Nonrelativistic pion interactions and the pionium lifetime

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We construct an effective Lagrangian for interacting pions with non-relativistic energies. The coupling constants can be expressed in terms of the different scattering lengths and slopes. When used in the calculation of the pionium decay rate, the scattering slope contribution gives a correction of about 8% compared with the lowest order contribution coming from the scattering lengths alone. [S0556-2821(98)07623-1]

Pionium is a hadronic atom of π^+ and π^- bound by the Coulomb force. It is highly unstable via the strong decay $\pi^+ + \pi^- \rightarrow \pi^0 + \pi^0$ which probes the low-energy interactions of the pions. As such it can be used to test more accurately the predictions of chiral perturbation theory which is an effective theory for QCD at low energies [1,2]. It was first constructed by Weinberg who used it at the tree level to calculate the $\pi\pi$ scattering amplitudes in agreement with current algebra results [3]. Since then the results have been improved with one-loop corrections by Gasser and Leutwyler [4] and are now carried to two-loop order [5]. On the other hand, the experimental values of these scattering amplitudes are still very uncertain. For instance, the isospin-zero S-wave scattering length is known with only 20% accuracy [6].

Recently a lot of interest has been generated by the possibility of a more accurate determination of scattering lengths from measurements of the hadronic decay of pionium [7,8]. In order for this to succeed, one must have a complete understanding of the different effects acting in the decay process. Since the pions in this hadronic atom are non-relativistic, they can be described by an effective theory expanded in terms of operators of increasing dimensions involving pion fields and their derivatives. By matching it to relativistic chiral perturbation theory or experiments, the *a priori* unknown coupling constants can be determined. It will be done in the following to order p^2 where p is the momentum of the pions. To this order the different scattering amplitudes are characterized by a scattering length a and a scattering slope b which then determine the coupling constants.

The same, effective Lagrangian also determines the dynamics of the bound pions in the pionium atom. Since we have a strictly non-relativistic system, we can use the ordinary Schrödinger equation to calculate the wavefunctions and there is no need for covariant formalisms like the Bethe-Salpeter equation or others. This is in the very spirit of non-relativistic QED (NRQED) established by Caswell and Lepage [9,10] and used with great success for muonium [11] and positronium [12]. In this non-relativistic framework one

can then systematically calculate corrections to the different energy levels. In particular, a complex contribution ΔE signals that the corresponding state is unstable with a decay rate given by $\Gamma = -2 \text{ Im } \Delta E$ and thus with lifetime $\tau = 1/\Gamma$.

The dominant part of the pionium decay comes from the constant part of the amplitude for $\pi^+ + \pi^- \rightarrow \pi^0 + \pi^0$, i.e. from the scattering length. In the following we will show that the energy dependence of the amplitude, or the scattering slope, gives an additional contribution which is around 8% of the leading term. It is an important correction and larger than typical electromagnetic corrections which have been considered until now [7,8].

Non-relativistic pions are described by the complex Schrödinger fields $\boldsymbol{\pi} = (\pi_+, \pi_0, \pi_-)$ where π_+ annihilates a π^+, π_-^* creates a π^- and so on. The free fields are described by the Lagrangian

$$\mathcal{L}_0(\pi_i) = \pi_i^* \left(i\frac{\partial}{\partial t} + \frac{1}{2m_i} \nabla^2 \right) \pi_i. \quad (1)$$

The masses of π_+ and π_- are the same and will be denoted by m_+ while π_0 has a slightly lower mass denoted by m_0. We could also include here the relativistic propagator correction $\propto \pi_i^* \nabla^4 \pi_i$, but we will ignore such small corrections in the following. In the same vein, we will not consider electromagnetic effects although they are in general important in the problem under consideration.

For the interacting part $\mathcal{L}_{int}$ we will assume exact isospin invariance and only S-wave interactions. We then find that the lowest order interaction can only involve two possible couplings:

$$\mathcal{L}_{int}(\boldsymbol{\pi}) = G_0(\boldsymbol{\pi}^* \cdot \boldsymbol{\pi})(\boldsymbol{\pi}^* \cdot \boldsymbol{\pi}) + H_0(\boldsymbol{\pi}^* \cdot \boldsymbol{\pi}^*)(\boldsymbol{\pi} \cdot \boldsymbol{\pi}). \quad (2)$$

Thus, we have the full Lagrangian $\mathcal{L} = \mathcal{L}_0(\pi_+) + \mathcal{L}_0(\pi_0) + \mathcal{L}_0(\pi_-) + \mathcal{L}_{int}(\boldsymbol{\pi})$. The interaction has dimension six and is thus not renormalizable in the ordinary sense. But considered as an effective theory, it can be renormalized to every order in the expansion of $\mathcal{L}_{int}$ in higher-dimensional operators. It has essentially the same form as a corresponding effective theory proposed for non-relativistic nucleons by

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Weinberg [13] and recently improved by Kaplan, Savage and Wise for np scattering [14] and the deuteron [15]. The divergent loop integrals can be regularized by a momentum cutoff, but as for most effective theories, it is much more efficient to use dimensional regularization with minimal subtraction. We will use this method in the following.

For dimensional reasons we know that the coupling constants G_0 and H_0 must be $\propto 1/m^2$ where the ''heavy mass'' m in our case is the pion mass. They can be obtained by matching to relativistic chiral perturbation theory or directly to experiments. Performing the matching in the first way, we find to lowest order in the expansion of the chiral Lagrangian the effective couplings $G_0 = -1/8f_\pi^2$ and $H_0 = +3/16f_\pi^2$ with the pion decay constant $f_\pi = 92.5$ MeV. The resulting couplings between pions in different isospin channels can now be deduced from Eq. (2) which takes the form

$$\mathcal{L}_{int}(\boldsymbol{\pi}) = \frac{1}{4}A_0(\pi_0^*\pi_0^*\pi_0\pi_0) + B_0(\pi_+^*\pi_-^*\pi_+\pi_-)$$

$$+ \frac{1}{2}C_0(\pi_+^*\pi_-^*\pi_0\pi_0 + \pi_0^*\pi_0^*\pi_+\pi_-)$$

$$+ \frac{1}{4}D_0(\pi_+^*\pi_+^*\pi_+\pi_+ + \pi_-^*\pi_-^*\pi_-\pi_-$$

$$+ 2\pi_+^*\pi_0^*\pi_+\pi_0 + 2\pi_-^*\pi_0^*\pi_-\pi_0) \qquad (3)$$

when written out. With the above tree-level values for the two fundamental coupling constants, we now have $A_0 = B_0/2 = C_0/3 = -D_0/2 = 1/4f_\pi^2$.

In order to compare with experiments, we calculate the S-wave scattering amplitude $T(\mathbf{p})$ where $\mathbf{p}$ is the c.m. momentum of the pions. The real part is usually defined by

$$\text{Re } T(\mathbf{p}) = \frac{8\pi}{m_\pi^2}\left(a + b\frac{\mathbf{p}^2}{m_\pi^2}\right) \qquad (4)$$

in terms of the scattering length a and the slope parameter b which gives the energy dependence of the amplitude to lowest order [6]. With the above values for the coupling constants, it is now straightforward to read off Weinberg's scattering lengths in the different isospin channels [3] from the Lagrangian Eq. (3).

Instead of using these results from chiral perturbation theory at tree level, we can instead match the coupling constants to the measured cross sections, i.e. to the observed S-wave scattering lengths a_0 and a_2 for isospin $I=0$ and $I=2$ respectively and the corresponding scattering slopes. In connection with pionium we will be especially interested in the process $\pi^+ + \pi^- \to \pi^0 + \pi^0$. To lowest order in perturbation theory the scattering amplitude is given by the Feynman diagram in Fig. 1 which gives

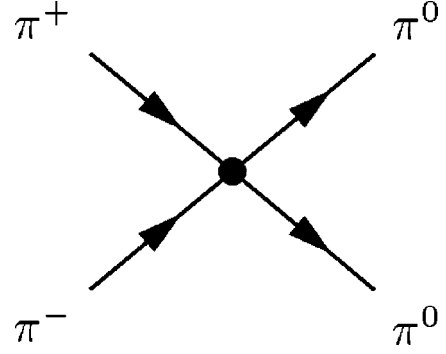$$T^{(0)}(\pi^+ + \pi^- \to \pi^0 + \pi^0) = \frac{8\pi}{3m_\pi^2}(a_0 - a_2) = C_0. \qquad (5)$$

FIG. 1. Tree-level contribution to the scattering process due to the contact term C_0.

We thus have C_0 directly expressed in terms of measured scattering lengths. Considering related processes, we can similarly obtain the other coupling constants,

$$A_0 = \frac{8\pi}{3m_\pi^2}(a_0 + 2a_2), \quad B_0 = \frac{8\pi}{3m_\pi^2}(a_0 + a_2/2), \quad D_0 = \frac{8\pi}{m_\pi^2}a_2 \qquad (6)$$

when we combine scattering amplitudes with definite isospin.

So far these relations are only valid at tree level of the effective theory. The scattering amplitudes are real and unitarity is thus not satisfied. This can be achieved by going to higher orders in perturbation theory. Again considering $\pi^+ + \pi^- \to \pi^0 + \pi^0$, we have two one-loop diagrams of the form shown in Fig. 2. One contains $\pi^+\pi^-$ and the other $\pi^0\pi^0$ in the intermediate state. Ignoring here the mass differences, they give the correction

$$T^{(1)}(\pi^+ + \pi^- \to \pi^0 + \pi^0) = -\left(B_0C_0 + \frac{1}{2}A_0C_0\right)I(\mathbf{p}) \qquad (7)$$

where the factor of 1/2 is due to the two identical particles in the $\pi^0\pi^0$ intermediate state.

The integral over intermediate momenta

$$I(\mathbf{p}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{E - \mathbf{k}^2/m_\pi + i\epsilon} \qquad (8)$$

where $E = \mathbf{p}^2/m_\pi$ is the total c.m. energy, is seen to be linearly divergent. Using now dimensional regularization with
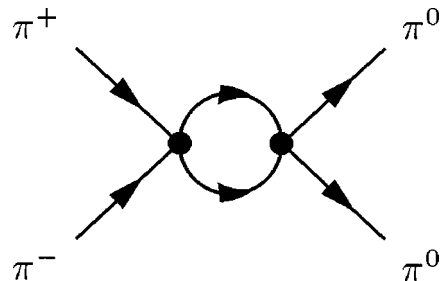

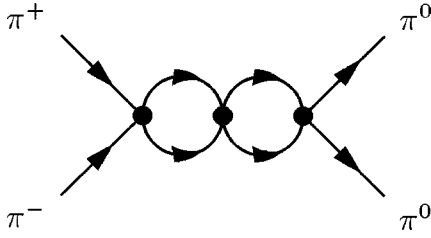FIG. 2. One-loop contributions to the scattering amplitude.

FIG. 3. Two-loop contributions to the scattering amplitude.

minimal subtraction [14], it is simply given by $I(\mathbf{p}) = -i|\mathbf{p}|m_\pi/4\pi$. Using instead a momentum cutoff Λ, it would contain a term proportional with Λ. This could then be absorbed by renormalization of the coupling constant C_0. The net result is either way a purely imaginary result which arises from the unitarity requirement, but does not contribute to the scattering length or slope parameter in Eq. (4).

However, going to two loops as in Fig. 3 we will obtain corrections to the tree-level results. The two intermediate bubbles can again contain a $\pi^+\pi^-$ or a $\pi^0\pi^0$ pair. Summing up the four contributions from combinations of different bubbles, we then have the next order correction

$$T^{(2)}(\pi^+ + \pi^- \rightarrow \pi^0 + \pi^0)$$

$$= \left(\frac{1}{2}C_0^3 + \frac{1}{2}A_0B_0C_0 + \frac{1}{4}A_0^2C_0 + B_0^2C_0\right)I^2(\mathbf{p})$$

$$= -\frac{8\pi}{3m_\pi^4}(a_0 - a_2)(a_0^2 + a_0a_2 + a_2^2)p^2 \qquad (9)$$

after regularization. It is seen to give an energy dependence of the scattering amplitude proportional to p^2 and thus contribute to the slope parameter in Eq. (4). But such an energy dependence at two-loop level can also result at tree level from an operator in the Lagrangian containing two derivatives. More specifically, for the $\pi^+ + \pi^- \rightarrow \pi^0 + \pi^0$ channel we consider here, we must include higher dimensional operators in the expansion of the effective Lagrangian. For S-wave interactions there is only one such possible operator to lowest order in the derivative expansion,

$$\mathcal{L}_{int}(\pi^+\pi^-\pi^0\pi^0)$$

$$= \frac{1}{2}C_0(\pi_+^*\pi_-^*\pi_0\pi_0)$$

$$+ \frac{1}{4}C_2(\pi_+^*\pi_-^*\pi_0\overleftrightarrow{\nabla}^2\pi_0 + \pi_+^*\overleftrightarrow{\nabla}^2\pi_-^*\pi_0\pi_0) + \text{H.c.}$$

$$\qquad (10)$$

Here the gradient is defined as $\overleftrightarrow{\nabla} = 1/2(\overrightarrow{\nabla} - \overleftarrow{\nabla})$. It corresponds to the vertex $\hat{V}_2$ in Fig. 4 with the value $\langle \mathbf{p}|\hat{V}_2|\mathbf{q}\rangle = \frac{1}{2}C_2(\mathbf{p}^2 + \mathbf{q}^2)$ in the c.m. reference frame.

Adding this contribution to the two-loop result Eq. (9) and matching to the definition of the full scattering amplitude in Eq. (4), we have
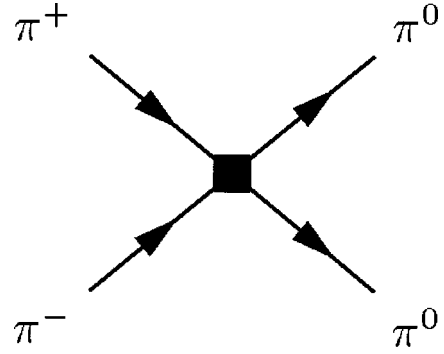
FIG. 4. Vertex due to the double-derivative coupling C_2.

$$T^{(2)} - C_2p^2 = \frac{8\pi}{3m_\pi^2}\frac{p^2}{m_\pi^2}(b_0 - b_2)$$

which gives

$$C_2 = -\frac{8\pi}{3m_\pi^4}[(b_0 - b_2) + (a_0 - a_2)(a_0^2 + a_0a_2 + a_2^2)]. \qquad (11)$$

Since this is a coupling constant in the effective Lagrangian, it can also be used for bound state problems. There are no ''off-shell'' problems in this approach.

We are now in the position to consider decay of pionium. The ground state with wave function $\Psi(r)$ has the energy $E = 2m_+(1 - \alpha^2/8)$. It will be perturbed by the different hadronic contact interactions in Eq. (3). For instance, at tree level we get a real energy shift from the elastic coupling B_0. Its magnitude is simply $-B_0|\Psi(0)|^2$ where the wave function at the origin is $|\Psi(0)|^2 = \gamma^3/\pi$ with $\gamma = \alpha m_+/2$. It is proportional to the scattering length $a_0 + a_2/2$. This is the hadronic energy level shift discussed first by Deser, Goldberger, Baumann and Thirring [16].

At next order in perturbation theory we must evaluate the diagram in Fig. 5 with a $\pi^+\pi^-$ in the intermediate state. It gives also a real, but smaller contribution proportional to $(a_0 + a_2/2)^2$. However, the same diagram, but now with a $\pi^0\pi^0$ in the intermediate state, is purely imaginary. In the bound state picture it is given by the matrix element

$$\Delta E = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\Psi^*(\mathbf{p})$$

$$\times C_0\frac{1}{2\Delta m - \mathbf{k}^2/m_0 + i\epsilon}C_0\Psi(\mathbf{q}) \qquad (12)$$
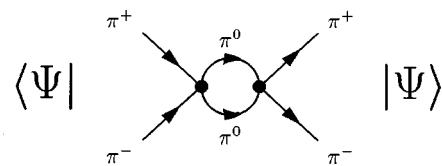

FIG. 5. One-loop correction to the ground state energy level which gives the decay rate.

as first shown by Labelle [8] using non-relativistic effective field theory. Here $\Delta m = m_+ - m_0$ gives the energy of the intermediate state and

$$\Psi(\mathbf{p}) = \frac{8\pi^{1/2}\gamma^{5/2}}{(\mathbf{p}^2 + \gamma^2)^2} \qquad (13)$$

is the Fourier transform of the ground-state wave function. It gives the probability to find the momentum p in this state. Using the regularized value of the integral Eq. (8), we obtain

$$\Delta E = -i\frac{m_0}{8\pi} C_0^2 \sqrt{2\Delta m m_0}|\Psi(0)|^2. \qquad (14)$$

This imaginary result signals that the ground state is unstable and will decay with a rate $\Gamma = -2\,\mathrm{Im}\,\Delta E$ induced by the hadronic coupling C_0. With the value given in Eq. (5) it becomes

$$\Gamma = \frac{16\pi}{9m_\pi^2}|\Psi(0)|^2 \frac{m_0^2}{m_\pi^2} \sqrt{\frac{2\Delta m}{m_0}} (a_0 - a_2)^2 \qquad (15)$$

which is the standard result.

However, there is some implicit uncertainty here in what value to use for the pion mass m_π. It comes from the definition of the scattering lengths for both charged and neutral pions. Taking it to be the charged mass m_+ for both of them, we can write the rate as $\Gamma = \Gamma_0(1 - 3\Delta m/2m_+)$ where

$$\Gamma_0 = \frac{16\pi}{9m_+^2}|\Psi(0)|^2 \sqrt{\frac{2\Delta m}{m_+}} (a_0 - a_2)^2. \qquad (16)$$

Since $\Delta m/m_+ = 0.033$, the last factor represents a 5% reduction of the main decay rate Eq. (16). Such kinematic corrections will be important in a future experimental determinations of the scattering lengths from the measured pionium decay rate.

Evaluating the two-loop correction to the ground-state energy, we find a purely real result since each bubble gives an imaginary contribution. Therefore in the order we are working at, the two-loop correction to the decay rate is zero.

The above standard result for the decay rate is due to the constant part of the $\pi^+ + \pi^- \to \pi^0 + \pi^0$ amplitude, i.e. the corresponding scattering length. But it has also an energy-dependent component parameterized by the scattering slope b in Eq. (4). We can now easily calculate this effect to lowest order in the corresponding derivative coupling C_2 in the Lagrangian Eq. (10). It results from evaluating the same diagram in Fig. 5 but with one of the C_0 vertices replaced with the C_2 vertex from Fig. 4. This gives the additional contribution

$$\Delta E^{(2)} = -\int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \Psi^*(\mathbf{p})$$

$$\times C_0 \frac{1}{2\Delta m - \mathbf{k}^2/m_0 + i\epsilon} \frac{1}{2} C_2(\mathbf{k}^2 + \mathbf{q}^2)\Psi(\mathbf{q})$$

$$(17)$$

to the ground-state level shift Eq. (12). The integrations over momenta $\mathbf{k}$ and $\mathbf{q}$ are now even more divergent than in the first contribution. But again we invoke dimensional regularization. We then find that the part coming from the $\mathbf{q}^2$ can be neglected since it is smaller by a factor $\alpha^2 m_0/\Delta m$. Thus the integral over $\mathbf{q}$ just gives $\Psi(0)$. Writing the integral over $\mathbf{k}$ as

$$\int \frac{d^3k}{(2\pi)^3} \frac{\mathbf{k}^2}{2\Delta m m_0 - \mathbf{k}^2 + i\epsilon}$$

$$= \int \frac{d^3k}{(2\pi)^3} \left[\frac{2\Delta m m_0}{2\Delta m m_0 - \mathbf{k}^2 + i\epsilon} - 1\right], \qquad (18)$$

we see that the last term is zero with dimensional regularization while the first part is just the previous integral Eq. (8), used in the calculation of the main level shift. In this way we obtain the finite result

$$\Delta E^{(2)} = i\frac{m_0}{4\pi} C_0 C_2 \Delta m m_0 \sqrt{2\Delta m m_0}|\Psi(0)|^2. \qquad (19)$$

Using now Eq. (11) for the coupling constant C_2, we thus obtain the corresponding correction $\Delta\Gamma^{(2)}$ to the decay rate. It can be written as

$$\frac{\Delta\Gamma^{(2)}}{\Gamma_0} = 2\frac{\Delta m}{m_+}\left[\frac{b_0 - b_2}{a_0 - a_2} + (a_0^2 + a_0 a_2 + a_2^2)\right] \qquad (20)$$

when we write $m_0 = m_+$ to leading order in the mass difference. The experimental values [6] of scattering lengths and slopes are $a_0 = 0.26 \pm 0.05$, $a_2 = -0.028 \pm 0.012$ and $b_0 = 0.25 \pm 0.03$, $b_2 = -0.082 \pm 0.008$. Using these, we find that the first term is more than an order of magnitude larger than the last. Combined, this amount to a $7.6\% + 0.4\% = 8.0\%$ correction to the main decay rate. On the other hand, using just the tree-level values [6], we obtain the very similar result $8.6\% + 0.1\% = 8.7\%$. With values from higher order chiral perturbation theory, the result is again not much different. The overall hadronic correction is sizable and larger than other known corrections of electromagnetic origin [7,8].

As a rough check of this rather large correction, we can try to estimate the decay rate directly from the matrix element Eq. (4) for $\pi^+ + \pi^- \to \pi^0 + \pi^0$ with $a = (a_0 - a_2)/3$ and $b = (b_0 - b_2)/3$. Taking $\Gamma \propto |T|^2$, we obtain to lowest order in the scattering slope the correction factor $1 + 2(b/a)(\langle\mathbf{p}^2\rangle/m_\pi^2)$ to the standard result Eq. (15). Dividing the average momentum $\langle\mathbf{p}^2\rangle$ equally between the initial state where it is γ^2 and can thus be neglected and the final state where it is $2\Delta m m_\pi$, we have exactly the dominant term in the more accurate result Eq. (20).

[1] S. Weinberg, Physica A **96**, 327 (1979).

[2] For a recent review, see G. Ecker, Prog. Part. Nucl. Phys. **36**, 71 (1996).

[3] S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966).

[4] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).

[5] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, and M. Saino, Phys. Lett. B **374**, 210 (1996).

[6] J. F. Donoghue, E. Golowich, and B. R. Holstein, *Dynamics of the Standard Model* (Cambridge University Press, Cambridge, England, 1992).

[7] V. E. Lyubovitskij, E. Z. Lipartia, and A. G. Rusetsky, JETP Lett. **66**, 783 (1997); V. E. Lyubovitskij, E. Z. Lipartia, and A. G. Rusetsky, hep-ph/9706244; H. Jallouli and H. Sazdjian, Phys. Rev. D **58**, 014011 (1998); M. Knecht and R. Urech, Nucl. Phys. **B519**, 329 (1998); M. A. Ivanov, V. E. Lyubovitskij, E. Z. Lipartia, and A. G. Rusetsky, Phys. Rev. D **58**,

094024 (1998); G. G. Bunatian, hep-ph/9805371.

[8] P. Labelle, hep-ph/9804201.

[9] W. E. Caswell and G. P. Lepage, Phys. Lett. **167B**, 437 (1985).

[10] T. Kinoshita and G. P. Lepage, in *Quantum Electrodynamics*, edited by T. Kinoshita (World Scientific, Singapore, 1990).

[11] T. Kinoshita and M. Nio, Phys. Rev. D **53**, 4909 (1996); M. Nio and T. Kinoshita *ibid.* **55**, 7267 (1997).

[12] A. H. Hoang, P. Labelle, and S. M. Zebarjad, Phys. Rev. Lett. **79**, 3387 (1997).

[13] S. Weinberg, Nucl. Phys. **B363**, 3 (1991); Phys. Lett. B **295**, 114 (1992).

[14] D. B. Kaplan, M. J. Savage, and M. B. Wise, nucl-th/9802075.

[15] D. B. Kaplan, M. J. Savage, and M. B. Wise, nucl-th/9804032.

[16] S. Deser, M. L. Goldberger, K. Baumann, and W. Thirring, Phys. Rev. **96**, 774 (1954).