm{Tr}\ \tau_0\bar{c}_i+\bar{\sigma}_i\ \mathrm{Tr}\ \tau_0c_i. \qquad (22)$$

This gauge fixing functional depends on two gauge parameters α and β. Using Eq. (7), Eq. (8), Eq. (9) and Eq. (19) one finds

$$S_{GF}=sW_{GF}=\sum_i \mathrm{Tr}\left[b_iF_i[U]-\bar{c}_iM_i[U,c]+\frac{\alpha}{2}b_i^2\right.$$
$$\left.+\beta b_i[\bar{c}_i,c_i]+\beta\bar{c}_i^2c_i^2\right]+(\beta-\alpha)\phi_i^0\ \mathrm{Tr}\ \tau_0\bar{c}_i^2$$
$$+\frac{1}{2}\bar{\sigma}_i\phi_i^0+\bar{\sigma}_i\ \mathrm{Tr}\ \tau_0c_i^2+\gamma_i\ \mathrm{Tr}\ \tau_0\bar{c}_i+\bar{\gamma}_i\ \mathrm{Tr}\ \tau_0b_i$$
$$+\sigma_i\ \mathrm{Tr}\ \tau_0c_i, \qquad (23)$$

where use has been made of the fact that $\omega_i=\omega_i^0\tau_0$ and $\phi_i=\phi_i^0\tau_0$ are fields with values in the Cartan sub-algebra only. If we only consider expectation values of functionals that do not depend on $\phi,\bar{\sigma},\gamma,\bar{\gamma}$ nor σ, these fields can be eliminated by their equations of motion.

The last three terms in Eq. (23) enforce that $\bar{c},c$ as well as b are orthogonal to the τ_0-direction, i.e. they eliminate the U(1)-neutral components of these fields. It follows that $\mathrm{Tr}\ b_i[\bar{c}_i,c_i]=0$ and that c_i^2 as well as $\bar{c}_i^2$ only have components in the τ_0-direction. The equations of motion for $\bar{\sigma}_i$ and ϕ_i^0 then give rise to a quartic ghost interaction which after a bit of algebra can be brought in the form $(\alpha-\beta)\mathrm{Tr}\ \bar{c}_i^2c_i^2$. These manipulations lead to a substantially simplified (on-shell) action

$$S_{GF}^{\text{on shell}}=\sum_i \mathrm{Tr}\left[b_iF_i[U]-\bar{c}_iM_i[U,c]+\frac{\alpha}{2}b_i^2+\alpha\bar{c}_i^2c_i^2\right], \qquad (24)$$

where b_i, $\bar{c}_i$ and c_i only have components that are charged under the U(1). Note that the on-shell action (24) no longer depends on the gauge parameter β of Eq. (22). The equations of motion have removed this parameter in favor of a quartic ghost interaction proportional to α. There is a quartic ghost interaction in any gauge $\alpha\neq 0$. It is a consequence of the equivariant BRST construction and does not depend on the employed gauge fixing function F_i. As we will see in the next section there is a good reason for this quartic ghost interaction. Let me comment here that Landau gauge with $\alpha=0$ is in a certain sense an *impossible* gauge on the lattice that can only be perturbatively defined. A non-perturbative definition of Landau gauge would require finding *all* solutions to the constraint $F_i[U^g]=0$ *exactly* for any configuration U. The problem is equivalent to finding all extrema of the Morse function $V[U]$ *exactly*—clearly an impossible task for any algorithm. Unlike the constraints on the fields that we solved to arrive at Eq. (24), the condition $F_i[U]=0$ is non-local and *cannot* be *solved* analytically for large lattices. Perhaps more important, the error in the estimation of an extremum of $V[U]$ to any finite numerical accuracy can be shown to grow rapidly with the number of lattice sites. In terms of the lattice renormalization group, $\alpha=0$ is an *unstable* fixed point.

For the proof of the next two sections that the partially gauge fixed lattice theory is physically equivalent to the SU(2) LGT, it is useful to also eliminate the charged Nakanishi-Lautrup field b_i. Due to Eq. (18) one obtains the effective gauge fixing action

$$S_{GF}^{\mathrm{eff}}=\sum_i \mathrm{Tr}\left[-\frac{1}{2\alpha}F_i[U]F_i[U]-\bar{c}_iM_i[U,c]+\alpha\bar{c}_i^2c_i^2\right], \qquad (25)$$

where c_i and $\bar{c}_i$ have only components that are charged under the U(1). Numerical integration of Grassmannian variables is not possible and the local action (25) so far is a mathematical construct. To explicitly perform the Grassmann integrals, Eq. (25) would have to be bilinear in the ghosts c and $\bar{c}$. Since $M_i[U,c]$ given by Eq. (20) is linear in the ghost c this objective is achieved by introducing an auxiliary site-variable ρ_i with vanishing ghost number to *linearize* the quartic ghost interaction. The action

$$S_{GF}^{\mathrm{linear}}=\sum_i \mathrm{Tr}\left[-\frac{1}{2\alpha}F_i[U]F_i[U]-\bar{c}_iM_i[U,c]\right.$$
$$\left.-\rho_i\tau_0[\bar{c}_i,c_i]\right]+\frac{1}{4\alpha}\rho_i^2, \qquad (26)$$

is equivalent to Eq. (25) upon using the equation of motion of ρ_i and the fact that $[\overline{c_i},c_i],c_i^2,\overline{c_i}^2$ all are in the Cartan sub-algebra. The Grassmann integrals of the partition function over c_i and $\overline{c_i}$ can now be performed analytically and give the determinant of a matrix that depends on the link configuration U *and* the auxiliary field ρ. In a numerical simulation of the partially gauge fixed LGT the Gaussian average of this determinant over ρ determines the measure for the link variables.

Before proving that the partition function of the partially gauge fixed LGT does not vanish, note that the gauge fixed action $S^{\mathrm{eff}}=S_{\mathrm{inv.}}+S_{GF}^{\mathrm{eff}}$ is invariant under the following relatively simple on-shell BRST symmetry $\tilde{s}$:

$$\tilde{s}U_{ij}=c_iU_{ij}-U_{ij}c_j\,,\quad \tilde{s}c_i=0,\quad \tilde{s}\overline{c_i}=-\frac{1}{\alpha}F_i[U],\quad (27)$$

where the ghost fields satisfy the constraints

$$\mathrm{Tr}\ \tau_0 c_i=\mathrm{Tr}\ \tau_0\overline{c_i}=0,\quad (28)$$

and $F_i[U]$ is given by Eq. (17). Note that the constraint (28) on $\overline{c_i}$ is consistent with Eq. (27) due to Eq. (18). Furthermore $\tilde{s}$ is on-shell nilpotent on functions that are invariant with respect to the U(1) gauge group. Thus

$$\tilde{s}^2U_{ij}=c_i^2U_{ij}-U_{ij}c_j^2,\quad (29)$$

effects an infinitesimal U(1) gauge transformation generated by $c^2\propto\tau_0$. Using the equation of motion for $\overline{c_i}$

$$M_i[U,c]=\alpha[\overline{c_i},c_i^2],\quad (30)$$

we have that

$$\tilde{s}^2\overline{c_i}=-\frac{1}{\alpha}M_i[U,c]\simeq[c_i^2,\overline{c_i}],\quad (31)$$

and thus on-shell is equivalent to an infinitesimal U(1) gauge transformation generated by c^2. We similarly obtain using Eq. (17) and Eq. (29) that

$$\tilde{s}^2F_i[U]=\tilde{s}M_i[U,c]=[c_i^2,F_i[U]].\quad (32)$$

The BRST-symmetry $\tilde{s}$ thus defines an *equivariant* cohomology on the (graded) Grassmann algebra of the set of U(1)-invariant functions

$$\mathcal{B}:=\{A[U,c]:A[U^h,c^h]=A[U,c]\forall h\in\mathcal{H}\},\quad (33)$$

of the link variables and ghost field c. The nontrivial observables of the partially gauge fixed LGT is the equivariant cohomology Σ,

$$\Sigma:=\{O\in\mathcal{B}:\tilde{s}O=0,O\neq\tilde{s}E,\forall E\in\mathcal{B}\}.\quad (34)$$

The functions in Σ with vanishing ghost number are the gauge-invariant functions of the links only, i.e. Wilson loops and their (linked) products. The physical observables of the original SU(2) LGT thus constitute the sector with vanishing ghost number of the equivariant cohomology Σ.

III. THE TOPOLOGICAL LATTICE THEORY (TLT)

I still have to show that the expectation value of a physical observable $O[U]\in\Sigma$

$$\langle O[U]\rangle:=\int\prod_{links}dU_{ij}\prod_{sites}d^2c_id^2\overline{c_i}O[U]\times\exp\{-S^{\mathrm{eff}}[u,c,\overline{c},\alpha]\},\quad (35)$$

with the action

$$S^{\mathrm{eff}}[U,c,\overline{c},\alpha]=S_{\mathrm{inv.}}[U]+S_{GF}^{\mathrm{eff}}[U,c,\overline{c},\alpha],\quad (36)$$

up to an overall (non-vanishing) normalization $\mathcal{N}(\alpha)$ is the expectation value of the observable in the original LGT with the gauge-invariant measure. We thus wish to show that

$$\langle O[U]\rangle=\mathcal{N}(\alpha)\int\prod_{links}dU_{ij}O[U]\exp\{-S_{\mathrm{inv.}}[U]\}=:\langle O[U]\rangle_{\mathrm{inv.}},\quad (37)$$

for *all* physical observables $O[U]$ and any *finite* lattice.

Since the volume of the SU(2) lattice gauge group of a finite lattice is a finite non-vanishing constant,

$$V_{\mathcal{G}}=\int\prod_{sites}dg_i<\infty,\quad (38)$$

we can multiply both sides of Eq. (35) by $V_{\mathcal{G}}$ and change the integration variables

$$U_{ij}=U_{ij}^{\prime g}=g_iU'_{ij}g_j^{\dagger}.\quad (39)$$

The Haar-measure $dU_{ij}=dU'_{ij}$ as well as the gauge invariant part of the lattice action and the observable $O[U]=O[U']$ are invariant under this (gauge) transformation and Eq. (35) becomes

$$V_{\mathcal{G}}\langle O[U]\rangle=\int\prod_{links}dU_{ij}O[U]\mathcal{Z}[U,\alpha]e^{-S_{\mathrm{inv.}}[U]},\quad (40)$$

where

$$\mathcal{Z}[U,\alpha]=\int\prod_{sites}dg_id^2c_id^2\overline{c_i}e^{-S_{GF}^{\mathrm{eff}}[U^g,c,\overline{c};\alpha]}.\quad (41)$$

Evidently $\mathcal{Z}[U^g,\alpha]=\mathcal{Z}[U,\alpha]$ is itself a gauge invariant observable. For Eq. (37) to hold for *all* observables $O[U]$, $\mathcal{Z}[U,\alpha]$ must be a constant that does not depend on the link configuration U at all. We therefore have to show that Eq. (41) is a non-vanishing constant on the configuration space, i.e. that the model defined by the partition function (41) is a TLT.

I will first determine the α-dependence of $\mathcal{Z}[U,\alpha]$ and then show that this partition function does not depend on a continuous deformation of the configuration U. The basis for these conclusions is that $S_{GF}^{\mathrm{eff}}[U^g,c,\bar{c};\alpha]$ is invariant with respect to a (on-shell nilpotent) BRST-symmetry $\hat{s}$ defined on the variables as

$$\hat{s}U_{ij}=0$$

$$\hat{s}g_i=c_ig_i\,,\quad \hat{s}g_i^\dagger=-g_i^\dagger c_i$$

$$\hat{s}c_i=0,\quad \hat{s}\bar{c}_i=-\frac{1}{\alpha}F_i[U^g]. \tag{42}$$

Note that the algebra (42) is very similar to the BRST-algebra Eq. (27) but does not transform the link configuration U. The third relation in Eq. (42) is a consequence of the second and $g_ig_i^\dagger=1$. The invariance of $S_{GF}^{\mathrm{eff}}[U^g,c,\bar{c};\alpha]$ follows immediately from the invariance of $S_{GF}^{\mathrm{eff}}[U,c,\bar{c};\alpha]$ under $\tilde{s}$ and

$$\hat{s}(g_iU_{ij}g_j^\dagger)=c_ig_iU_{ij}g_j^\dagger-g_iU_{ij}g_j^\dagger c_j=\tilde{s}U_{ij}|_{U_{ij}\to g_iU_{ij}g_j^\dagger}. \tag{43}$$

Since $\hat{s}c=0$, the measure $dg_id^2c_id^2\bar{c}_i$ in Eq. (41) is evidently $\hat{s}$-invariant if dg_i is the Haar-measure of the structure group.

To simplify notation I define the (not normalized) expectation value of any function X of the fields $g,c,\bar{c}$ in the TLT

$$\langle X\rangle_{U,\alpha}:=\int\prod_{sites}dg_id^2c_id^2\bar{c}_iXe^{-S_{GF}^{\mathrm{eff}}[U^g,c,\bar{c};\alpha]}. \tag{44}$$

The function X can itself depend parametrically on the configuration U and the gauge parameter α. In this notation, $\mathcal{Z}$ of Eq. (41) is just $\langle 1\rangle_{U,\alpha}$. Using Eq. (25), the definition (41) implies that

$$\begin{aligned}\alpha\frac{\partial}{\partial\alpha}\mathcal{Z}[U,\alpha]&=-\left\langle\sum_{sites}\mathrm{Tr}\,\frac{1}{2\alpha}F_i[U^g]F_i[U^g]+\alpha\bar{c}_i^2c_i^2\right\rangle_{U,\alpha}\\&=\left\langle\hat{s}\sum_{sites}\frac{1}{2}\,\mathrm{Tr}\,\bar{c}_iF_i[U^g]\right\rangle_{U,\alpha}\\&\quad+\left\langle\sum_{sites}\mathrm{Tr}\,\frac{1}{2}\,\bar{c}_iM_i[U^g,c]-\alpha\bar{c}_i^2c_i^2\right\rangle_{U,\alpha}\\&=NZ[U,\alpha],\end{aligned} \tag{45}$$

for a lattice with N sites. The last equality in Eq. (45) is a consequence of the $\hat{s}$-invariance of S_{GF}^{eff} and the measure and of the equation of motion (30). Due to Eq. (45),

$$\tilde{\mathcal{Z}}[U]=\alpha^{-N}\mathcal{Z}[U,\alpha], \tag{46}$$

is a gauge invariant functional of the configuration U that does not depend on α.

Similar reasoning shows that $\mathcal{Z}[U,\alpha]$ does not change under a *continuous* deformation of the orbit. Since $\hat{s}U=0$, we symbolically have

$$\begin{aligned}&\delta_UZ[U,\alpha]\\&\quad=\left\langle\delta_U\sum_i\mathrm{Tr}\left[\frac{1}{2\alpha}F_i[U^g]F_i[U^g]+\bar{c}_iM_i[U^g,c]\right]\right\rangle_{U,\alpha}\\&\quad=-\left\langle\hat{s}\,\delta_U\sum_i\mathrm{Tr}\,\frac{1}{2}\,\bar{c}_iF_i[U]\right\rangle_{U,\alpha}\\&\qquad+\left\langle\delta_U\sum_i\mathrm{Tr}\left[\frac{1}{2}\,\bar{c}_iM_i[U^g,c]-\alpha\bar{c}_i^2c_i^2\right]\right\rangle_{U,\alpha}\\&\quad=0,\end{aligned} \tag{47}$$

where the last equality again makes use of the equation of motion (30). [In Eq. (47) the variation δ_U of the link variables of course respects $U_{ij}\in\mathrm{SU}(2)$.]

The property Eq. (47) that $\mathcal{Z}[U,\alpha]$ (and thus also $\tilde{\mathcal{Z}}[U]$) is constant on a connected set of link configurations greatly simplifies our task. To determine the value of $\tilde{\mathcal{Z}}[U]$ we need only consider a particular link configuration in each connected sector of the orbit space. In a LGT *every* link configuration is connected to the trivial one with $U_{ij}=1$ on all links. Thus Eq. (47) implies that $\tilde{\mathcal{Z}}[U]$ is a constant that does not depend on the link configuration. To show that this constant does not *vanish*, it is sufficient that

$$\tilde{\mathcal{Z}}[U=1]\neq 0, \tag{48}$$

for any finite lattice. Equation (47) and Eq. (46) together with Eq. (40) imply that the expectation value (35) of *any* physical observable O in the partially gauge fixed LGT is proportional to the expectation value of the same observable in the original SU(2) LGT. The proportionality constant furthermore does not depend on the observable and does not vanish when Eq. (48) holds.

Equation (47) together with Eq. (46) establish that the model described by the partition function $\tilde{\mathcal{Z}}[U]$ is the lattice version of a TQFT (of Witten type) on the space $\mathcal{G}/\mathcal{H}$. The partition function of this TLT is some topological characteristic of the coset space. In the next section we will explicitly demonstrate that $\tilde{\mathcal{Z}}$ is proportional to the Euler characteristic $\chi(\mathcal{G}/\mathcal{H})$. Since $\chi(\mathcal{G}/\mathcal{H})=\chi[\otimes_{\mathrm{sites}}\mathrm{SU(2)/U(1)}\simeq S_2]=[\chi(S_2)]^N=2^N\neq 0$, this will prove that $\tilde{\mathcal{Z}}$ indeed does not vanish. Note that the basic reason for only partially gauge fixing the SU(2) LGT using an equivariant BRST construction was that $\chi(\mathcal{G})=0$—the partition function of a TLT that is proportional to the Euler character of the compact lattice gauge group would have vanished no matter what Morse potential one chooses.

IV. SEMI-CLASSICAL EVALUATION OF $\tilde{\mathcal{Z}}[U=1]$

Although multi-dimensional, a LGT is nevertheless only a statistical mechanical system. Even more importantly, the variables of this system are compact. Consequently the lattice action $S_{\mathrm{inv.}}$, and also $V[U]$ defined by Eq. (14) are *bounded* functions for any finite lattice. We are in the fortunate position that almost all requirements of Morse theory (which generally applies to compact spaces and bounded functions) are satisfied for the TLT. At this point we could therefore simply cite the Poincaré-Hopf theorem and known results from topological quantum mechanical models [11] to assert that the partition function $\tilde{\mathcal{Z}}[U]$ is proportional to the Euler characteristic $\chi(\mathcal{G}/\mathcal{H})$ of the manifold that is the domain of the *bounded* Morse-function $V_U[g]$

$$V_U[g]=V[U^g]:\mathcal{G}/\mathcal{H}\to R, \tag{49}$$

when $V[U^g]$ is considered as a function of the gauge transformation for fixed link configuration U. Since $\chi(\mathcal{G}/\mathcal{H})=2^N\neq 0$ this would prove our assertions.

The TLT on the other hand is a sufficiently simple model for us to *explicitly* see these topological theorems at work. The following computation of $\tilde{\mathcal{Z}}[1]$ also shows which ''pure gauge'' configurations give a vanishing contribution to $\tilde{\mathcal{Z}}[1]$ in the limit $\alpha\to 0$ and which don't. In Sec. VI this gives us greater certainty in the evaluation of correlation functions in the *critical limit* $g^2\to 0$ of the gauge fixed model since only a certain class of saddle points contributes in the limit $\alpha\to 0$. In the course of the calculation we will furthermore characterize *all* Gribov copies of the vacuum configuration $U=1$ to the gauge condition $F_i[1^g]=0$. Perhaps the most interesting aspect of the computation is the important role of the quartic ghost interaction in Eq. (25).

Using the result of the previous section that $\tilde{\mathcal{Z}}$ does not depend on the gauge parameter α, we may choose α sufficiently small for a saddle point approximation to the integral (41) to be as accurate as we please. Although I will not explicitly compute the errors of the saddle point approximation, it is quite obvious that the evaluation becomes exact in the limit $\alpha\to 0$ for a lattice with $N<\infty$ sites, because Σ_i Tr $F_i[1^g]F_i[1^g]$ in this case is a bounded function on a *finite* dimensional space of gauge transformations.

To compute

$$\tilde{\mathcal{Z}}[1]=\lim_{\alpha\to 0_+}\tilde{\mathcal{Z}}[1], \tag{50}$$

with the action (25) in the definition (41) of $\mathcal{Z}$, we need to consider *all* solutions $\tilde{g}$ to the equations

$$F_i[1^{\tilde{g}}]=0\forall\ \text{sites}\ \ i. \tag{51}$$

Because $F_i[U]$ is the Lie-derivative (12) of the Morse-potential (14), Eq. (51) in principle requires us to determine all *extrema* of

$$V[g]=V[U_{ij}=g_i g_j^\dagger]=\sum_{\text{links}}|\text{Tr}\ \tau_+ g_i g_j^\dagger|^2, \tag{52}$$

in the space of lattice gauge transformations. By construction, Eq. (52) is invariant with respect to left-handed U(1) gauge transformations $h\in\mathcal{H}$,

$$V[hg]=V[g]\forall h\in\mathcal{H}. \tag{53}$$

We can use the invariance (53) to parametrize the SU(2)/U(1) coset element g_i at each site by only *two real* angles,

$$g_i=\begin{pmatrix}\cos(\theta_i/2) & \sin(\theta_i/2)e^{i\varphi_i}\\ -\sin(\theta_i/2)e^{-i\varphi_i} & \cos(\theta_i/2)\end{pmatrix}, \tag{54}$$

with $\theta_i\in[0,\pi]$ and $\varphi_i\in[0,2\pi)$. We can always choose $h_i\in U(1)$ to eliminate the phase in the diagonal elements of g_i. At $\theta_i=\pi$ the diagonal elements of g_i in Eq. (54) vanish and the phase of the off-diagonal elements can be arbitrarily changed by an U(1)-transformation. Identifying all the points $(\theta=\pi,\varphi)$ we see that there is a one-to-one correspondence

$$g_i(\theta_i,\varphi_i)\in \text{SU(2)/U(1)}\leftrightarrow \hat{s}_i\in S_2, \tag{55}$$

between $g_i\in$ SU(2)/U(1) and unit ''spins'' $\hat{s}_i=(\sin\theta_i\cos\phi_i,\sin\theta_i\sin\varphi_i,\cos\theta_i)$ describing a two-dimensional sphere. This is of course just the statement that the coset manifold SU(2)/U(1)$\simeq S_2$. Using the parametrization (54), Eq. (52) after a bit of algebra can be seen to be the energy of the Heisenberg model,

$$V[g]=\frac{1}{4}\sum_{i\sim j}(\hat{s}_i-\hat{s}_j)^2. \tag{56}$$

The relation (56) helps to visualize and classify the extrema of $V[g]$. $V[g]$ possesses a *continuous global* SO(3) invariance corresponding to a coherent rotation of all the spins $\hat{s}_i$. The extrema of $V[g]$ are thus characterized by the subgroup of SO(3) under which they are invariant. There are only two kinds of extrema:

(I) extrema that are invariant under an SO(2) subgroup of SO(3). In this case all the spins are collinear. There are *two* zero modes associated with any extremum of this kind, corresponding to the broken generators of the coset space SO(3)/SO(2). These zero modes correspond to infinitesimal global rotations of the collinear spins, whereas an SO(2)-rotation along the axis of any particular spin does not change these extremal configurations. Thus type I extrema fall in classes $[\tilde{g}]_I$ modulo global rotations of all the spins. One can select a unique representative of such a class by specifying the direction of any particular spin. There is a one-to-one correspondence between configurations in $[\tilde{g}]_I$ and points on a two-dimensional sphere S_2. Since all the spins are collinear, there are exactly 2^{N-1} classes $[\tilde{g}]_I$ on a lattice with N sites.

(II) extrema that are *not* invariant under any continuous subgroup of SO(3). In this case the spins are not *all* collinear. An example of this kind of extrema are the solitons of the 1-dimensional periodic spin chain [12]. By Goldstone's theorem there are *three* zero modes corresponding to the generators of SO(3). Their action on a particular configuration can be visualized as follows: two generators correspond to global rotations of the extremal configuration. The third effects an infinitesimal SO(2)-rotation of the configuration along the axis of a *particular* spin. Thus type II extrema fall into classes $[\tilde{g}]_{II}$ modulo SO(3) rotations. A particular representative of such a class is selected by specifying the direction of one of the spins, say $\hat{s}_0$ *and* the direction of $\hat{s}_0\times\hat{s}_j$, where s_j is a specific spin of the configuration that is *not* collinear to s_0. There is thus a one-to-one correspondence between configurations in $[\tilde{g}]_{II}$ and the points of a 3-dimensional sphere S_3.

The above classification of the extrema of $V[g]$ is complete in the sense that there are no other *continuous* symmetries relating extremal configurations.

Expanding $V[g]$ and $F_i[1^g]$ to quadratic-, respectively linear-, order near an extremum $\tilde{g}$ one has

$$V[g=\tilde{g}e^{\theta}]=V[\tilde{g}]-\sum_i \text{ Tr } \theta_i^{\dagger}M_i[1^{\tilde{g}},\theta]+O(\theta^3)$$

$$F_i[1^g]=M_i[1^{\tilde{g}},\theta]+O(\theta^2). \tag{57}$$

For the saddle point evaluation it is useful to expand in terms of eigenvectors of the $2N$ linear equations

$$M_i[1^{\tilde{g}},\phi^{(n)}]=\lambda_{(n)}\phi_i^{(n)}, \quad n=1,2,\ldots,2N \tag{58}$$

where the eigenvalues $\lambda^{(n)}$ and eigenvectors $\phi^{(n)}$ implicitly depend on the extremum $\tilde{g}$. Since $V[g]$ in Eq. (56) is a globally SO(3) invariant real function of the spins, Eq. (57) implies that the eigenvalues $\lambda_{(n)}$ are real and depend only on the class $[\tilde{g}]$ of the extremal configuration (and not on the particular representative of that class). Since the quadratic form in Eq. (57) is real, the eigenvectors $\phi^{(n)}$ furthermore can be chosen to form a complete orthonormal set with respect to the inner product

$$\langle n|m\rangle=\sum_{sites} \text{ Tr } \phi_i^{(n)\dagger}\phi_i^{(m)}=\delta_{nm}. \tag{59}$$

In the vicinity of an extremal configuration $\tilde{g}$, the action S_{GF}^{eff} is of the form [using the expansions (57)],

$$S_{GF}^{\text{eff}}[\alpha;\tilde{g}e^{\theta},c,\bar{c}]\sim\sum_i \text{ Tr}\Bigg[\frac{1}{2\alpha}M_i^{\dagger}[1^{\tilde{g}},\theta]M_i[1^{\tilde{g}},\theta] - \bar{c}_iM_i[1^{\tilde{g}},c]+\alpha\bar{c}_i^2c_i^2\Bigg], \tag{60}$$

up to terms of order θ^3, respectively $\theta c\bar{c}$. Since we omitted terms of order θ^3 and $\theta\bar{c}c$ in the expansion (60), retaining the quartic ghost interaction could appear questionable. We will however soon see that the sole purpose of the quartic ghost interaction to leading order in α is to absorb Grassmannian zero-modes. The neglected terms are higher order variations of the Morse-potential and therefore do not couple to the zero-modes. The *leading* contribution in α can thus be calculated using Eq. (60). Note also that $M_i[U,\theta]$ given by Eq. (20) is anti-Hermitian.

To diagonalize the quadratic form in Eq. (60) we expand θ,c and $\bar{c}$ in the complete set of orthonormal eigenvectors of Eq. (58)

$$\theta_j=i\sum_n \xi^{(n)}\phi_j^{(n)}, \quad c_j=\sum_n c^{(n)}\phi_j^{(n)}, \quad \bar{c}_j=\sum_n \bar{c}^{(n)}\phi_j^{(n)\dagger}, \tag{61}$$

with real coefficients $\xi^{(n)}$ and Grassmannian variables $c^{(n)},\bar{c}^{(n)}$. In terms of these coefficients the action (60) in the vicinity of the extremum $\tilde{g}$ takes the form

$$S_{GF}^{\text{eff}}[\tilde{g},\alpha;\{\xi^{(n)},c^{(n)},\bar{c}^{(n)}\}] \sim\sum_n\Bigg[\frac{1}{2\alpha}\xi^{(n)}\lambda_{(n)}^2\xi^{(n)}-\bar{c}^{(n)}\lambda_{(n)}c^{(n)} +\alpha\sum_{klm}R_{klmn}\bar{c}^{(k)}\bar{c}^{(l)}c^{(m)}c^{(n)}\Bigg], \tag{62}$$

with

$$R_{klmn}=\sum_{sites} \text{ Tr } \phi_i^{(k)\dagger}\phi_i^{(l)\dagger}\phi_i^{(m)}\phi_i^{(n)}. \tag{63}$$

The change of basis (61) diagonalizes the quadratic part of the action near an extremum. The remaining quartic ghost interaction is irrelevant for the semi-classical evaluation *except* for Grassmannian zero-modes that do not enter quadratically.

As noted above, extrema of type I are characterized by *two* zero-modes with vanishing eigenvalues. I will denote these eigenvectors by $\phi^{(1)},\phi^{(2)}$ in the following $(\lambda_{(1)}=\lambda_{(2)}=0)$. The SO(2) symmetry of type I extrema also implies that the dimension of the space of solutions to a given eigenvalue is *even* [there are no SO(2)-invariant eigenmodes in this case]. We thus can arrange matters so that $\lambda_{(2m)}=\lambda_{(2m-1)}$, $m=1,\ldots,N$.

There are on the other hand *three* zero-modes for type II extrema, which I will label $\phi^{(1)},\phi^{(2)},\phi^{(3)}$, with $\lambda_{(1)}=\lambda_{(2)}=\lambda_{(3)}=0$.

In a semi-classical evaluation of $\tilde{\mathcal{Z}}[1]$ the zero-modes have to be handled with care. The introduction of collective coordinates for the bosonic zero modes is standard [13]:

(i) The representatives in a class $[\tilde{g}]_I$ of type I extrema, are described by two collective angles θ,φ which (for instance) denote the direction of $\hat{s}_0$, the spin at a particular site.

(ii) a particular representative in a class $[\tilde{g}]_{II}$ of type II extrema is specified by three collective angles θ,φ,ψ. θ,φ again give the direction of $\hat{s}_0$, while $\psi\in[0,\pi]$ can be chosen to denote the direction of $\hat{s}_0\times\hat{s}_j$, where $\hat{s}_j$ is a particular spin that is *not* collinear to $\hat{s}_0$. The range of ψ is restricted to $[0,\pi]$, since $\psi\in[\pi,2\pi]$ are equivalent configurations (as can be seen by interchanging the meaning of $\hat{s}_0$ and $\hat{s}_j$ in the above definitions of the angles). These three collective angles parametrize an S_3.

In terms of the angles θ_i,φ_i parameterizing the coset SU(2)/U(1) as in Eq. (54), the Haar-measure dg_i is proportional to

$$\int dg_i \to \int dh_i \int_{S_2} d\Omega_i = \int dh_i \int_0^{\pi} \sin\,\theta_i d\theta_i \int_0^{2\pi} d\varphi_i , \tag{64}$$

where dh_i is the Haar-measure of the U(1) group. After the change of variables (61) the bosonic measure in Eq. (41) for sufficiently small fluctuations $\xi^{(n)}$ near an extremal solution $\tilde{g}$ thus becomes

$$\prod_{sites} dg_i \bigg|_{\tilde{g}} = V_{\mathcal{H}} \prod_{n=1}^{2N} d\xi^{(n)}. \tag{65}$$

For $\lambda_{(n)} \neq 0$ the fluctuation $\xi^{(n)}$ is of order $\sqrt{\alpha}$ and the approximation (65) in this case is valid for sufficiently small α. The coefficients of bosonic zero-modes on the other hand are not suppressed. These fluctuations generate (small) SO(3) rotations of the extremal configuration $\tilde{g}$ and are replaced by integrations over the collective coordinates of the extremal configurations in the corresponding class $[\tilde{g}]$. The correct semi-classical measure for the integration of bosonic fluctuations around a class of type I extrema thus is

$$\prod_{sites} dg_i \bigg|_{[\tilde{g}]_I} = V_{\mathcal{H}} \left(\int_{S_2} d\Omega_2 \right) \prod_{n=3}^{2N} d\xi^{(n)}, \tag{66}$$

where $d\Omega_2 = \sin\,\theta d\theta d\phi$ is the parametrization of the S_2 in terms of the collective coordinates. Similarly the semi-classical measure for a class of type II extrema is

$$\prod_{sites} dg_i \bigg|_{[\tilde{g}]_{II}} = V_{\mathcal{H}} \left(\int_{S_3} d\Omega_3 \right) \prod_{n=4}^{2N} d\xi^{(n)}, \tag{67}$$

where $d\Omega_3 = \sin\,\theta d\theta d\phi \sin^2\,\psi d\psi$ is the parametrization of S_3 in terms of the collective angles. The Jacobian for the change of basis (61) is a constant and the measure for the Grassmann-coefficients $c^{(n)}, \bar{c}^{(n)}$ thus can be written

$$\prod_{sites} d^2 c_i d^2 \bar{c}_i = \prod_{n=1}^{2N} dc^{(n)} d\bar{c}^{(n)}. \tag{68}$$

Using Eq. (46), Eq. (62) and the appropriate semi-classical measures (66), (67) and (68), the saddle point evaluation of $\tilde{\mathcal{Z}}[1]$ gives

$$\begin{aligned}\tilde{\mathcal{Z}}[1] &= \lim_{\alpha\to 0_+} \alpha^{-N} \int \prod_{sites\ \ i} dh_i d\Omega_i d^2 c_i d^2 \bar{c}_i e^{-S_{GF}^{\rm eff}[1^g, c, \bar{c}; \alpha]} \\ &= V_{\mathcal{H}} \left\{ \sum_{[\tilde{g}]_I} Z_I([\tilde{g}]_I) + \sum_{[\tilde{g}]_{II}} Z_{II}([\tilde{g}]_{II}) \right\},\end{aligned} \tag{69}$$

with the semi-classical weights

$$\begin{aligned} Z_I([\tilde{g}]_I) = \lim_{\alpha\to 0_+} \alpha^{-N} \int_{S_2} d\Omega_2 \prod_{n=3}^{2N} d\xi^{(n)} \prod_{n=1}^{2N} dc^{(n)} d\bar{c}^{(n)} \\ \times \exp\{-S_{GF}^{\rm eff}[\tilde{g}(\Omega_2), \alpha; \{\xi^{(n)}, c^{(n)}, \bar{c}^{(n)}\}]\},\end{aligned} \tag{70}$$

and

$$\begin{aligned} Z_{II}([\tilde{g}]_{II}) = \lim_{\alpha\to 0_+} \alpha^{-N} \int_{S_3} d\Omega_3 \prod_{n=4}^{2N} d\xi^{(n)} \prod_{n=1}^{2N} dc^{(n)} d\bar{c}^{(n)} \\ \times \exp\{-S_{GF}^{\rm eff}[\tilde{g}(\Omega_3), \alpha; \{\xi^{(n)}, c^{(n)}, \bar{c}^{(n)}\}]\},\end{aligned} \tag{71}$$

of a class of extrema of type I, respectively type II. The crucial observation that enables us to actually compute $\tilde{\mathcal{Z}}[1]$ is that the weight Z_{II} vanishes. It vanishes due to the 3^{rd} zero-mode of type II extrema. The argument goes as follows. In Eq. (71) we may perform the bosonic and fermionic integrations of all modes except the zero-modes corresponding to $n=1$, 2 or 3. The integrals are Gaussian and the quartic ghost interaction in Eq. (62) to leading order in α does not contribute to these integrations. The Grassmann integration of a pair $c^{(n)}, \bar{c}^{(n)}$ and the corresponding bosonic integral over $\xi^{(n)}$ for $n \neq 1$, 2 or 3 results in a factor proportional to

$$\lambda_{(n)} \sqrt{\frac{\alpha}{(\lambda_{(n)})^2}} = \pm \sqrt{\alpha}, \tag{72}$$

depending on whether $\lambda_{(n)}$ is a positive or negative eigenvalue ($\lambda^{(n)} \neq 0$ for $n \neq 1,2,3$). We can perform $2N-3$ integrals in this fashion and the expression for Z_{II} to leading order in α (up to an irrelevant finite and α-independent normalization) becomes

$$\begin{aligned} Z_{II}([\tilde{g}]_{II}) &= \lim_{\alpha\to 0_+} \pm \alpha^{-3/2} \int_{S_3} d\Omega_3 \int \prod_{n=1}^{3} dc^{(n)} d\bar{c}^{(n)} \\ &\quad \times \exp\left[-\alpha \sum_{klmn} R_{klmn} \bar{c}^{(k)} \bar{c}^{(l)} c^{(m)} c^{(n)} \right] \\ &= \lim_{\alpha\to 0_+} \pm 8\pi^2 \alpha^{-1/2} R_{1212} \int dc^{(3)} d\bar{c}^{(3)} \\ &= 0. \end{aligned} \tag{73}$$

The coefficient of the leading term in the loop expansion of Z_{II} vanishes due to two uncompensated Grassmann modes. The integration over the corresponding bosonic zero-modes is *finite* because S_3 is compact. [This is in agreement with the Poincaré-Hopf theorem which states that the contribution of such a class of extrema is proportional to $\chi(S_3)=0$.] The objection that we only computed the coefficient of the term of order $1/\sqrt{\alpha}$ and that higher orders of the loop expansion could lead to a finite result does not hold, because the parameter α in this calculation *is* the loop parameter. Corrections to the above result thus are at least of order $\sqrt{\alpha}$ and vanish in the limit $\alpha \to 0_+$. The weight Z_{II} of a single class of type II extrema therefore indeed vanishes. With a finite number of spins one furthermore expects only a finite number of such classes. In this case the total contribution of type II extrema to $\tilde{\mathcal{Z}}[1]$ also certainly vanishes. Thus the number of classes of type II extrema for the 1-dimensional periodic spin chain is given by its length. Since none of our arguments explicitly depend on the dimensionality of the lattice, it is safe to conclude that type II extrema give a vanishing contribution to $\tilde{\mathcal{Z}}[1]$ on any finite periodic lattice.

The semi-classical weight of a class $[\tilde{g}]_I$ of type I extrema on the other hand does *not* vanish. These are solutions of Eq. (51) where all the spins are collinear. As noted before, the SO(2)-invariance of such an extremal configuration implies that for every eigenvector $\phi^{(2m)}$, there is also an orthogonal one $\phi^{(2m-1)}$ to the *same* eigenvalue. The latter is just an SO(2) rotation by 90° around the common spin axis of the first. Eigenvalues thus come in pairs. Proceeding as before and performing the $2(N-1)$ bosonic and Grassmann integrals over $\xi^{(n)}$, $c^{(n)}$ and $\bar{c}^{(n)}$ with $n\neq 1,2$ in Eq. (70), one obtains (again up to an irrelevant finite and α-independent overall normalization)

$$\begin{aligned} Z_I([\tilde{g}]_I) = & \lim_{\alpha\to 0_+} \alpha^{-1}\int_{S_2} d\Omega_2 \int \prod_{n=1}^{2} dc^{(n)} d\bar{c}^{(n)} \\ & \times \exp\left[-\alpha\sum_{klmn} R_{klmn}\bar{c}^{(k)}\bar{c}^{(l)}c^{(m)}c^{(n)}\right] \\ = & 4\pi(4R_{1212}) = 8\pi/N, \end{aligned} \tag{74}$$

for the weight of any class of type I extrema. To evaluate R_{1212} in Eq. (74) I used the zero-modes $\phi^{(1)}$ and $\phi^{(2)}$ corresponding to global rotations of the extremal configuration and normalized by Eq. (59). For a collinear spin configuration of type I these zero-modes are readily found and the result for R_{1212} defined by Eq. (63) does not depend on the (collinear) configuration. Note that the semi-classical weight Z_I of each class is the same. By suitably normalizing the Haar-measure, we can thus set $Z_I([\tilde{g}]_I)=2$ for any class $[\tilde{g}]_I$. [I choose this normalization of the weight in accordance with the Poincaré-Hopf theorem, where the contribution of an S_2 manifold of extremal solutions is normalized to $\chi(S_2)=2$.] Relative to the direction of one of the spins, the other collinear spin can be either parallel or anti-parallel. There are thus $2^{(N-1)}$ classes of extremal configurations of type I and we finally obtain (with the conventional normalization)

$$\tilde{\mathcal{Z}}[1] = 2^N = \chi((S_2)^N) = \chi(\mathcal{G}/\mathcal{H}) \neq 0, \tag{75}$$

in complete agreement with the Poincaré-Hopf theorem.

V. GAUGE FIXING OF THE RESIDUAL U(1) GAUGE GROUP

In the last three sections we have shown that the partially gauge fixed U(1)-invariant LGT is normalizable and reproduces the expectation values of gauge-invariant physical observables of the original SU(2) LGT. The lattice action (36) is local, invariant with respect to the Abelian lattice gauge group $\mathcal{H}$ and preserves the space-time symmetries of the lattice. This model could be of considerable interest in the numerical investigation of LGT because its structure group is Abelian. Following the procedures of [14] one perhaps can also derive a corresponding *dual* lattice model.

A perturbative evaluation however requires a further reduction of the U(1) structure group to a discrete one. Forcrand and Hetrick [15] presented an elegant algorithm to *uniquely* and covariantly fix the gauge of an Abelian LGT by Hodge decomposition. Their procedure solves the problem of covariant Abelian gauge fixing from a numerical point of view. The algorithm is however non-local and I have not been able to derive the corresponding effective gauge fixed action it generates. Recently an alternative solution [16] was suggested that corresponds to a certain coherent superposition of Sharpe's gauges [2]. To apply these gauge-fixing ideas to the Abelian subgroup of SU(2) is not entirely trivial nor very transparent and will not be pursued here. It has been argued [2,17] that a BRST-symmetric local ''covariant'' lattice action of the link-variables that is physically equivalent to a U(1) LGT with well-defined lowest order continuum propagators does not exist. This is in agreement with our topological considerations. From the topological point of view this problem is a consequence of the fact that $\chi[\mathrm{U}(1)/Z_n]=\chi[\mathrm{U}(1)]=0$ for any (finite) discrete subgroup $Z_n\subset \mathrm{U}(1)$. The partition function of a TLT that localizes on a gauge fixing surface derivable from a Morse potential in the U(1) case thus vanishes (and consequently also the partition function of the ''gauge-fixed'' BRST-invariant model). Unfortunately the ''linear'' covariant gauge condition that gives well-defined continuum propagators *is* the Lie-derivative of a Morse function [18].

Requiring that a non-Abelian gauge-fixed local lattice action leads to well-defined propagators in the (naive) continuum limit could however simply be too much to ask—the continuum model is after all related to the continuum gauge group, which is non-compact and topologically quite different from the compact structure group of the lattice. We in fact *can* demand that a loop expansion of the gauge-fixed *non-Abelian* LGT makes sense although the naive continuum propagators are ill-defined. One should stress in this context that a loop expansion of lattice correlators coincides with the conventional perturbative expansion only for vanishing (bare) coupling g^2. The loop expansion is obtained by expanding the *full* bosonic *lattice* measure in the vicinity of its maximum for small but *finite* bare coupling. Additional quadratic terms of sub-leading order in g^2 arise in such an expansion of the effective action from the Haar-measure as well as the ghost- (and possibly the fermionic-) determinants. The quadratic terms from the ghost-determinant and the Haar-measure generally are not transverse in a non-Abelian LGT and thus lead to well-defined *lattice propagators*. For sufficiently small coupling g^2, the transverse (physical) part of these lattice propagators is dominated by the naive continuum expression whereas the longitudinal part is formally of order $1/g^2$. The loop expansion should nevertheless result in an *analytic* g^2-expansion of gauge invariant (physical) lattice correlators. The loop expansion of unphysical correlation functions generally will not be analytic in g^2. This systematic expansion of the bosonic lattice measure is perhaps rather similar to the phenomenologically successful *tadpole improved* lattice perturbation theory [19].

One *can* isolate the classical configuration maximizing the measure with respect to (lattice) gauge transformations by reducing the gauge symmetry of the Abelian LGT covariantly while preserving a BRST-symmetry. The partition function of the corresponding TLT simply *must not* be proportional to the Euler characteristic of the U(1) structure group. One has to choose some *non-vanishing* topological invariant of the group manifold—such as the number of connected components. The construction of such a TLT below is based on a nilpotent BRST-symmetry δ and proves that a local Z_2 LGT is physically equivalent to the original SU(2) LGT. A numerical simulation of this Z_2 LGT is furthermore hardly more complicated than the simulation of the partially fixed U(1)-invariant model.

The key to a further reduction of the continuous gauge symmetry of the U(1)-invariant lattice theory in our case is the Nakanishi-Lautrup field b_i that was introduced by the equivariant BRST-construction. It is a Hermitian scalar that is *charged* under the U(1). At every site it is of the form

$$b_i = B_i \tau_+ + B_i^* \tau_- , \tag{76}$$

where B_i is a complex number and B_i^* its complex conjugate. Parametrizing $h_i = e^{2i\varphi_i \tau_0} \in \mathrm{U}(1)$, with $\varphi \in [0,2\pi)$, one observes that B_i transforms as

$$B_i^h = e^{2i\varphi_i} B_i , \tag{77}$$

under the residual U(1). We thus have the option to fix the phase of the Nakanishi-Lautrup field and thereby reduce the gauge invariance of the model to the discrete gauge group $Z_2 \subset \mathrm{U}(1)$ (we cannot do better, since the Nakanishi-Lautrup field is oblivious to Z_2 gauge transformations of the lattice configuration). We can for instance require that B is a real and *positive* field. The corresponding measure in the gauge fixed functional integral becomes

$$\int d^2 b_j \xrightarrow{U(1)\to Z_2} \int_0^\infty B_j dB_j . \tag{78}$$

Note that the integration is over *positive* real variables B_j only. (An unconstrained integration over all real values of B_j would lead to a vanishing partition function of the corresponding TLT which can then be shown to be proportional to the Euler character of S_1.)

We can perform the integration of the Nakanishi-Lautrup field b_i also in the Z_2 LGT. Due to Eq. (18) the anti-Hermitian field $F_j[U]$ at each site is given by a complex number $f_j[U]$ and may be written

$$F_j[U] = \tau_+ f_j[U] - \tau_- f_j^*[U]. \tag{79}$$

Using Eq. (78) and Eq. (24) the B-dependent bosonic part of the partition function for the Z_2 LGT is local and proportional to

$$\prod_{sites} \int_0^\infty B_i dB_i e^{-2iB_i \,\mathrm{Im}\, f_i[U] - \alpha B_i^2} \propto \prod_{sites} \mathcal{P}\left(\frac{\mathrm{Im}\, f_i[U]}{\sqrt{\alpha}} \right), \tag{80}$$

with the complex weight function $\mathcal{P}$,

$$\mathcal{P}(x) = 1 - x e^{-x^2} \left(i\sqrt{\pi} + \int_{-x}^{x} e^{t^2} dt \right), \tag{81}$$

related to the error function on the imaginary axis. Note that $\mathcal{P}(-x) = \mathcal{P}^*(x)$ and that $f_i[U] = -f_i[U^\dagger]$, i.e. f_i changes sign when the direction of all the links is reversed. The expectation value of Hermitian observables of the link variables is thus real. From Eq. (81) we observe that the local measure $\mathcal{P}(x\sim\infty) \sim 1/(2x^2)$ does not vanish exponentially for large values of $x = \mathrm{Im}\, f_i[U]/\sqrt{\alpha}$. $|\mathcal{P}(x)|$ however decreases monotonically and peaks at $x=0$. For sufficiently large lattices $\mathcal{P}(x)$ can be approximated by $\mathcal{P}(x) \sim \exp[-i\sqrt{\pi}x - (2-\pi/2)x^2]$ up to terms that are irrelevant in the critical limit of the model. Note that the total phase of the measure in this limit is proportional to $\Sigma_i\, \mathrm{Im}\, f_i[U]$ and vanishes on a periodic lattice. Expanding near the trivial configuration $U=1$, it is readily seen that $\mathrm{Im}\, f_i[U\sim 1]$ is *not* linear in all longitudinal fluctuations. In agreement with the previous discussion, the modification of the lattice action by a term proportional to $(\mathrm{Im}\, f_i[U])^2$ does not lead to well defined (naive) continuum propagators.

To exhibit the BRST-structure of this U(1) gauge fixing and relate it to a TLT, we note that it can be obtained by inserting

$$\mathcal{Z}_{U(1)}[B] = \prod_{sites} Z[B_i], \tag{82}$$

in the functional integral. Here the local TLT ‘‘partition function’’ at each site is simply the integral

$$Z[x] = \oint_{|z|=1} z^* dz |xz + x^* z^*| \times \delta(i(xz - x^* z^*)). \tag{83}$$

The integration of the U(1)-group element $h_i \in \mathrm{U}(1)$ in Eq. (83) is here written as a contour integral of z along the unit circle in the complex plane. Equation (77) shows that one in principle has to integrate twice over the unit circle. This however just introduces an irrelevant factor of 2 in Eq. (83).

If $Z[x]$ is a non-vanishing constant one can insert Eq. (82) in the functional integral and change variables to factorize the group volume $V_{\mathcal{H}}$ and arrive at the effective measure (78) of the gauge-fixed model. Performing the integrations in Eq. (83) one explicitly finds that $Z[x]$ does not vanish and furthermore does not depend on x. The integral $Z[x]$ is thus a topological invariant of the circle S_1. The corresponding topological model with local ‘‘action’’ can be explicitly constructed by exponentiating all the factors in Eq. (83). Introducing real bosonic variables u,v and Grassmann variables $\eta, \bar{\eta}, \nu, \bar{\nu}$ one can rewrite $Z[x]$ as the integral

$$Z[x] = \oint_{|z|=1} z^* dz \int \int_{-\infty}^{\infty} du\, dv\, d\eta\, d\bar{\eta}\, d\nu\, d\bar{\nu} \times e^{-S[xz,u,v,\eta,\bar{\eta},\nu,\bar{\nu}]}, \tag{84}$$

with the local ‘‘action,’’

$$S[a,u,v,\eta,\overline{\eta},\nu,\overline{\nu}] = u(a-a^*) + (a+a^*)(v^2(a+a^*) + \eta\overline{\eta} + \nu\overline{\nu}). \tag{85}$$

Performing the Grassmann integrals over $\overline{\eta},\eta,\overline{\nu},\nu$ and the ordinary Gaussian integral over v in Eq. (84) gives $|a+a^*|$. The integration over u leads to the constraint $a=a^*$. To show that Eq. (84) with action (85) is a topological integral of Witten type, we verify that S is exact with respect to a nilpotent symmetry δ defined on the (local) variables as

$$\delta x = 0$$

$$\delta z = z\eta, \quad \delta z^* = -z^*\eta, \quad \delta\eta = 0$$

$$\delta a = \delta xz = a\eta, \quad \delta a^* = -a^*\eta$$

$$\delta\overline{\eta} = u + \nu\overline{\nu} + v^2(a-a^*),$$

$$\delta u = -\nu\overline{\nu}\eta - v^2(a+a^*)(\nu+\eta)$$

$$\delta\overline{\nu} = \overline{\nu}(\nu-\eta) - 2v^2 a, \quad \delta v = -v\nu/2, \quad \delta\nu = 0. \tag{86}$$

Using the algebra (86) it is straightforward to show that δ is nilpotent, $\delta^2=0$. Note that $zz^*=1$ and (consequently) aa^* are invariants. I obtained Eq. (86) by demanding $\delta x=0$ and $\delta z = z\eta$, i.e. by *ghostifying* the U(1) transformation. Together with the nil-potency of δ these assumptions imply the first three lines of relations in Eq. (86). The remainder of the algebra was found by demanding that the action (85) is δ-closed. I unfortunately can offer no further insight for the construction of the algebra (86) which appears to be far too involved for the problem it solves. At the end of the day one obtains that the action (85) is the δ-exact expression

$$S[a,u,v,\eta,\overline{\eta},\nu,\overline{\nu}] = \delta(\overline{\eta}(a-a^*) - 2\overline{\nu}a^*). \tag{87}$$

This shows that Eq. (84) can be interpreted as a topological invariant of Witten type. This invariant does not vanish and is *not* proportional to the Euler character of U(1). The only independent topological invariant of a circle is the number of its connected components [the lowest Betty number $b_0(S_1) = 1 = b_1(S_1)$]. This topological characteristic, like the Euler character of a manifold, is multiplicative, i.e. $b_0(M_1 \times M_2) = b_0(M_1) b_0(M_2)$ for any two compact manifolds M_1 and M_2. We can evidently choose to normalize Eq. (83) so that $Z[x] = b_0(S_1) = 1$ and then interpret the partition function $\mathcal{Z}_{U(1)}[B]$ of Eq. (82) as the number of connected components of the U(1) gauge group of the lattice. The TLT construction supports the claim [8] that gauge fixing is equivalent to the construction of a TQFT on the gauge group whose partition function does *not* vanish.

VI. THE AUXILIARY FIELD ρ

In the Appendix the Grassmannian fields are integrated in favor of a non-local measure (A19) for the gauge fixed LGT. It is thus in principle possible to numerically simulate the Z_2 LGT. We have shown that such a simulation would reveal nothing new for *gauge invariant* physical observables. The reason for constructing a physically equivalent covariant LGT with a much smaller invariance group was to gain a better *analytical* understanding of the model in the critical limit. The best we could do was a reduction of the continuous SU(2) gauge symmetry of the original LGT to a discrete Z_2-structure group. The natural question to ask is whether this discrete gauge group is spontaneously broken. Although there is no gauge invariant physical order parameter, whether or not this symmetry is broken could shed some light on the dynamics of the model.

Here I will however only discuss the role of the auxiliary scalar field ρ_i introduced to linearize the quartic ghost interaction. I will show that the bosonic measure is maximal at a non-trivial (constant) ρ_i. This is crucial for a loop expansion of physical observables of the lattice model. I first show that the effective measure for the field ρ_i is non-trivial and gauge invariant. Consider the weight of Eq. (A17) as a function of the link configuration U and the auxiliary variable ρ integrated over the gauge group $\mathcal{G}$:

$$\mathcal{Q}[U,\rho;\alpha] = \int \prod_{sites} dg_i \mathcal{W}[U^g,\rho;\alpha]. \tag{88}$$

By construction the observable $\mathcal{Q}$ is gauge invariant

$$\mathcal{Q}[U^g,\rho;\alpha] = \mathcal{Q}[U,\rho;\alpha], \tag{89}$$

for any configuration ρ. We furthermore know from the previous sections that on any finite lattice,

$$\mathcal{N}(\alpha) = \int \prod_{sites} d\rho_i \mathcal{Q}[U,\rho;\alpha], \tag{90}$$

is a non-vanishing (finite) normalization constant that does not depend on the link configuration U. The two results [Eq. (89) and Eq. (90)] imply that one can define a normalizable measure $W[\rho]$ for the scalar field ρ

$$W[\rho;\alpha,g^2] = \langle \mathcal{Q}[U,\rho;\alpha] \rangle_{inv.}. \tag{91}$$

Here the expectation value on the RHS is with the original SU(2)-invariant measure. Equation (90) implies that $W[\rho]$ is normalizable and does not vanish identically. The measure (91) for the configurations ρ is therefore non-trivial. Upon changing variables $U \to U^g$ in the gauge-invariant RHS of Eq. (91) we can decouple the integration over the gauge group $\mathcal{G}$ and equivalently write

$$W[\rho;\alpha,g^2] = \langle 1 \rangle_\rho, \tag{92}$$

where $\langle \mathcal{O} \rangle_\rho$, given by

$$\langle \mathcal{O} \rangle_\rho = \int \prod_{links} dU_{ij} \mathcal{O}[U,\rho] \mathcal{W}[U,\rho;\alpha] e^{-S_{\rm inv.}[U]}, \tag{93}$$

is the expectation value of a *gauge invariant* function $\mathcal{O}[U,\rho] = \mathcal{O}[U^g,\rho]$ for a given configuration ρ of the *gauge-*

fixed model. Due to Eq. (88) computing Eq. (93) for gauge invariant observables $\mathcal{O}$ is entirely equivalent to evaluating the gauge-invariant correlator

$$\langle \mathcal{O} \rangle_\rho = \langle \mathcal{Q}[U,\rho;\alpha]\mathcal{O} \rangle_{inv.}, \tag{94}$$

with the SU(2)-invariant measure of the SU(2) LGT.

A perturbative evaluation of the Z_2 LGT should at least retain those configurations $\overline{\rho}$ for which $\langle \mathcal{O} \rangle_{\overline{\rho}}$ (and in particular $\langle 1 \rangle_{\overline{\rho}}$) *do not* vanish in the limit $g^2 \to 0$. I argue that these are configurations in the vicinity of non-trivial constant configurations $\overline{\rho_i} = \text{const}$ only.

In the limit $g^2 \to 0$, the gauge invariant action $S_{\text{inv.}}[U]$ on a *finite* lattice constrains the configuration space U to the subset of pure gauge configurations $U_{ij} = g_i g_j^\dagger$. As we have seen in Sec. IV there are at least two zero modes of the quadratic form in Eq. (25) in this case: they correspond to *global* rotations of the gauge spins $\hat{s}_i$. These zero-modes were shown to be absorbed by the quartic ghost interaction of Eq. (25). In the linearized version (26) of the model these zero-modes couple to the auxiliary field ρ only. The determinant in Eq. (A17) and consequently Eq. (93) for a finite lattice thus vanish in the limit $g^2 \to 0$ for configurations ρ that satisfy

$$\sum_i \rho_i \ \text{Tr}[\tau_0, \phi_i^{(1)\dagger}]\phi_i^{(2)} = 0, \tag{95}$$

where $\phi_i^{(1)}, \phi_i^{(2)}$ are the two global zero modes of a pure gauge configuration. Since the *global* zero-modes rotate all the spins equally, the vector v_i

$$\text{Tr}[\tau_0, \phi_i^{(1)\dagger}]\phi_i^{(2)} = v_i = \text{const}, \tag{96}$$

has constant entries irrespective of the pure gauge configuration being considered. Equation (95) implies that $\langle 1 \rangle_\rho \to 0$ in the limit $g^2 \to 0$ on any *finite* lattice if the configuration ρ is orthogonal to v_i, i.e. has no constant component. Note that this is true for any value of $\alpha \neq 0$.

The argument above is however true only in the *wrong* limit where one takes $g^2 \to 0$ *before* considering the thermodynamic limit $N \to \infty$. In the thermodynamic limit one can only say that the relevant configurations in the limit $g^2 \sim 0$ are (in a statistical sense) in the vicinity of pure gauge configurations. One nevertheless would expect that configurations ρ with large contributions to $\langle 1 \rangle_\rho$ are also in some sense close to nontrivial constant ones. More precisely, the argument above and the considerations of Sec. III indicate that a loop expansion of the gauge fixed Z_2-model on a *finite* lattice in the vicinity of a (particular) pure gauge configuration is sensible only for $\rho_i = \overline{\rho_i} = \text{const} \neq 0$. We otherwise would expand about a configuration that has vanishing weight.

The covariant loop expansion of the Z_2-model can in fact be examined in more detail and also gives some insight into the critical limit of the model on an infinite lattice. Consistency requires that the value of $\overline{\rho}$ be determined order by order of the loop expansion by

$$\langle \rho_i \rangle = \overline{\rho}, \tag{97}$$

with $\langle \rho_i \rangle$ given by Eq. (A19). At the ''tree''-level of the loop expansion for the *bosonic* measure (97) implies that the unique maximum of the measure is at $\overline{\rho}^{tree}$ which is the solution of

$$\left. \frac{\partial}{\partial \overline{\rho}} \mathcal{W}[U^{tree}, \overline{\rho}; \alpha] \right|_{\overline{\rho}^{tree}} = 0, \tag{98}$$

for a pure gauge configuration U^{tree} that satisfies the gauge condition,

$$\text{Im } f_i[U^{tree}] = 0. \tag{99}$$

As far as the perturbative expansion of gauge invariant observables is concerned, we may choose any *one* of the 2^N (gauge equivalent) configurations U^{tree} that contribute to the partition function. These were obtained in Sec. IV and correspond to collinear gauge spins. We may then solve Eq. (98) to obtain the appropriate value of $\overline{\rho}^{tree}$. In fact, $\overline{\rho}^{tree}$ is the same for any one of the 2^N discrete ''vacua'' due to the gauge invariance (91) of $W[\rho]$.

The simplest (perturbative) vacuum configuration for the links, and the only one that leads to a covariant perturbative expansion, is $U_{ij}^{tree} = 1$. With periodic boundary conditions for the lattice, det $\mathcal{M}[u=1, v=0, \rho]$ is readily calculated for constant $\rho_i = \overline{\rho}$. One can diagonalize $\mathcal{M}[1,0,\overline{\rho}]$ in the basis of eigenvectors $X_i(\vec{n})$ of the Hermitian matrix $A[1]$ given by Eq. (A14),

$$A[1] \cdot X(\vec{n}) = \Delta_{\vec{n}} X(\vec{n}), \tag{100}$$

where

$$\Delta_{\vec{n}} = -4 \sum_{\mu=1}^{D} \sin^2(\pi n_\mu / L), \quad n_\mu = 1, \ldots, L \tag{101}$$

are the eigenvalues of the Laplace-operator of a D-dimensional hyper-cubic lattice with $L^D = N$ sites. The eigenvalues of $\mathcal{M}[1,0,\overline{\rho}]$ are

$$\lambda_\pm^{(\vec{n})} = \Delta_{\vec{n}} \pm i\overline{\rho}, \tag{102}$$

and det $\mathcal{M}$ for the vacuum configuration therefore is

$$\det \mathcal{M}[1,0,\overline{\rho}] = \prod_{\vec{n}} (\Delta_{\vec{n}}^2 + \overline{\rho}^2). \tag{103}$$

As expected, the determinant (103) vanishes for $\overline{\rho} \to 0$ like $\overline{\rho}^2$ on any finite lattice. The determinant (103) is furthermore a monotonically increasing positive function of $\overline{\rho}^2$. For $\overline{\rho}^2 \gg D$, the determinant behaves as $\mathcal{M}[1,0,\overline{\rho} \gg D] \sim \overline{\rho}^{2N} = e^{N \ln \overline{\rho}^2}$. Comparing with the monotonically decreasing exponent $e^{-N\overline{\rho}^2/(4\alpha)}$ in $\mathcal{W}$, we see that the maximum is *unique* and of order $\overline{\rho}^2 \sim 4\alpha$ for large α. For a perturbative expansion we are however interested in the value for the maximum at (arbitrary) small $\alpha \sim 0$ for which gauge fluctuations are suppressed. In this limit we expect $\overline{\rho}^2 \sim 0$ also. Using Eq. (103), Eq. (A17) and the definition (91), the unique value $\overline{\rho}(\alpha)$ where the weight $W[\overline{\rho}(\alpha); \alpha, g^2 \sim 0]$ is maximal to lowest order in the loop expansion is the solution of

$$\sum_{\vec{n}} \frac{1}{\Delta_{\vec{n}}^2+\overline{\rho}^2}=\frac{N}{4\alpha}. \tag{104}$$

For a *finite* lattice this gap equation would have to be solved numerically. In the thermodynamic limit, the summations in Eq. (104) can be performed. In this limit (104) on a periodic D-dimensional lattice becomes

$$\int_0^\infty dx \frac{\sin(x\overline{\rho})}{\overline{\rho}}(e^{-2x}I_0(2x))^D=\frac{1}{4\alpha}, \tag{105}$$

where $I_0(x)$ is the Bessel function of zeroth order at imaginary arguments. Equation (105) is obtained by exponentiating the summand in Eq. (104) and using the identity

$$\lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{L} e^{-2x\,\sin^2(\pi n/L)}=\frac{e^{-x}}{2\pi}\int_0^{2\pi} e^{x\cos\phi}d\phi=e^{-x}I_0(x). \tag{106}$$

The asymptotic behavior of $I_0(x\sim\infty)\sim e^x/\sqrt{2\pi x}$ shows that the integral in Eq. (105) converges for $\overline{\rho}\neq 0$ in any dimension. In $D<4$ dimensions the integral behaves like $\overline{\rho}^{(D-4)/2}$ for $\overline{\rho}\sim 0$. At the upper critical dimension $D=4$ its behavior is logarithmic. For sufficiently small α, Eq. (105) in $D=4$ is well approximated by

$$\ln\frac{\overline{\rho}^2}{\kappa^4}=-\frac{2\pi^2}{\alpha}+O(\overline{\rho}), \tag{107}$$

where the constant κ is given by

$$\begin{aligned}\ln\frac{\kappa^2}{4\pi}&=1-\gamma_E+4\pi^2\int_0^\infty dx x\left[(I_0(x)e^{-x})^4-\frac{1}{1+4\pi^2x^2}\right]\\&=2.26098\ldots.\end{aligned} \tag{108}$$

For $\alpha\sim 0$, Eq. (107) determines the optimal $\overline{\rho}$ of the corresponding vacuum configuration $(U=1,\overline{\rho})$ for the perturbative expansion in $D=4$ dimensions. Note that the relation (107) is not affected by physical fermions (i.e. quarks). The corresponding fermionic determinant is gauge invariant, does not depend on the auxiliary field ρ and does not vanish at the pure gauge configuration $U=1$. This contribution to lowest order in the loop expansion can thus be absorbed in the normalization of W. The critical limit $g^2,\alpha\to 0$, of the LGT can be interpreted as a quantum field theory by assigning a spacing a to the lattice and defining continuum fields. The continuum field $\tilde{\rho}(x)$ has canonical dimension 2 and is related to ρ by $\rho_i=a^2\tilde{\rho}(x_i)$. In the critical limit (107) is the tree-level statement

$$\langle\tilde{\rho}(x)\rangle=(\kappa^2/a^2)e^{-\pi^2/\alpha}, \tag{109}$$

for the continuum field $\tilde{\rho}(x)$. Because of Eq. (91), $\langle\tilde{\rho}\rangle$ is a physical gauge invariant quantity and we could regard the LHS of Eq. (109) as a constant physical scale of the model. In this case Eq. (109) is the expression for dimensional transmutation of the gauge parameter α to lowest order. On the other hand the physical asymptotic scale parameter Λ_L of the lattice to this order in the loop expansion is related to the coupling g^2 and the lattice spacing a by

$$a^2\Lambda_L^2=e^{-24\pi^2/(11-n_f)g^2+O(\ln g^2)}. \tag{110}$$

Λ_L is a finite physical scale in the critical limit of the SU(2) LGT with n_f fermionic flavors. Equation (109) singles out a *particular* gauge $\overline{\alpha}(g)=g^2(11-n_f)/24+O(g^4)$ in which $\langle\tilde{\rho}(x)\rangle$ scales like a physical quantity in the critical limit, that is

$$\langle\tilde{\rho}\rangle=\kappa^2\Lambda_L^2\quad\text{for}\quad\alpha=\overline{\alpha}=g^2(11-n_f)/24+O(g^4). \tag{111}$$

In this *critical* gauge the loop expansion of gauge invariant correlators automatically produces power corrections (in $\overline{\rho}\neq 0$) that scale correctly in the critical limit. Note that the power corrections of the loop expansion vanish exponentially compared to Λ_L in the critical limit for gauges $\lim_{g\to 0}\alpha/g^2<(11-n_f)/24$ and dominate the correlations in gauges $\lim_{g\to 0}\alpha/g^2>(11-n_f)/24$—a sign that the asymptotic expansion does not make much sense in such gauges. We know on the other hand that *physical* power corrections do arise in the full theory. In critical gauges with $\lim_{g\to 0}\alpha/g^2=(11-n_f)/24$ they also arise in the loop expansion of the gauge-fixed model. It is justified to call these gauges *critical* because they delineate the domain of validity of the loop expansion.

To check the assertion that $\overline{\rho}$ scales like a physical quantity in critical covariant gauges, one should evaluate the anomalous dimension of $\tilde{\rho}$. From the foregoing one expects this anomalous dimension to vanish to leading order in the gauge $\overline{\alpha}(g)=g^2(11-n_f)/24+O(g^4)$. It is then possible to adjust the critical value $\overline{\alpha}(g)$ order by order in the loop expansion so that the anomalous dimension of $\tilde{\rho}$ vanishes to all orders. I only wish to stress here that the existence of such a critical gauge is a direct consequence of the fact that the ghost determinant $M[U=1,\rho]$ *vanishes* for $\rho=0$. If it were *finite* at $\rho=0$, we would have been justified to expand perturbatively around $\overline{\rho}=0$ for sufficiently small values of α. In actual fact Eq. (105) has *no solution* for $\alpha\to 0$ in $D>4$ dimensions: in the thermodynamic limit the weight $W[\rho,\alpha]$ peaks at $\overline{\rho}=0$ for α less than some critical value in $D>4$ dimensions. In $D\leqslant 4$, $\overline{\rho}=0$ is however only approached as $\alpha\to 0$ and the weight W is maximal at a nontrivial value of $\overline{\rho}$ for *any* non-vanishing α.

VII. SUMMARY AND COMMENTS

In the foregoing we constructed a LGT with a discrete structure group Z_2 that is physically equivalent to the standard SU(2) LGT. The Z_2-model possesses all of the space-time symmetries of the original LGT. The reduction of the gauge group was shown to be equivalent to the formulation of a TLT on a coset, respectively group, manifold. Care was taken to ensure that the partition function of the TLT's (and consequently the partially gauge-fixed LGT's) are normalizable. On a lattice, using Morse theory to construct a TLT whose partition function is proportional to the Euler characteristic of a compact manifold is a mathematically rigorous procedure. We saw that this method by itself does not suffice to fix the gauge completely because the Euler character of the lattice gauge group $\mathcal{G}$ vanishes. To partially fix the original SU(2)-gauge symmetry to a discrete Z_2 gauge symmetry we proceeded in two steps.

The gauge invariance was first reduced to the Abelian U(1) gauge group using an equivariant BRST-construction. We showed that this procedure is equivalent to the formulation of a TLT on the coset space $\mathcal{G}/\mathcal{H}=[\mathrm{SU}(2)/\mathrm{U}(1)]^N=S_2^N$ and explicitly proved that the partition function of this TLT is proportional to the Euler characteristic of the coset manifold. Since this Euler number does not vanish, the TLT is normalizable and the partially gauge fixed U(1) LGT is physically equivalent to one with non-Abelian structure group SU(2). Although we here only considered an SU(2) LGT, the procedure can be generalized to fix the gauge of an SU(n) LGT to the maximal Abelian subgroup $\mathcal{H}=[\mathrm{U}(1)^{n-1}]^N$. This follows by induction from $\mathrm{SU}(n+1)/[\mathrm{SU}(n)\times\mathrm{U}(1)]\simeq S_{2n+1}/S_1\simeq CP_n$ and the fact that the Euler character $\chi(CP_n)=n+1$ does not vanish.[1] In the case I considered, the action Eq. (36) of the U(1) LGT is local but depends on Grassmannian ghosts and includes a 4-ghost interaction. To my knowledge it is the first example of a (partially) gauge-fixed lattice model with an (equivariant) BRST-symmetry that is proven to be physically equivalent to the original SU(2) LGT also non-perturbatively. It could be considered the first concrete realization of non-Abelian BRST-symmetry in a non-perturbative setting. The construction of the corresponding TLT and the proof in Sec. IV show how the Gribov-ambiguity associated with the covariant gauge fixing is circumvented: although *there are* many Gribov copies (and even whole manifolds of them) associated with any orbit, they conspire to give a topological invariant [in our example the Euler character $\chi(S_2^N)=2^N$] that does not depend on the orbit within a connected sector. Since the orbit space of a LGT is connected, the existence of Gribov copies in covariant gauges does *not* invalidate the gauge-fixing procedure if the topological invariant the TLT computes does not vanish. This is in contradistinction to conventional Dirac-quantization of first class constraints [20], which in principle is valid only if the solution to the gauge condition is *unique*. The formulation of gauge-fixing as a topological model on the moduli-space of the gauge theory perhaps also clarifies the dispute [2,16,17] concerning the non-perturbative validity of covariantly gauge fixed models with BRST-symmetry. I believe this procedure in general permits one to handle Gribov ambiguities.

The U(1)-invariant lattice model was subsequently reduced to one with a Z_2-structure group by using the Nakanishi-Lautrup field of the previous partial gauge fixing. This U(1)-gauge fixing is entirely local and the constraints can be *solved* explicitly. The gauge fixing can again be related to a corresponding local TLT. The partition function of this TLT is however proportional to the number of connected components of the U(1)-gauge group rather than its Euler character (which vanishes). The gauge fixing is *not linear* and naive continuum propagators do not exist. I argued that this should not prevent one from considering the *loop* expansion of the Z_2 LGT since the lattice propagators *do exist* for any finite value of the coupling.

The maxima of the measure of the Z_2 LGT are isolated and a loop expansion of gauge invariant observables in the vicinity of *any* gauge-equivalent vacuum configuration of this model is feasible. The price one pays is the considerably more complicated non-local and generally complex measure of the resulting bosonic Z_2 LGT after integration of the Grassmannian variables. This bosonic partition function depends on the link variables U_{ij} *as well as* a local *gauge invariant* scalar field ρ_i and on *two* coupling constants g^2 and α. The former is inherited from the original SU(2) LGT whereas the latter was introduced by the gauge fixing.

The expectation values of physical gauge invariant observables of the original LGT (gauge invariant functions of the link variables only) do not depend on α by construction. The expectation value of gauge invariant functions of the auxiliary field ρ_i and especially $\langle\rho_i\rangle$ however generally *do* depend on α as well as Λ_L. We found that the maximum of the bosonic measure for $\alpha\neq 0$ occurs at $\rho_i=\overline{\rho(\alpha)}\neq 0$ and derived the *gap equation* (104) relating $\overline{\rho}$ to α in lowest order. In the thermodynamic limit of a 4-dimensional lattice the relation is given by Eq. (107) in the limit of very small $\alpha\sim 0$. The most interesting result of this analysis is that the expectation value $\langle\tilde{\rho}(x)\rangle$ of the corresponding continuum field $\rho_i=a^2\tilde{\rho}(x_i)$ is proportional to the asymptotic scale parameter Λ_L^2 in a particular *critical* gauge $\overline{\alpha}(g)=g^2(11-n_f)/24+O(g^4)$. *Non-perturbative* power corrections to physical observables proportional to Λ_L^{2k}/p^{2k} (with $k\geqslant 2$) appear *computable* in this critical gauge. In effect this would imply that the non-perturbative expectation values of Wilson's operator-product-expansion for the asymptotic behavior of physical correlators are *part of* the asymptotic loop expansion in the *critical* gauge. Although a direct (numerical) evaluation of physical correlations was shown to be independent of the gauge parameter, I argued that the *accuracy* of the asymptotic perturbative expansion may, and generally *does*, depend on the gauge. The analysis of the SU(2) LGT tends to support the conjecture that power corrections are accessible by the loop-expansion in certain covariant gauges. A similar mechanism was previously observed in the continuum theory [21]. In this case the expectation value of a scalar moduli-parameter also was related [22] to the scale anomaly of the model.

If power corrections to physical correlators are indeed computable in critical covariant gauges, the loop expansion in conjunction with dispersion relations could be a powerful tool to obtain information on the spectrum of the model. The phenomenological success of QCD sum-rules [23] suggests that it might be worth pursuing this possibility.

Apart from these speculations, the topological approach to gauge fixing of a LGT has shown that

(i) gauge fixing of a LGT is equivalent to the construction of a certain TLT of Witten type;

[1] I would like to thank A. Rozenberg for this remark.

(ii) the gauge-fixed lattice model is normalizable only if the topological invariant computed by the partition function of the associated TLT does not vanish;
(iii) the BRST-symmetry of the gauge fixed LGT is inherited from the associated TLT and is also realized non-perturbatively;
(iv) covariant and BRST-invariant gauge fixing of a LGT is possible and the Gribov ambiguity of these gauges can be controlled;
(v) quartic ghost interactions arise naturally in the non-Abelian case due to residual global invariances and are perhaps unavoidable in covariant gauges.

At present this approach appears to be the only systematic method that guarantees that the gauge-fixed model is *covariant, local and physically equivalent* to the original non-Abelian gauge invariant theory also non-perturbatively.

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APPENDIX: SOME CALCULATIONS SPECIFIC TO AN SU(2) LGT

The link matrices $U_{ij}\in \mathrm{SU}(2)$ of an SU(2) LGT,

$$U_{ij}=u_{ij}(1/2+\tau_0)+u_{ij}^*(1/2-\tau_0)+v_{ij}\tau_+-v_{ij}^*\tau_-, \tag{A1}$$

can be parametrized by two complex numbers u_{ij} and v_{ij} that satisfy the constraint

$$|u_{ij}|^2+|v_{ij}|^2=1. \tag{A2}$$

This parametrization facilitates some calculations in the SU(2) LGT [24]. Below I give expressions for some of the quantities of the main text in terms of the u_{ij}'s and v_{ij}'s.

The Morse-potential (14) can be written

$$V[u,v]=\sum_{links}|v_{ij}|^2=\sum_{links}(1-|u_{ij}|^2). \tag{A3}$$

U(1) gauge transformations change the phases of u_{ij} and v_{ij} but not their lengths. An infinitesimal transformation $g_i\sim 1 \in \mathrm{SU}(2)/\mathrm{U}(1)$ is of the form

$$g_i=1+\theta_i\tau_+-\theta_i^*\tau_-, \tag{A4}$$

where the θ_i are infinitesimal complex numbers. To first order, the parameters u_{ij} and v_{ij} of a link change by

$$\Delta u_{ij}=v_{ij}\theta_j^*-\theta_i v_{ij}^*$$
$$\Delta v_{ij}=\theta_i u_{ij}^*-u_{ij}\theta_j. \tag{A5}$$

The constraint (A2) to this order is invariant under the transformation (A5). From Eq. (A5) we obtain that the Morse-function (A3) changes by

$$\Delta V[u,v]=\sum_{links}\theta_i v_{ij}^* u_{ij}^*-v_{ij}^* u_{ij}\theta_j+\text{c.c.}$$
$$=\sum_i\sum_{j\sim i}\theta_i v_{ij}^* u_{ij}^*+\text{c.c.}, \tag{A6}$$

where $u_{ji}=u_{ij}^*$ and $v_{ji}=-v_{ij}$ was used to rewrite the second term. On the other hand Eq. (12) together with Eq. (79) imply

$$\Delta V=-\sum_i (f_i^*[U]\theta_i+\text{c.c.}). \tag{A7}$$

Comparing Eq. (A7) with Eq. (A6) one obtains

$$f_i[u,v]=-\sum_{j\sim i}v_{ij}u_{ij}. \tag{A8}$$

We can compute the linear operator $M_i[U,c]$ in analogous fashion by considering the variation of $f_i[u,v]$ under infinitesimal transformations of the form (A5). One gets

$$sf_i[u,v]|_{\omega=0}=\sum_{j\sim i}C_i(|v_{ij}|^2-|u_{ij}|^2)+u_{ij}^2C_j-v_{ij}^2C_j^*, \tag{A9}$$

where the Grassmann variables C_i, C_i^* are defined by the decomposition

$$c_i=C_i\tau_+-C_i^*\tau_-. \tag{A10}$$

Similarly decomposing $\bar{c}$ as

$$\bar{c}_i=\bar{C}_i\tau_++\bar{C}_i^*\tau_-, \tag{A11}$$

one obtains for the quadratic form

$$\sum_i \mathrm{Tr}\ \bar{c}_i M_i[U,c]$$
$$=\sum_i \mathrm{Tr}\ \bar{c}_i sF_i[U]|_{\omega=0}$$
$$=-\sum_i(\bar{C}_i sf_i[u,v]|_{\omega=0}-\text{c.c.})$$
$$=\sum_i\sum_{j\sim i}\bar{C}_i(|u_{ij}|^2-|v_{ij}|^2)C_i$$
$$+\bar{C}_i v_{ij}^2 C_j^*-\bar{C}_i u_{ij}^2 C_j-\text{c.c.} \tag{A12}$$

Here ''complex conjugation'' for Grassmann variables is the substitution $C,\bar{C}\leftrightarrow C^*,\bar{C}^*$ at each site.

Using Eq. (A10) and Eq. (A11) the interaction with the real scalar field ρ_i in Eq. (26) is written

$$\sum_i \rho_i\ \mathrm{Tr}\ \tau_0[\bar{c}_i,c_i]=-\sum_i\rho_i(\bar{C}_iC_i^*+\bar{C}_i^*C_i). \tag{A13}$$

Defining the two complex $N\times N$ matrices with entries,

$$A_{ij}[u] = u_{ij}^2 + \delta_{ij} \sum_{k \sim i} (1 - 2|u_{ik}|^2)$$

$$B_{ij}[v,\rho] = \delta_{ij}\rho_i - v_{ij}^2, \tag{A14}$$

the integration of the Grassmannian variables in the gauge fixing part of the action (26) results in a weight proportional to

$$\det \mathcal{M}[u,v,\rho], \tag{A15}$$

for the remaining bosonic functional integral. The $2N\times 2N$ complex matrix $\mathcal{M}$ is

$$\mathcal{M}[u,v,\rho] = \begin{pmatrix} A[u] & B[v,\rho] \\ B^\dagger[v,-\rho] & A^\dagger[u] \end{pmatrix}. \tag{A16}$$

Note the dependence on the auxiliary field ρ in Eq. (A16). For purely imaginary ρ_i the matrix $\mathcal{M}$ is Hermitian and its eigenvalues (and thus its determinant) are real. The weight (A15) of a given bosonic configuration with real ρ_i is however generally complex. One can also corroborate that $A[u_{ij}=1]$ is the lattice Laplacian with exactly *one* vanishing eigenvalue on a periodic lattice. $\mathcal{M}[1,0,0]$ thus has exactly two vanishing eigenvalues corresponding to the two zero-modes of this vacuum configuration that were found in Sec. IV. The same reasoning shows that, det $\mathcal{M}[u,v,0]$ in fact vanishes for any pure gauge configuration.

Collecting these results, the gauge-fixing weight $\mathcal{W}[u,v,\rho;\alpha]$ of a given link configuration can be written

$$\mathcal{W}[u,v,\rho;\alpha] = \det \mathcal{M}[u,v,\rho] \prod_{sites} e^{-\rho_i^2/(4\alpha)} \times P\left(\frac{f_i[u,v]}{\sqrt{\alpha}}\right), \tag{A17}$$

where the local weight $P(x)$ depends on whether the SU(2) LGT is partially gauge fixed to the Abelian U(1) or the discrete Z_2 structure group

$$P(x) = \begin{cases} e^{-|x|^2} & \text{for } U(1) \\ \mathcal{P}(\text{Im } x) \text{ of Eq. (81)} & \text{for } Z_2. \end{cases} \tag{A18}$$

The expectation value of operators $\mathcal{O}[u,v,\rho]$ that only depend on the link variables and the auxiliary field ρ can now be found by (numerically) evaluating the remaining bosonic integrals in

$$\langle \mathcal{O}[u,v,\rho] \rangle = \int \prod_{links} d^2u_{ij} d^2v_{ij} \delta(1 - |u_{ij}|^2 - |v_{ij}|^2) \times \prod_{sites} d\rho_i \mathcal{O}[u,v,\rho] \mathcal{W}[u,v,\rho;\alpha] \times \exp\{-S_{\text{inv.}}[u,v]\}. \tag{A19}$$

[1] G. Dell' Antonio and D. Zwanziger, Commun. Math. Phys. **138**, 291 (1991).
[2] B. Sharpe, J. Math. Phys. **25**, 3324 (1984); for a concise review of the problem see H. Neuberger, Phys. Rev. D **58**, 057502 (1998).
[3] H. Neuberger, Phys. Lett. B **183**, 337 (1987).
[4] I. M. Singer, Commun. Math. Phys. **60**, 7 (1978).
[5] V. N. Gribov, Nucl. Phys. **B139**, 1 (1978).
[6] D. H. Adams, Phys. Lett. B **417**, 53 (1998).
[7] P. Hirschfeld, Nucl. Phys. **B157**, 37 (1979); see also K. Fujikawa, *ibid.* **B468**, 355 (1996); See R. Friedberg, T. D. Lee, Y. Pang, and H. C. Ren, Ann. Phys. (N.Y.) **246**, 381 (1996), for a soluble model with Gribov copies.
[8] L. Baulieu and M. Schaden, Int. J. Mod. Phys. A **13**, 985 (1998).
[9] L. Baulieu, A. Rozenberg, and M. Schaden, Phys. Rev. D **54**, 7825 (1996).
[10] H. Neuberger, Phys. Lett. B **175**, 69 (1986).
[11] For a review of TQFT see D. Birmingham, M. Blau, M. Rakowski, and G. Thompson, Phys. Rep. **209**, 129 (1991).
[12] A. Jevicki and N. Papanicolaou, Ann. Phys. (N.Y.) **120**, 107 (1979).
[13] R. Rajaraman, *Solitons and Instantons: an Introduction to Solitons and Instantons in Quantum Field Theory* (Elesevier, New York, 1987).
[14] R. Savit, Rev. Mod. Phys. **52**, 453 (1980).
[15] P. de Forcrand and J. E. Hetrick, Nucl. Phys. B (Proc. Suppl.) **42**, 861 (1995).
[16] M. Testa, Phys. Lett. B **429**, 349 (1998).
[17] W. Bock, M. Golterman, and Y. Shamir, Phys. Rev. D **58**, 097504 (1998).
[18] T. Maskawa and H. Nakajima, Prog. Theor. Phys. **60**, 1526 (1978); **63**, 641 (1980); D. Zwanziger, Nucl. Phys. **B209**, 336 (1982); M. A. Semenov-Tyan-Shanskii and V. A. Franke, *Proceedings of the Seminars of the Leningrad Mathematics Institute* (Plenum, New York, 1986).
[19] G. P. Lepage and P. Mackenzie, Phys. Rev. D **48**, 2250 (1993).
[20] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); Proc. R. Soc. London **A246**, 326 (1958).
[21] M. Schaden and A. Rozenberg, Phys. Rev. D **57**, 3670 (1998).
[22] M. Schaden, Phys. Rev. D **58**, 025016 (1998).
[23] For a review see S. Narison, *QCD Spectral Sum Rules*, edited by S. Narison, Lecture Notes in Physics Vol. 26 (World Scientific, Singapore, 1989).
[24] M. J. Creutz, *Quarks, Gluons and Lattices* (Cambridge University Press, Cambridge, England, 1983).