# Two-loop QCD corrections to semileptonic $b$ decays at an intermediate recoil

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We present complete $\mathcal{O}(\alpha_s^2)$ corrections to the decay $b\to c l \nu_l$ at the point where the invariant mass of the leptons $\sqrt{q^2}$ equals the $c$ quark mass. We use this result, together with previously obtained corrections at the ends of the $q^2$ spectrum, to estimate the total width of the semileptonic $b\to c$ decay with $\mathcal{O}(\alpha_s^2)$ accuracy, essential for the $|V_{cb}|$ determination. [S0556-2821(99)06501-7]

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Precise determination of the Cabibbo-Kobayashi-Maskawa (CKM) matrix parameters is one of the main goals of the present and upcoming high energy experiments. One of the three parameters which can be measured with the highest accuracy is $|V_{cb}|$, for which there are two complementary methods (for a recent review, see e.g. [1,2]). The first one is based on studying the exclusive decay $B\to D^\star$ + leptons near the zero recoil end of the phase space. The other, so-called inclusive method examines the total semileptonic width of the $B$ meson, $\Gamma_{\rm sl}(B\to X_c l \nu_l)$. For both determinations the experimental precision has reached a few percent level and will be further improved in future $B$ factories. To fully exploit those measurements, the theoretical prediction for the $b$ quark semileptonic width must be known with comparable precision. This requires a study of both short-distance (perturbative) and long-distance (non-perturbative) QCD effects. For the exclusive method the $\mathcal{O}(\alpha_s^2)$ corrections were calculated in [3]. Technically, the most challenging part of that calculation was the QCD correction to the axial $b\to c$ current normalization at zero recoil, obtained in [4,5] and confirmed in a recent study [6]. The theoretical accuracy of the exclusive method is limited by the errors in the non-perturbative matrix elements which are enhanced due to the not so large mass of the $c$ quark.

In the inclusive method the non−perturbative corrections are somewhat smaller, with suppression by $1/m_b^2$. They are estimated [7] to decrease the meson semileptonic width by approximately 5% compared to the free quark decay rate. With the non-perturbative corrections under control, the perturbative corrections should be carefully analyzed. Ideally one would like to know the two loop QCD correction for an arbitrary invariant mass squared of the leptons ($q^2$) in the decay $b\to c l \nu_l$. This has been achieved for only a subset of corrections, enhanced by a large factor describing the running of the strong coupling constant [8]; these so-called Brodsky-Lepage-Mackenzie (BLM) corrections [9] have been even resummed to all orders [10]. However, in the absence of a complete calculation of the two-loop QCD effects, the related uncertainty in $|V_{cb}|$ determined from the inclusive method had to be guessed and various estimates have been given in the literature. It is clear that the problem can be only solved by an explicit calculation of the complete $\mathcal{O}(\alpha_s^2)$ effects and therefore the need for such a calculation was often emphasized. In the present paper we demonstrate that, with the new result presented here, perturbative effects can be estimated with sufficient accuracy.

For an arbitrary value of $q^2$ (and, by the same token, for the total width), the calculation of the second order corrections is a very difficult technical task. In earlier works we presented such results for the two extreme points, $q^2=0$ [11] and $q^2=(m_b-m_c)^2$ [4,5]. While at both points the non-BLM corrections were below 1% of the total semileptonic width, it was not clear what happens at intermediate values of $q^2$, where a complete calculation appeared infeasible. Nevertheless, from the results in zero and maximal recoil points it was conjectured in [11] that the second order non-BLM corrections to the semileptonic decay width $b\to c l \nu_l$ do not exceed 1%.

Since then, we have found another kinematical point, $q^2=m_c^2$, where the complete calculation of the second order QCD corrections is possible. It is fortunate that for the physical value of $m_c$ this roughly corresponds to the middle of the $q$ range, so that together with the end points we have nicely distributed information (see Fig. 1).

Thus, the purpose of this paper is twofold. We first describe a calculation of the $\mathcal{O}(\alpha_s^2)$ correction to the differential semileptonic width of the $b$ quark for $q^2=m_c^2$. We then use the results obtained in this paper, as well as in [4,5,11], to estimate the $\mathcal{O}(\alpha_s^2)$ correction to the semileptonic decay width of the $b$ quark.

We start with a short description of the new calculation of the $\mathcal{O}(\alpha_s^2)$ correction for $q^2=m_c^2$. A complete description of the technical details of the present calculation will be presented elsewhere. The basic idea is the expansion of the decay amplitude in the velocity of the massive quark in the final state.

The Feynman diagrams which describe this process can be divided up into three classes according to the number of real gluons emitted. In the first group there are the purely virtual two-loop corrections. Their value is known at the point $m_c=q=m_b/2$ from our previous study on the zero recoil line [5]. Since the actual value of $m_c$ is not equal $m_b/2$, we expand the diagrams in the parameter $\beta$, related to the difference of these two quantities (see below). The coefficients of this expansion are given by diagrams with higher powers of propagators. By solving a system of recurrence relations, obtained by integration by parts, we found an algorithm with which all these Feynman integrals can be expressed in terms of four non-trivial functions, given below in Eq. (4), and some simpler (single-scale) Feynman integrals. This is the most challenging part of this calculation and it limits the number of terms we have been able to obtain using the present computing facilities.

In the other two groups there are diagrams with two real gluons or one real gluon and a virtual loop. It is relatively easy to perform calculations in these cases using (slightly modified) computational techniques described in detail in [12]. This possibility is related to the fact that for $q^2=m_c^2$ the three- and four-particle phase space integrals can be expressed in terms of the Euler $\Gamma$ function.

We write the differential semileptonic decay width of the decay $b\to c l\nu$ at $q^2=m_c^2$ as

$$\left[\frac{\mathrm{d}\Gamma_{\mathrm{sl}}}{\mathrm{d}q^2}\right]_{q^2=m_c^2}=\frac{G_F^2 m_b^3}{96\pi^3}|V_{cb}|^2 \times\left[\Delta_{\mathrm{Born}}+\frac{\alpha_s}{\pi}C_F\Delta_1+\left(\frac{\alpha_s}{\pi}\right)^2 C_F\Delta_2\right], \quad (1)$$

where $\Delta_{\mathrm{Born},1,2}$ describe the $m_c/m_b$ dependence in various orders in the strong coupling constant. Both $\Delta_{\mathrm{Born}}=(1-m_c^2/m_b^2)\sqrt{1-4m_c^2/m_b^2}$ and $\Delta_1$ are known in a closed analytical form [13,14]. $\Delta_2$ is calculated in the present paper. For the purpose of presentation we divide it up into four contributions according to the color factors:

$$\Delta_2=\sqrt{1-4m_c^2/m_b^2} \times[(C_F-C_A/2)\Delta_F+C_A\Delta_A+T_R N_L\Delta_L+T_R\Delta_H]. \quad (2)$$

The last term, $\Delta_H$, describes the contributions of the massive $b$ and $c$ quark loops. The top quark contribution is suppressed by a factor $\sim m_b^2/m_t^2$ and has been neglected. For the SU(3) group the color factors are $C_A=3$, $C_F=4/3$, $T_R=1/2$. Here $N_L=3$ is the number of the quark flavors whose masses have been neglected ($u$, $d$, and $s$).

We have computed 5 terms of the expansion in $\beta\equiv(1-4m_c^2/m_b^2)$ of $\Delta_{F,A,L,H}$. In order to save space, in the formulas presented here we give only numerical values of the coefficients of the second and higher powers of $\beta$ [these coefficients contain $\ln(\beta)$; we use $\beta=0.64$ to evaluate them]. We use the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme for the strong coupling constant and normalize it at the scale $\sqrt{m_b m_c}$, which has been proposed as optimal for the semileptonic $b\to c$ transitions, in view of the proximity of the small velocity limit [15]. Using the pole mass of the $b$ and $c$ quarks and expressing the one-loop corrections in terms of $\alpha_s(\sqrt{m_b m_c})$ we find

$$\begin{aligned}\Delta_A={}&\frac{49}{96}-\frac{19}{96}\pi^2\ln(2)+\frac{121}{384}\pi^2-\frac{591}{128}\ln(2)+\frac{179}{64}\ln^2(2)-\frac{51}{64}\zeta_3\\&+\beta\left[-\frac{89669}{6912}-\frac{17}{288}\pi^2\ln(2)-\frac{1}{12}\pi^2\ln(\beta)+\frac{43}{96}\pi^2-\frac{7}{6}\ln(2)\ln(\beta)+\frac{10225}{1152}\ln(2)\right.\\&\left.+\frac{523}{192}\ln^2(2)+\frac{89}{64}\zeta_3+\frac{253}{72}\ln(\beta)-\frac{11}{24}\ln^2(\beta)\right]-2.2631\beta^2-1.1747\beta^3-0.3171\beta^4,\end{aligned}$$

$$\begin{aligned}\Delta_F={}&\frac{117}{16}-\frac{19}{128}f_1-\frac{3}{8}f_3-\frac{5}{64}f_4+\frac{13}{12}\pi^2\ln(2)-\frac{173}{384}\pi^2-\frac{957}{64}\ln(2)+\frac{409}{64}\ln^2(2)-\frac{57}{32}\zeta_3-\frac{291}{32}R_2\\&+\beta\left[\frac{313}{128}-\frac{37}{192}f_1-\frac{1}{16}f_3+\frac{7}{96}f_4+\frac{55}{144}\pi^2\ln(2)-\frac{6071}{27648}\pi^2+\frac{9}{4}\ln(2)\ln(\beta)\right.\\&\left.-\frac{1825}{144}\ln(2)+\frac{179813}{13824}\ln^2(2)+\frac{25}{32}\zeta_3+\frac{465}{256}R_2-2\ln(\beta)\right]+0.11065\beta^2+0.54858\beta^3+0.78239\beta^4,\end{aligned}$$

$$\begin{aligned}\Delta_L={}&\frac{11}{12}-\frac{15}{16}\ln(2)+\beta\left[\frac{2699}{864}-\frac{1}{9}\pi^2+\frac{5}{6}\ln(2)\ln(\beta)-\frac{517}{144}\ln(2)+\ln^2(2)-\frac{13}{9}\ln(\beta)+\frac{1}{6}\ln^2(\beta)\right]\\&+0.72295\beta^2+0.62301\beta^3+0.26442\beta^4,\end{aligned}$$

$$\Delta_H = \frac{1379}{96} + \frac{355}{3072}\pi^2 + \frac{195}{64}\ln(2) + \frac{3317}{512}\ln^2(2) + \frac{7371}{256}R_2 + \beta\left[\frac{1579}{216} + \frac{1891}{1024}\pi^2 + \frac{299}{192}\ln(2) + \frac{6343}{512}\ln^2(2) + \frac{11385}{256}R_2\right] - 0.14557\beta^2 - 0.075047\beta^3 - 0.061437\beta^4. \tag{3}$$

In the above formulas we use the notation $f_{1,3,4}$ and $R_2$ for the values at $\omega = 1/2$ of the so-denoted functions, for which analytical formulas are given in [5]. Numerically they give

$$f_1 \approx 3.24460, \quad f_3 \approx 12.3201,$$
$$f_4 \approx 7.83195, \quad R_2 \approx -0.72946. \tag{4}$$

We now would like to estimate the uncalculated remainder of the series given in Eqs. (3) and the error in the final result. In a series $\Sigma a_n \beta^n$ the remainder is less than the last known term multiplied by $\beta/(1-\beta)$, provided that $a_n$ is a decreasing sequence, which we assume here. Therefore we estimate the final result by adding to the known terms half of $\beta/(1-\beta)$ times the last term. This additional term also gives a conservative estimate of the error. This procedure does not apply directly to $\Delta_F$, where the coefficients appear to grow. From previous experience with similar calculations we think that this is because the series describing $\Delta_F$ has not yet achieved its asymptotic behavior $\sim 1/n$. For example, in the maximal recoil case [12] the first five terms of the expansion for $\Delta_F$ behave rather wildly; nevertheless, they approximate the final result with an accuracy of about 25% (fortunately, the final result for $\Delta_2$ is rather insensitive to this error). We assign a similar error bar to $\Delta_F$. For $\beta = 0.64$ we find

$$\Delta_A = -2.37(5), \quad \Delta_F = 1.78(50),$$
$$\Delta_L = 1.09(4), \quad \Delta_H = -0.29(1). \tag{5}$$

Finally, we get, for the correction defined in Eq. (2) (we add the errors in quadrature),

$$\Delta_2 = -4.72(14). \tag{6}$$

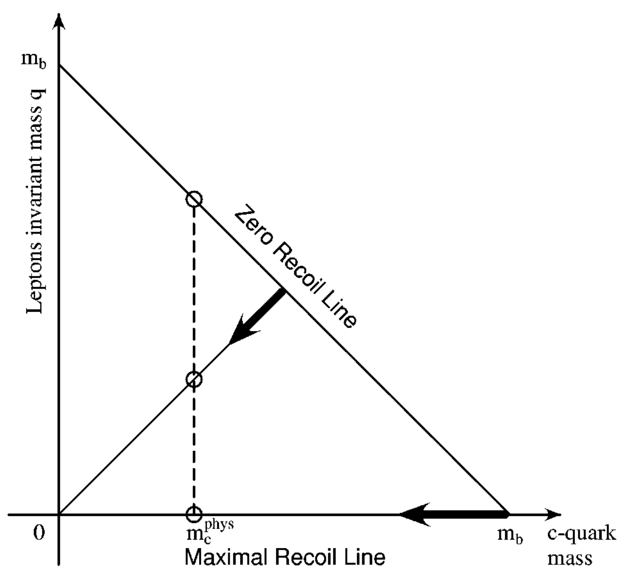


FIG. 1. Status of the two-loop QCD corrections to the decay $b \to c +$ leptons. The dashed line denotes the physical region for the actual $c$ quark. Points where the full corrections are known are circled. An analytical formula is known along the whole zero recoil line [5,6]. The other two points are found from expansions from the base points of the two arrows: at maximal recoil [11] and at the intersection with the diagonal (present work).

We now summarize the information about perturbative corrections to $b \to c$ transitions and estimate the $\mathcal{O}(\alpha_s^2)$ correction to the total semileptonic decay width of the $b$ quark. We use the results presented in this paper, as well as in our previous papers [4,5,11]. For arbitrary $q^2$, we define

$$\frac{d\Gamma_{sl}}{dq^2} = \frac{G_F^2 m_b^3}{96\pi^3}|V_{cb}|^2 \times\left[A_{\rm Born} + \frac{\alpha_s(\sqrt{m_b m_c})}{\pi}C_F A_1 + \left(\frac{\alpha_s}{\pi}\right)^2 C_F A_2\right] \tag{7}$$

where $m_b$ refers to the pole mass of the $b$ quark.

The BLM part of $A_2$ was obtained in [8]. The difference between the complete $A_2$ and $A_2^{\rm BLM}$ is called a non-BLM correction. We introduce a quantity $\xi$,

$$\xi(q^2) = \frac{A_2(q^2) - A_2^{\rm BLM}(q^2)}{A_{\rm Born}(q^2)}, \tag{8}$$

which describes the size of the non-BLM correction relative to the Born term as a function of $q^2$. The values of $\xi(q^2)$ for $q^2=0$ [11], $q^2 = m_c^2$ (this paper) and $q^2 = q^2_{\rm max} \equiv (m_b - m_c)^2$ [4,5] are, respectively, 0.65, 1.0, and 0.06 (for $m_c/m_b = 0.3$).

These three numbers permit us to estimate the $\mathcal{O}(\alpha_s^2)$ correction to the semileptonic decay width. For this purpose, we interpolate the non-BLM corrections with the function $\xi(q^2) = a_2 q^4 + a_1 q^2 + a_0$, where the coefficients $a_i$ are determined from the above values of $\xi(0)$, $\xi(m_c^2)$, $\xi(q^2_{\rm max})$. The function $\xi(q^2)$ is then integrated over $q^2$, using the Born differential rate as the weight. As the result one gets an estimate of the non-BLM correction for the total semileptonic decay rate:

$$\frac{\int_0^{(m_b-m_c)^2} dq^2 A_{\rm Born}(q^2)\xi(q^2)}{\int_0^{(m_b-m_c)^2} dq^2 A_{\rm Born}(q^2)} \simeq 1.1. \tag{9}$$

The validity of this estimate can be checked by using the same procedure to obtain an estimate of the BLM correction for the total decay rate using the known results at $q^2=0, m_c^2, (m_b-m_c)^2$ as an input.

Such a fit results in the value of the BLM correction $-8.6(\alpha_s/\pi)^2$, to be compared with the exact result [8] $-9.8(\alpha_s/\pi)^2$.[1] We conclude that the accuracy of our simple fit is not worse than 30%, which also includes the errors in the three input data points. We therefore obtain the following formula for the total semileptonic decay rate $\Gamma_{\rm sl}(b\to c l \nu_l)$:

$$\Gamma_{\rm sl}=\frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2 F\left(\frac{m_c^2}{m_b^2}\right)\times\left[1-1.67\,\frac{\alpha_s(\sqrt{m_b m_c})}{\pi}+(-9.8+1.4\pm 0.4)\left(\frac{\alpha_s}{\pi}\right)^2\right], \tag{10}$$

where $F(x)=1-8x-12x^2\ln(x)+8x^3-x^4$ and we used $m_c/m_b=0.3$. For the sake of clarity we separated the BLM and the non-BLM parts of the second order corrections. We also explicitly indicated the uncertainty in our estimate of the second order non-BLM correction.

In principle, Eq. (10) provides the result for the semileptonic decay width $b\to c l \nu_l$, when expressed through the pole $b$ and $c$ quark masses, valid to $\mathcal{O}(\alpha_s^2)$. One notices that Eq. (10) contains a large second order correction due to the BLM effects. For a long time this observation seemed to seriously limit the precision in $|V_{cb}|$, as obtained from the inclusive method. It was, however, noticed [16,17] that the large value of the second order BLM corrections is correlated with the fact that the pole quark masses were used in Eq. (10). It is well established that the pole quark masses cannot be defined when nonperturbative corrections are addressed. A hint that this is really the case is given by the bad behavior of the perturbation series itself. In the case of the semileptonic $b$ decays the problem is enhanced, since the decay width is proportional to the fifth power of the $b$-quark mass.

It was suggested in [16] and then further elaborated in [15] that the most appropriate masses, to be used in the expression for the decay width, are the so-called low-scale running quark masses normalized at a scale $\mu\sim 1-2$ GeV. On the one hand, such masses can be defined on the nonperturbative level, on the other (and this is related to the first point) their use is expected to minimize the perturbative corrections to the semileptonic decay width. To $\mathcal{O}(\alpha_s^2)$ a (perturbative) relation between the pole and the low–scale quark masses was obtained in [18] and reads

$$m_{\rm pole}=m(\mu)+[\Lambda(\mu)]_{\rm pert}+\frac{1}{2m(\mu)}[\mu_\pi^2(\mu)]_{\rm pert}, \tag{11}$$

where

$$[\Lambda(\mu)]_{\rm pert}=\frac{4}{3}C_F\mu\frac{\alpha_s(\mu_0)}{\pi}\left\{1+\frac{\alpha_s}{\pi}\left[\left(\frac{4}{3}-\frac{1}{2}\ln\frac{2\mu}{\mu_0}\right)\beta_0-C_A\left(\frac{\pi^2}{6}-\frac{13}{12}\right)\right]\right\},$$

$$[\mu_\pi^2(\mu)]_{\rm pert}=C_F\mu^2\frac{\alpha_s(\mu_0)}{\pi}\left\{1+\frac{\alpha_s}{\pi}\left[\left(\frac{13}{12}-\frac{1}{2}\ln\frac{2\mu}{\mu_0}\right)\beta_0-C_A\left(\frac{\pi^2}{6}-\frac{13}{12}\right)\right]\right\}.$$

It is important that, in contrast to the pole masses, the low-scale masses do not have any significant numerical ambiguity. We use Eq. (11) to rewrite the expression for the semileptonic decay width (10) through the low-scale masses normalized at $\mu=1$ GeV. In the BLM approximation such a calculation was performed in [16]. As a result, we find that the perturbative coefficients decrease noticeably:

$$\Gamma_{\rm sl}(b\to c l \nu_l)=\frac{G_F^2\tilde{m}_b^5|V_{cb}|^2}{192\pi^3}F\left(\frac{\tilde{m}_c^2}{\tilde{m}_b^2}\right)\times\left[1-1.14\,\frac{\alpha_s(\sqrt{\tilde{m}_b\tilde{m}_c})}{\pi}-(2.65\pm 0.40)\left(\frac{\alpha_s}{\pi}\right)^2\right], \tag{12}$$

where we have used the values of the low scale running quark masses $\tilde{m}_b=4.64(5)$ GeV and $\tilde{m}_c=1.25(10)$ GeV, as suggested in [1]. Though in [1] the errors assigned to the $b$ and $c$ quark low scale running mass were considered conservative, in our opinion this issue is not completely clear and a dedicated analysis is required. It is, however, rather certain that in contrast to the pole mass, the accuracy of $\tilde{m}_Q$ can *in principle* be reliably estimated.

We see that the perturbative series for the inclusive width appears to behave better when the low scale masses are used, in accordance with the theoretical arguments [16,15,1].

To sum, we have estimated the second order QCD corrections to the width of the semileptonic $b\to c$ decay. The small value of these corrections shows that the perturbative series is not likely to cause any significant uncertainty in the $|V_{cb}|$ extracted using the inclusive method, provided that the decay width is calculated using the low scale mass definition. Further improvement in the theoretical predictions for $\Gamma_{\rm sl}(B\to X_c l \nu_l)$ will be possible when more precise quark mass values and the non-perturbative matrix elements have been determined.

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[1]Actually, Ref. [8] gives $-15.1(\alpha_s/\pi)^2$. We modify that value by using 4, rather than 3, massless flavors for the $\alpha_s$ evolution, and by using $\sqrt{m_b m_c}$, rather than $m_b$, as its normalization point.

[1] I. Bigi, M. Shifman, and N. Uraltsev, Annu. Rev. Nucl. Part. Sci. **47**, 591 (1997).
[2] M. Neubert, in *Heavy Flavours*, 2nd ed., edited by A. Buras and M. Lindner (World Scientific, Singapore, in press), hep-ph/9702375.
[3] A. Czarnecki, K. Melnikov, and N. Uraltsev, Phys. Rev. D **57**, 1769 (1998).
[4] A. Czarnecki, Phys. Rev. Lett. **76**, 4124 (1996).
[5] A. Czarnecki and K. Melnikov, Nucl. Phys. **B505**, 65 (1997).
[6] J. Franzkowski and J. B. Tausk, Eur. Phys. J. C **5**, 517 (1998).
[7] I. I. Bigi, N. G. Uraltsev, and A. I. Vainshtein, Phys. Lett. B **293**, 430 (1992); I. I. Bigi *et al.*, Phys. Rev. Lett. **71**, 496 (1993).
[8] M. Luke, M. J. Savage, and M. B. Wise, Phys. Rev. Lett. **345**, 301 (1995).
[9] S. J. Brodsky, G. P. Lepage, and P. B. Mackenzie, Phys. Rev. D **28**, 228 (1983).
[10] P. Ball, M. Beneke, and V. M. Braun, Phys. Rev. D **52**, 3929 (1995).
[11] A. Czarnecki and K. Melnikov, Phys. Rev. Lett. **78**, 3630 (1997).
[12] A. Czarnecki and K. Melnikov, Phys. Rev. D **56**, 7216 (1997).
[13] M. Jeżabek and J. H. Kühn, Nucl. Phys. **B314**, 1 (1989).
[14] A. Czarnecki, Phys. Lett. B **252**, 467 (1990).
[15] I. Bigi, M. Shifman, N. Uraltsev, and A. Vainshtein, Phys. Rev. D **56**, 4017 (1997).
[16] N. G. Uraltsev, Int. J. Mod. Phys. A **11**, 515 (1996).
[17] M. Beneke and V. M. Braun, Phys. Lett. B **348**, 513 (1995).
[18] A. Czarnecki, K. Melnikov, and N. Uraltsev, Phys. Rev. Lett. **80**, 3189 (1998).