Confining properties of the homogeneous self-dual field and the effective potential in SU(2) Yang-Mills theory

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We examine in non-Abelian gauge theory the heavy quark limit in the presence of the (anti-)self-dual homogeneous background field and see that a confining potential emerges, consistent with the Wilson criterion, although the potential is quadratic and not linear in the quark separation. This builds upon the well-known feature that propagators in such a background field are entire functions. The way in which deconfinement can occur at finite temperature is then studied in the static temporal gauge by calculation of the effective potential at high temperature. Finally we discuss the problems to be surmounted in setting up the calculation of the effective potential nonperturbatively on the lattice. [S0556-2821(99)00801-2]

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I. INTRODUCTION

Over the years, various characterizations have been proposed for ''confinement,'' the property that colored degrees of freedom are undetectable at present-day collider energies. Certainly, the Wilson criterion that static color sources cannot be separated arbitrarily far apart [1] has lead to many insights both into lattice simulations and into analytical calculations. In particular, based on the Wilson criterion lattice simulations have established a confinement-deconfinement phase transition at finite temperature. There are however alternate characterizations for confinement which may be more directly relevant for dynamical quarks and gluons, and which are based on the analytic properties of the nonperturbative quark or gluon propagators. In this paper, we shall focus on the suggestion that the absence of poles in the complex energy plane of field propagators is consistent with confinement of quarks and gluons, in other words that propagators are entire functions. That this can be correctly described as ''confinement'' is easy to see: the absence of poles means that no colored degrees of freedom can appear in physical asymptotic states. This characterization of confinement is not necessarily in conflict with the Wilson criterion. Indeed, one of our aims will be to show that, in the static quark limit, entire quark propagators lead to the Wilson criterion.

A quite simple mechanism for rendering quark and gluon propagators entire in the complex energy plane is to apply a homogeneous background gluon field which satisfies the key property that it be either self-dual or anti-self-dual. Such a background gauge field is characterized by

$$B^a_\mu(x)\tau^a \equiv \frac{1}{2} n^a \tau^a B_{\mu\nu} x_\nu \tag{1}$$

$$\tilde{B}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda} B_{\rho\lambda} = \pm B_{\mu\nu} \tag{2}$$

$$B_{\mu\nu}B_{\mu\rho} = B^2 \delta_{\nu\rho}, \quad B = \text{const},$$
$$B_{ij} = -\epsilon_{ijk} B_k, \quad B_{j4} = \pm B_j. \tag{3}$$

The positive and negative signs in Eqs. (2),(3) correspond, respectively, to the self-dual and anti-self-dual cases. The color vector n^a points in some fixed direction which can be chosen such that $n^a\tau^a$ is diagonal; n^a picks out the Cartan subalgebra of the color group. Various properties of this field in SU(2) gauge theory were investigated originally in [2,3]. For example, in contradistinction to the chromomagnetic background field [4] the self-dual background is stable. Moreover, it was observed that this field leads to entire functions for the charged scalar field propagator. In the sense described above, then, this field can provide for confinement of quarks and gluons. Diagonal components of the gluon field [such as SU(N) algebra elements] are not confined at least at the level of the lowest order propagator in the background field. A self-dual homogeneous field is at least then a possible source for confinement in QCD if it can be shown that such a field is a dominant configuration in the QCD functional integral.

This verification can come from a computation of the effective potential for the candidate background field and the demonstration that the potential has a minimum at a nonzero value for the background field. The effective potential was calculated to one-loop in [2,5]. These results however were inconclusive in the sense that the quantum corrections to the

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potential were as large as the zeroth order classical term. To our knowledge, despite several attempts to study the effective potential for the Savvidy chromomagnetic background [4] on the lattice [6–10], an analogous nonperturbative computation for the self-dual homogeneous background has not been attempted. Nonetheless, with the assumption that the effective potential for a self-dual homogeneous background field has a nontrivial minimum and using just those quark and gluon propagators which exhibit confinement in the sense of entire functions, some successful phenomenological investigations for SU(3) have been carried out [11,12]. Quantitatively, experimental data for the spectrum of light, heavy-light and heavy quarkonium systems can be reproduced to within 10% in this effective description.

In this work we concentrate on the problem of confinement and the effective potential for the SU(2) gauge theory. Our goal is first to describe the confining properties of the self-dual background field in the more familiar terms of the Wilson picture [1]. Second, we seek to show that, even if we cannot prove the existence of a nontrivial minimum in the effective potential for this background field at zero temperature and strong coupling, nonetheless deconfinement at *high temperature* can occur. Namely, we will show that at high temperature the effective potential for a self-dual background field acquires a minimum at zero field value.

In the first instance we illustrate the confining properties of the self-dual homogeneous background by studying the problem of heavy particles and anti-particles in this background field. We thus examine the nonrelativistic limit. We indeed find that a confining potential for static charges emerges: the stationary trajectories of particles and anti-particles in the background field (1) separated by distance $|\vec{X}|$ and held apart for time T are suppressed by a factor

$$\exp\left(-iT\frac{B^2}{32\mu}\vec{X}^2\right),$$

where μ is the reduced mass of the two-particle system. This result differs from that seen in most lattice simulations because of the different long-range properties of the field considered here as compared to those normally implemented in lattice gauge theory. The oscillator binding potential arises here effectively due to an interaction of the charges with the background field, but not by virtue of quantum gluon exchange between these charges. The self-duality and homogeneity of the background field are of crucial importance. The oscillator nonrelativistic potential is not inconsistent with the phenomenology of Regge trajectories in the hadronic spectrum since the latter is a feature of light quark systems. In the approach to the relativistic bound state problem of [12] based on the bosonization of the one-gluon exchange interaction between quark currents in the presence of the vacuum field (1) it is seen that the property that quark and gluon propagators be entire precisely gives rise to Regge behavior in light-quark systems.

It is appropriate to mention here the evident fact that a vacuum field such as Eq. (1) would lead to a breaking of the range of symmetries such as CP, color and O(3). A satisfactory restoration of these symmetries at the hadronic scale assumes the inclusion of domain structures in the vacuum. In a given domain the vacuum field has a specific direction and is either self-dual or anti-self-dual, but this is uncorrelated with the specific realization of Eq. (1) in another domain configuration. The idea of domains in the QCD vacuum was discussed in application to various homogeneous fields [2,13,14]. In the effective meson Lagrangian of [12] this idea was realized as the prescription that different quark loops (namely, those separated by the meson lines) in a diagram must be averaged over different configurations of the vacuum field (1) independently of each other. In the present paper we do not consider this problem, and only wish to note that the above formula for the contribution of stationary trajectories does not depend on directions and is the same for both self-dual and anti-self-dual homogeneous fields.

In the second instance, although we cannot compute the effective potential nonperturbatively, we nonetheless seek to show that at high temperature, where asymptotic freedom should set in, the effective potential does actually acquire a minimum at zero external field consistent with deconfinement. This is not just a trivial consequence of perturbation theory. Lattice simulations have confirmed the picture that high-temperature Yang-Mills theory, though deconfined, shows significant signals of nonperturbative structure [15]. In order to account for some of these properties we have used the recent developments in temporal and axial type gauges at finite extension or temperature by Lenz and co-workers [16,17]. Here a complete gauge fixing of Yang-Mills theory was formulated, accompanied by an integration out of certain zero mode fields which themselves are related intimately to the Polyakov loop order parameter [18] for the confinement-deconfinement phase transition in pure Yang-Mills theory. The integration out of these variables generates for off-diagonal gluon fields a temperature dependent mass $M(T)$ which diverges with increasing temperature, T. In [16] it was checked that, despite the gluon mass, renormalization at the one-loop order was standard, leading to the correct one-loop beta function for SU(2) consistent with gauge invariance. Moreover, this mass was shown to be related to the string constant in a linearly confining potential. Though the actual mechanism for confinement in our study is quite independent of that in [16], this gluon mass generation is of crucial importance for us. It defines a scale $M=M(T)$ in the running coupling constant $g_R(M)$ so that at high temperature the coupling is small. We are thus able to perform a controlled calculation and find that at high temperature the effective potential takes the form

$$U_{\text{eff}}(B^2)=\frac{B^2}{g_R^2(M)}+\frac{29}{525\pi^2}\frac{B^4}{M^4(T)}+\mathcal{O}(B^6/M^8(T))+\mathcal{O}(g_R^2(M))$$

which has a minimum at zero field $B=0$. If non-zero B can generate confinement at zero and low temperatures, then our result shows that deconfinement at high temperature can occur.

In the following section we demonstrate that the self-dual homogeneous field provides simultaneously for the Wilson confinement criterion and the property that propagators of off-diagonal (charged) fields in a self-dual homogeneous gauge field are entire functions. Following that we consider the high temperature limit in the effective potential. The paper concludes with a summary of results and a discussion of the problem of computing the effective potential on the lattice. Much of the detail of explicit calculations is relegated to four appendixes.

II. SELF-DUAL HOMOGENEOUS FIELD AND THE WILSON CRITERION

To illustrate the relationship between confinement and the property that Green's functions in an (anti-)self-dual background field are entire functions it suffices to consider a simple charged scalar field of mass m coupled to the background gauge field $B_\mu = B_{\mu\nu}x_\nu$ defined by Eqs. (1)–(3). The relationship between this and the original Yang-Mills theory can be understood as follows: by assumption, the effective potential for the configuration, Eqs. (1)–(3), exhibits a minimum at $B^2 \neq 0$ which itself is proportional to the fundamental scale of the theory, $\Lambda_{\rm YM}$. By shifting the fields, we study the coupling of small fluctuations to this non-vanishing background. Thus the ϕ-fields are those components of the gluon field which couple in the leading order to the background. We are thus lead to the effective Lagrangian

$$\mathcal{L}(x) = -\phi^\dagger(x)\{-[\partial_\mu + iB_\mu(x)]^2 + m^2\}\phi(x),$$

$$B_\mu = \frac{1}{2}B_{\mu\nu}x_\nu,$$

and work, initially at least, in Euclidean space. Because we seek to approach the Wilson criterion, we consider the analogous Green's function describing a particle-antiparticle loop. Thus the object we are interested in is the four-point function

$$G(x,y|B) = \langle :\phi^\dagger(x)\phi(x)::\phi^\dagger(y)\phi(y):\rangle_B = S(x,y|B)S(y,x|B). \tag{4}$$

The normal ordering is taken to exclude the disconnected diagram. The two-point function $S(x,y|B)$ is itself a solution to the equation

$$\{-[\partial_\mu + iB_\mu(x)]^2 + m^2\}S(x,y|B) = \delta(x-y).$$

The propagator in the external field transforms under translations ($x \to x+a,\ y \to y+a$) as

$$S(x,y|B) = e^{i/2x_\mu B_{\mu\nu}a_\nu/2}S(x+a,y+a|B)e^{-i/2y_\rho B_{\rho\sigma}a_\sigma/2}. \tag{5}$$

The Green's function (4) is gauge invariant and, hence, translation invariant. By means of transformation (5) with $a = -(x+y)/2$ we rewrite the function (4) in a manifestly translation invariant form

$$G(x,y|B) = G(x+a,y+a|B) = G((x-y)/2,(y-x)/2|B) = W(x-y|B).$$

Using the proper time method the propagator can be represented in the form of a path integral over a one-dimensional field ξ [19],

$$S(x,y|B) = e^{i/2x_\mu B_{\mu\nu}y_\nu/2}\int_0^\infty d\alpha\frac{e^{-\alpha/2m^2/2}}{8\pi^2\alpha^2}\int D\xi \times\exp\left\{-\int_0^\alpha d\tau\frac{1}{2}[\dot{\xi}^2(\tau) + i\dot{\xi}_\mu(\tau)B_{\mu\nu}\xi_\nu(\tau)]\right\} \tag{6}$$

with the boundary conditions $\xi(0) = -(x-y)/2,\ \xi(\alpha) = (x-y)/2$, and the normalization

$$\int D\xi\,\exp\left\{-\int_0^\alpha d\tau\frac{\dot{\xi}^2(\tau)}{2}\right\} = \exp\{-(x-y)^2/2\alpha\}.$$

Let us first review the confining properties of these fields in terms of the analytical properties of the propagator. It is instructive to consider first the case of arbitrary constant $B_{\mu\nu}$. Since the vectors $\vec{H}\pm\vec{E}$ ($H_i = \epsilon_{ikj}B_{kj}/2,\ E_i = B_{i4}$) are rotated independently of each other under Euclidean O(4) transformations, the tensor $B_{\mu\nu}$ can be put into the configuration $B_{34} = E,\ B_{12} = H,\ B_{13} = B_{14} = B_{23} = B_{24} = 0$, and $H > 0,\ -H \leqslant E \leqslant H$ [2]. The path integral in Eq. (6) can be easily performed with the result

$$S(x,y|B) = e^{i/2x_\mu B_{\mu\nu}y_\nu/2}\frac{H|E|}{16\pi^2}\int_0^\infty\frac{d\alpha e^{-m^2\alpha}}{\sinh(H\alpha)\sinh(|E|\alpha)} \times\exp\left\{-\frac{1}{4}H[(x_1-y_1)^2 + (x_2-y_2)^2]\coth(H\alpha) - \frac{1}{4}|E|[(x_3-y_3)^2 + (x_4-y_4)^2]\coth(|E|\alpha)\right\}.$$

This leads to a Fourier transform of the translation invariant part:

$$\tilde{S}(p|B) = \int_0^\infty\frac{d\alpha e^{-m^2\alpha}}{\cosh(H\alpha)\cosh(|E|\alpha)} \times\exp\left\{-\frac{1}{H}(p_1^2 + p_2^2)\tanh(H\alpha) - \frac{1}{|E|}(p_3^2 + p_4^2)\tanh(|E|\alpha)\right\}. \tag{7}$$

When E is nonzero this function is finite for any complex $p_1^2 + p_2^2$ and $p_3^2 + p_4^2$ and thus is an entire analytical function. When $E = 0$ this representation exhibits a pole in the physical region $p_4^2 = -(p_3^2 + m^2 + H)$, which corresponds to a free propagation along the third axis with the energy equal to the lowest Landau level of spinless particle. In the (1-2) plane the particle is confined.

Thus, for $E \neq 0$, no physical particle corresponding to the field $\phi(x)$ can appear in the spectrum. The charged particles are, in other words, confined. However, as has been shown in

[2], such an Abelian constant field is unstable against small quantum fluctuations *except in the case that it is self-dual or antiself-dual*: $H=B$, $E=\pm B$. In the following, we concentrate precisely on this configuration. In this case Eq. (7) takes the simple form $[t=\tanh(B\alpha)]$

$$\tilde{S}(p^2|B)=\frac{1}{B}\int_0^1 dt\left(\frac{1-t}{1+t}\right)^{m^2/2B}\exp\left\{-\frac{p^2}{B}t\right\} \tag{8}$$

which represents an entire function in the complex p^2 plane. A special case is that of $m=0$: the Fourier transform of the massless propagator turns out to be

$$\tilde{S}(p^2|B)|_{m=0}=(1-e^{-p^2/B})/p^2. \tag{9}$$

This is manifestly an entire analytical function in the complex p^2-plane: the apparent massless pole at $p^2=0$ simply cancels out, illustrating most cleanly the confinement property. As a matter of fact, entire propagators mean that the quantum field theory is nonlocal. It should be noted here that at the axiomatic level nonlocal quantum field theory was successfully constructed some time ago [20–22]. In particular, causality and unitarity of the S-matrix were proved, a procedure for canonical quantization of nonlocal field theories was constructed and, recently, Froissart type bounds on cross-sections at high energy were obtained [23]. But to summarize this brief review of known results for constant fields, we can say that confinement in the sense of entire propagators is a property of any Euclidean Abelian constant field configuration with non-zero magnetic *and* electric components, but the (anti-)self-dual case is distinguished by being stable against quantum fluctuations.

To see how this property can relate to the Wilson criterion, we now approach the problem of static charges. We consider heavy particles, with $m^2 \gg B$. In this limit Eq. (6) can be represented in the form of a quantum mechanical path integral (see Appendix A)

$$S(x,y|B)\propto e^{-mT}\int D\vec{\eta}\,\exp\left\{-\int_0^T d\beta L(\eta(\beta))\right\}, \tag{10}$$

where

$$L=\frac{m\dot{\vec{\eta}}^2}{2}-\frac{i}{2}\vec{B}[\dot{\vec{\eta}}\times\vec{\eta}]+\frac{1}{2m}(\vec{\eta}\cdot\vec{E})^2,$$

$$T=x_4-y_4,\quad \vec{\eta}(0)=-(\vec{x}-\vec{y})/2,$$

$$\vec{\eta}(T)=(\vec{x}-\vec{y})/2.$$

Here, $E_j=B_{4j}$ is the electric component of the tensor $B_{\mu\nu}$, and $B_i=-\frac{1}{2}\epsilon_{ijk}B_{jk}$ is the magnetic component. We will implement the (anti-)self-duality condition $E_j=\pm B_j$ below. For the present, we insert the representation (10) into Eq. (4), introduce the center of mass coordinates $\vec{R}=(\vec{\eta}_1+\vec{\eta}_2)/2$, $\vec{r}=\vec{\eta}_1-\vec{\eta}_2$, $\vec{R}(0)=\vec{R}(T)=0$, $\vec{r}(0)=-\vec{r}(T)=\vec{y}-\vec{x}$, and integrate out the center of mass coordinate $\vec{R}$. The integral over $\vec{R}$ obviously does not depend on $\vec{x}$ and $\vec{y}$, which is simply a consequence of the translation invariance of the function W. After continuation to physical time $(T=iT,\beta=it)$ the result for W is

$$W(\vec{x}-\vec{y},T|B)\propto e^{-2imT}\int D\vec{r}\,\exp\left\{i\int_0^T dt L(\vec{r}(t))\right\},$$

$$L=\frac{\mu\dot{\vec{r}}^2}{2}+\frac{1}{4}\dot{\vec{r}}[\vec{r}\times\vec{B}]-\frac{1}{8\mu}(\vec{r}\vec{B})^2,$$

where $\mu=m/2$ is the reduced mass of the two-particle system. One sees that the conjugate momentum and the Hamiltonian are

$$\vec{p}=\mu\dot{\vec{r}}+\frac{1}{4}[\vec{r}\times\vec{B}]$$

$$H=\frac{\vec{p}^2}{2\mu}-\frac{1}{4\mu}\vec{p}[\vec{r}\times\vec{B}]+\frac{1}{32\mu}[\vec{r}^2\vec{B}^2+3(\vec{r}\vec{B})^2], \tag{11}$$

and that the function W can be reexpressed as a phase-space functional integral,

$$W(\vec{x}-\vec{y},T|B)$$
$$\propto e^{-2imT}\int D\vec{r}D\vec{p}\,\exp\left\{-i\int_0^T dt[H(\vec{r},\vec{p})-\vec{p}\cdot\dot{\vec{r}}]\right\}. \tag{12}$$

Equations (11) and (12) show that the massive charged particle and anti-particle in the external self-dual field are bounded by an oscillator potential. Now, consistent with Wilson [1], we extract from the path integral the contribution to the phase space of the stationary trajectory $(\vec{p}=0,|\vec{r}|=|\vec{x}-\vec{y}|)$. Equation (11) indicates that this trajectory corresponds to uniform circular movement of the particle-antiparticle pair on a circle with radius $|\vec{x}-\vec{y}|$ in the plane perpendicular to the direction of field $\vec{B}$. We find that the contribution is exponentially suppressed,

$$\exp\left(-iT\frac{B^2}{32\mu}(\vec{x}-\vec{y})^2\right). \tag{13}$$

The Wilson criterion for confinement is indeed satisfied. However, here we have a ‘‘volume law’’ rather than an area law. The relationship between this result and that in standard lattice gauge theory will be discussed in the final section. For now, we stress that the confining potential has appeared due to the background field, and not due to an interaction between particles via gauge boson exchange. Such effects will generate additional potential terms to the Hamiltonian, and will thus affect the energy spectrum of the system. But gauge boson exchange will not change the basic confining properties of the background field.

This picture of bound state formation seems strange at first sight. However, an analogy with the quantum dots (or artificial atoms) of solid state physics can be recognized [24]. Quantum dots are quasi-zero-dimensional electron systems in semiconductor nanostructures in which three-dimensional confinement of small numbers of electrons is achieved by a combination of band offsets and electrostatic means. The simplest model Hamiltonian for the few-electron quantum dot was obtained by solving the Schrödinger and Poisson equations self-consistently within the Hartree approximation [25]. It was found that the oscillator confining potential has nearly circular symmetry. The difference in our case is the origin of the confining potential. The Hamiltonian (11) has appeared due to the background gauge field which may arise in the vacuum as a result of gluon self-interactions.

In QCD, this picture of confinement and bound state formation in the static quark limit will be basically the same. Thus Eqs. (11),(12) give illustrative insight into the basic nature of confinement provided for by the self-dual field. But as Eq. (9) indicates, the significance of the property of entireness of Green's functions as a characterization of confinement applies to dynamical fields and thus is relevant to the fully relativistic bound state spectrum of QCD, the physically relevant problem. Thus the qualitative basis for investigation into the impact of confinement on the relativistic bound state spectrum are equations like Eqs. (8) and (9), as has been carried out in [12]. Here an effective meson theory based on the bosonization of nonlocal quark currents has been developed. The background field has been taken into account both in quark and gluon propagators. Within this effective theory the ground and excited state spectra of light, heavy-light mesons and heavy quarkonia have been calculated, with the only parameters being quark masses, the background field strength and the gauge coupling constant. Agreement with experimental data is obtained to within 10%. Regge behavior within this approach is recovered precisely by the fact that gluon and quark propagators are entire functions. The relationship between this mechanism of confinement and flavor chiral symmetry breaking is analyzed in [11].

Having explored again the confining properties of the self-dual homogeneous background field in QCD, we now turn to the problem of the effective potential for this field at finite temperature and the question of deconfinement.

III. SELF-DUAL FIELD AND FINITE TEMPERATURE

In this section we compute the one-loop effective potential for the self-dual background field at finite temperature in SU(2) Yang-Mills theory. This enables us to study its presence or absence at high temperatures where perturbation theory should become reliable.

Since we are already in Euclidean space in order to define the self-dual field, it is convenient to introduce finite temperature T by working in the imaginary time formalism. The x_4 direction is now a finite interval of length $\beta=1/T$ and boundary conditions must be imposed on the gluon fields, which we shall come to below. We work in a completely gauge-fixed formalism within which we will introduce the external field. At zero temperature, the background gauge is most convenient. However, in the present case, the breaking of manifest Lorentz invariance (by the heat bath) suggests that the temporal (axial) gauge is a natural gauge choice. Specifically, we choose

$$\partial_4 A_4^a(x)=0 \qquad (14)$$

followed by a diagonalization of the surviving zero mode $a_4^a(\vec{x})\tau^a=(1/\beta)\int_0^\beta A_4^a(x)\tau^a dx_4$. This gauge is a special case of the static temporal gauge. Here one encounters the problem of the nontrivial Haar measure in the functional integral quantization of the theory [27]. Concomitantly, the diagonalized variable $a_4^{\rm diag}(\vec{x})$ is compact. The functional integral over this variable is thus non-Gaussian. Progress on the computation of this integral for SU(2) Yang-Mills theory was made recently in [16] wherein, using a lattice regularization, it was shown that the integration out of $a_4^{\rm diag}(\vec{x})$ leads to an effective action for the remaining degrees of freedom. In the absence of external fields, the key features of this effective theory were that off-diagonal, namely charged, components of the gluon fields acquired a temperature dependent mass $M(T)$. Second, the boundary conditions in x_4 of these fields were changed from periodic to antiperiodic.

We rederive this effective theory in Appendix B, and show that the presence of the self-dual background field does not force major modifications. In particular, a rigorous result for the mass, expected to be valid at low but non-zero temperatures [16,17], is reproduced even in the presence of the homogeneous field, namely,

$$M(T)=\sqrt{(\pi^2/3-2)}T. \qquad (15)$$

In [17] it was argued that stability with respect to chromomagnetic fluctuations means that the mass term in the deconfined phase should take the form

$$M(T)=\frac{11}{12\pi}Tg^2(T),\quad T\to\infty, \qquad (16)$$

where $g(T)$ is the perturbative running coupling constant. The important consequence of this result is that at high temperature the mass itself diverges but the ratio $M(T)/T$ vanishes in this limit. This latter property is sufficient to guarantee the recovery of the Stefan-Boltzmann law in the high-temperature regime.

Now we consider the self-dual external field and choose it to point in the same color direction as $a_4^a\tau^a$. It is important to note that this corresponds to a distinct physical choice since gauge freedom does not allow both $B_\mu^a\tau^a$ and $a_4^a\tau^a$ to be simultaneously diagonal.

We come to the question of the gluonic boundary conditions. Here care is required as, unlike the chromomagnetic choice [4,26,17], the self-dual field involves a component pointing in the, now compact, time direction. We are therefore no longer free to impose the usual periodic boundary condition. Instead, the choice must be consistent now with parallel transport in the presence of an external field. Specifically, the appropriate boundary condition in the spatial directions $\vec{x}$ is the usual vanishing one. For the direction x_4 which is finite, $x_4\in[0,\beta]$, one usually chooses periodic boundary conditions in the absence of external fields. This can be represented in the form

$$e^{\beta\partial_4}A_\mu^a(x_4,\vec{x})=A_\mu^a(x_4,\vec{x}). \qquad (17)$$

In the presence of an external field B^a_μ the natural generalization of this for the fluctuating gauge fields Q^a_μ is obtained via parallel transport, namely,

$$(e^{\beta D_4})^{ab} Q^b_\mu(x_4,\vec{x}) = Q^a_\mu(x_4,\vec{x})$$
$$D_4^{ab} = \delta^{ab}\partial_4 - \epsilon^{3ab} B_4. \tag{18}$$

This boundary condition will, in the simplest way, preserve the periodicity of observable, gauge invariant quantities. We shall refer to this position-dependent twisted boundary condition as *quasiperiodic*. When the considerations of Appendix B are carried out and the zero mode of the *fluctuating* gauge field, q_4^{diag}, is integrated out, Eq. (18) becomes a quasi-*anti*periodic boundary condition

$$(e^{\beta D_4})^{ab} Q^b_\mu(x_4,\vec{x}) = -Q^a_\mu(x_4,\vec{x}). \tag{19}$$

To summarize what will be important then for the following calculation, there are two key features: first, that boundary conditions are modified to being quasi-antiperiodic, and second that the off-diagonal gluon components have a temperature dependent mass $M(T)$ which diverges as T increases. It is precisely this which gives us a well-controlled high temperature regime specified by $T \gg \Lambda_{\mathrm{SU}(2)}$ and $B < T^2$.

To calculate the effective potential now, it is convenient to bring the field-strength tensor to the form (taking the field $\vec{B}$ to be directed along the third spatial axis)

$$(B_{\mu\nu})_{\mu,\nu=1,2,3,4} = \begin{pmatrix} 0 & -B & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm B \\ 0 & 0 & \mp B & 0 \end{pmatrix}, \tag{20}$$

where the upper (lower) sign corresponds to the self-dual (anti-self-dual) field. The effective potential is defined in the usual way using the functional integral:

$$\mathcal{Z} = \mathcal{N}\int DQ\,\exp\left\{\int d^4x\,\mathcal{L}_{\mathrm{eff}}[Q_i^A, Q_i^3, B_\mu^3]\right\}$$
$$= \exp\{-\beta V U_{\mathrm{eff}}(B,\beta,g)\} \tag{21}$$

where $i,j,k,l = 1,2,3$, $A,B = 1,2$ denote spatial and off-diagonal field components for gluons respectively, V is the three-dimensional spatial volume, and, as derived in Appendix B, the effective Lagrangian can be written as

$$\mathcal{L}_{\mathrm{eff}}[Q_i^A, Q_i^3, B_\mu^3] = \mathcal{L}_{\mathrm{YM}}[Q_\mu, B_\mu]|_{Q_4=0} - \frac{1}{2}M^2(T) Q_i^A Q_i^A, \tag{22}$$

with $\mathcal{L}_{\mathrm{YM}}$ the standard Yang-Mills action. The functional integral is defined on the space of quasi-antiperiodic fields satisfying Eq. (19). The normalization in Eq. (21) is chosen so that $U_{\mathrm{eff}}(0,\beta,g) = 0$. To the action, a gauge-fixing term involving the neutral zero mode gluons, $Q_i^3(\vec{x}) = (1/\beta)\int_0^\beta dx_4 Q_i^3(x)$, can be added, but which decouples at one-loop after the normalization at zero field, $B=0$. Dropping terms in the Lagrangian higher than quadratic in the fluctuating fields Q_i^A, we can extract from $\mathcal{L}_{\mathrm{YM}}$ the following piece relevant for the one-loop effective potential:

$$\mathcal{L} = -\frac{1}{2} Q_i^A(x)[-(\nabla^2)^{AB}\delta_{ij} + M^2(T)\delta^{AB}\delta_{ij} + (D_iD_j)^{AB}$$
$$+ 2B_{ij}\varepsilon^{3AB}] Q_j^B(x) \tag{23}$$

where $\nabla^2 = D_4^2 + D_i^2$. The quadratic operator in Eq. (23) has zero modes for the case $M=0$ which are called chromons [2]. A correct calculation of the chromon contribution to the effective potential at zero temperature requires an extension of the one-loop approximation: an interaction (or mixing) between zero modes and normal modes has to be taken into account. However, we take the temperature to be sufficiently large so that $M^2(T)$ is correspondingly large compared to B. Large M means that the contribution of chromons is regular at one-loop order so that the mixing between them and normal modes can be neglected. The one-loop effective potential is thus given by

$$U_{\mathrm{eff}} = \frac{B^2}{g_0^2} + \frac{1}{V\beta}\,\mathrm{Tr}\,\ln\left[\frac{-\nabla^2\delta_{ij} + D_iD_j - 2iB_{ij} + M^2(T)\delta_{ij}}{-\partial^2\delta_{kl} + \partial_k\partial_l + M^2(T)\delta_{kl}}\right]. \tag{24}$$

Here B^2/g_0^2 comes from the classical action with g_0 is the bare coupling constant. The effective potential can be rewritten in the form

$$U_{\mathrm{eff}} = \frac{B^2}{g_0^2} - \int_V \frac{d^3x}{V}\int_0^\beta \frac{dx_4}{\beta}$$
$$\times \int_{M^2(T)}^\infty dm^2 [D_{ii}^\beta(x,x|B,m) - D_{ii}^\beta(x,x|0,m)]. \tag{25}$$

We have to calculate the trace of the propagator $D_{ij}^\beta(x,y|B,m)$ satisfying quasi-antiperiodic boundary conditions, Eq. (19):

$$D_{ij}^\beta(x_4+\beta,\vec{x};y|B,m) = -D_{ij}^\beta(x_4,\vec{x};y|B,m)\exp[-i\beta B_4(\vec{x})],$$
$$D_{ij}^\beta(x;y_4+\beta,\vec{y}|B,m) = -D_{ij}^\beta(x;y_4,\vec{y}|B,m)\exp[i\beta B_4(\vec{y})]. \tag{26}$$

This can be implemented by first solving for the Green's function $\Delta_{ij}(x|B,m)$ relevant to the zero temperature or infinite volume and then building up the Green's function satisfying the finite temperature boundary condition via (see also [28] and references therein)

$$D_{ij}^\beta(x;y|B,m) = \sum_{n=-\infty}^{\infty}(-1)^n \Delta_{ij}(x_4 - y_4 + n\beta; \vec{x}-\vec{y}|B,m)$$
$$\times \exp\left(\frac{i}{2}x_\mu B_{\mu\nu} y_\nu + \frac{i}{2} n\beta B_4(\vec{x}+\vec{y})\right). \tag{27}$$

It should be stressed that Eq. (27) implies the existence of an orthogonal complete set of eigenfunctions of the operator ∇^2

satisfying the quasi-antiperiodic boundary conditions. The existence of such a set of functions is demonstrated in Appendix C.

The infinite volume or zero temperature Green's function Δ_{ij} is a solution to the equation

$$[(-\nabla^2+m^2)\delta_{ij}+D_iD_j-2iB_{ij}]\Delta_{jk}(x|B,m)=\delta_{ik}\delta(x). \tag{28}$$

A complete solution of this system of equations is quite involved, but the trace of the propagator is tractable as is shown in Appendix D. One comment is in order though: the summation over n in the space-time trace of Eq. (27) is suppressed in the infinite volume limit $V\to\infty$ due to the electric field component of the self-dual field. So in fact the only relevant contribution of Eq. (27) to the effective potential (25) is that from $n=0$. Using this fact and results (D5) and (D2) derived in Appendix D, we arrive at the relation

$$\Delta_{ii}(0|B,m) = \sum_k \left[F_k(x,x)-\int d^4z F_k(z,x)\tilde{D}_k^2(x)\Delta_4(x,z)\right] + 2B^2\int d^4z\int d^4z' F_0(x,z)\Delta_4(z,z')F_0(z',x), \tag{29}$$

where

$$F_k(x,y)=\exp\left(\frac{i}{2}x_\mu B_{\mu\nu}y_\nu\right)\frac{B^2}{16\pi^2}\int_0^\infty \frac{dr}{\sinh^2(Br)} \times\exp\left[-m^2r+2B\xi_k r-\frac{1}{4}(x-y)^2B\coth(Br)\right],$$

$$\Delta_4(x,z)=\frac{1}{2\sqrt{\pi}}\delta^{(3)}(\vec{x}-\vec{z})\int_0^\infty\frac{dt}{\sqrt{t}} \times\exp\left[-m^2t-\frac{(z_4-x_4)^2}{4t}-\frac{i}{2}(z_4-x_4)B_{4j}x^j\right]. \tag{30}$$

According to Eqs. (25), (29) and (30), the effective potential can be expressed as the combination

$$U_{\rm eff}(B^2)=\frac{B^2}{g_R^2(M)}+U_1(B^2)+U_2(B^2)+U_3(B^2), \tag{31}$$

where

$$U_1=-\frac{B^2}{16\pi^2}\int_0^\infty\frac{ds}{s^3}\exp\left(-\frac{M^2}{B}s\right) \times\left\{\frac{s^2}{\sinh^2 s}[1+2\cosh(2s)]-3s^2-3\right\},$$

$$U_2=-\frac{B^2}{32\pi^2}\int\int_0^\infty\frac{ds\,dt}{s^2(s+t)}\exp\left(-\frac{M^2}{B}(s+t)\right) \times\left\{\frac{s^2}{\sinh^2 s}\,\frac{2\sinh(2s)-\coth s[1+2\cosh(2s)]}{\sqrt{1+t\coth s}} +\frac{3}{\sqrt{s(s+t)}}-\frac{ts^2}{2(s+t)\sqrt{s(s+t)}}\right\},$$

$$U_3=-\frac{B^2}{8\pi^2}\int\int\int_0^\infty\frac{ds\,dr\,dt}{(s+r+t)}\exp\left(-\frac{M^2}{B}(s+r+t)\right) \times\{[\sinh(s+r)]^{-3/2}[\sinh(s+r)+t\cosh(s-r)]^{-1/2} -(s+r)^{-3/2}(s+r+t)^{-1/2}\}. \tag{32}$$

The functions U_1, U_2 and U_3 correspond to ultraviolet finite contributions of the three terms in Eq. (29). The renormalized coupling constant g_R^2 runs with the scale defined by the mass $M=M(T)$ and is

$$\frac{1}{g_R^2}=\frac{1}{g_0^2}\left(1-b_0\int_{s_0}^\infty\frac{ds}{s}e^{-M^2s}\right), \tag{33}$$

where we have used (gauge invariant) Schwinger regularization. The renormalization procedure we use corresponds to the zero momentum subtraction scheme. Taking the parameter $s_0\to0$ generates the ultraviolet divergence. The constant b_0 is nothing but the coefficient of the beta function to lowest order. Its value arises as a sum of the divergent parts of the three terms in Eq. (29) which give for $U_1^{\rm div}$, $U_2^{\rm div}$ and $U_3^{\rm div}$ contributions 3/16, 1/48 and 1/4, respectively, so that

$$b_0=11/24$$

correctly arises. That we get this is another check on the consistency of our formalism. In particular, by renormalizing in this way we have combined the $\mathcal{O}(B^2)$ quantum corrections with the classical term, so that the next corrections begin at $\mathcal{O}(B^4)$.

We thus obtain our final result for the effective potential at high temperature:

$$U_{\rm eff}(B^2)=\frac{B^2}{g_R^2(M)}+\frac{29}{525\pi^2}\frac{B^4}{M^4(T)}+\mathcal{O}(B^6/M^8(T)) +\mathcal{O}(g_R^2(M)). \tag{34}$$

Since as $T\to\infty$, $M(T)\to\infty$, we have $g_R(M)\ll1$, and our calculation is reliable in this regime. So the effective potential acquires a minimum at zero value of the external field. The background field switches off at high temperature, and we can characterize the high temperature phase as exhibiting deconfinement.

IV. DISCUSSION

The central results of this work are expressed in Eqs. (13),(34). From these we understand that if confinement is due to fields in the QCD vacuum which are long-range (homogeneous in our case) and satisfy self-duality or anti-self-duality, then this is in conflict neither with the Wilson criterion for static quark systems nor with the natural expectation that with increasing temperature there is a transition from confinement to deconfinement. It will be immediately noticed that we have not obtained a linear heavy quark potential as has been observed in lattice simulations. The reason for this discrepancy is straightforward: lattice calculations normally implement periodic boundary conditions from the very outset. As shall be repeatedly seen in the following, the existence of fields non-vanishing at infinity entails significant problems for incorporation on the lattice due to the quasi-periodic boundary conditions. It seems that there is strong evidence that lattice calculations of the heavy quark potential have *quite correctly* not seen a quadratic potential because the effects of the vacuum field we consider have not been built in. How to build these effects in is a problem we discuss below.

On the other hand, the self-dual field, at least at the level of the lowest order propagators in this background, does not immediately account for all aspects of confinement: diagonal gluons in SU(2) have poles in the propagator and this is a consequence of the fact that they do not couple directly to the diagonal background configuration. As mentioned in the Introduction, the simple self-dual homogeneous configuration is not the entire story, and there is room for local effects which can complete the picture of confinement. The vacuum field breaks spontaneously CP, color and O(3) symmetries. There is a continuum of degenerate vacua corresponding to different directions of the vacuum field. This implies [29] the existence of soliton-like field configurations under the homogeneous background field, which could play the role of topologically nontrivial local defects in the QCD vacuum such as domain walls. In the absence of explicit solutions we can only speculate on the robustness of our results against inclusion of such effects. But insofar as the confining properties of the self-dual homogeneous field depend only on the strength of the field and not the direction (in real and internal space), it seems plausible that domains distinguished only by changes in direction will not disrupt the confinement we observe.

However, all of this rests on the assumption that at zero and low temperatures the effective potential for this background has a minimum at non-zero field value and there is no substitute for a genuine nonperturbative calculation. The only realistic choice for this is the formalism of lattice QCD. We thus discuss now in some detail the problems to be confronted with setting up the calculation on the lattice and some insights our preliminary investigation into this offers.

The essential question we need to answer is what the contribution of the homogeneous field configuration, Eq. (1), is to the partition function of lattice SU(2) gauge theory:

$$Z=\int_{\mathcal{U}} DU\ \exp\{-S[U]\}. \tag{35}$$

Here, S is now the standard Wilson action and U is shorthand for

$$U_{n,\mu}=\exp\{iaA_\mu(an)\}\in \mathrm{SU}(2), \quad \forall n,\mu, \tag{36}$$

the link variable, and DU is a functional Haar measure. The lattice spacing is a. Link variables are functions of n and are subject to some boundary conditions. They thus belong to some functional space $\mathcal{U}$. Usually, with N representing the size of the lattice in a given direction, periodic boundary conditions

$$U_{n+N,\mu}=U_{n,\mu}$$

are imposed in order to implement the translation invariance of the theory in the thermodynamic limit. However, the field $B_\mu(an)=aB_{\mu\nu}n_\nu$ is evidently not translation-invariant. In principle, there are two ways to proceed, both of which have been used in applications to the Savvidy chromomagnetic background [4]. The first choice is to force the long range modes to be simply periodic on the lattice. This can be done by "quantization" of the field strength [6–8]

$$a^2B_{\mu\nu}=2\pi\frac{b_{\mu\nu}}{N}, \tag{37}$$

where the matrix elements $b_{\mu\nu}$ are integers. This certainly provides for periodicity of the corresponding link variable, but rewriting Eq. (37) as

$$B_{\mu\nu}=2\pi\frac{b_{\mu\nu}}{aL}, \quad L=aN,$$

and going to the thermodynamic ($L\to\infty$) continuum ($a\to 0$) limit one obtains

$$B_{\mu\nu}=2\pi b_{\mu\nu}/C,\ 0\leqslant C\leqslant\infty,$$

so that the field strength is discretized into multiples of $2\pi/C$ *even in the continuum thermodynamic limit*. Moreover, the constant C itself depends on our choice of limiting prescription. These outcomes render this approach rather unappealing. A second approach is to change boundary conditions. Free boundary conditions have been advocated in the approach of the authors of [10] who apply it to the lattice calculation of the effective potential for the chromomagnetic field in three-dimensional SU(2) theory.

In our case, Eq. (18) suggests the following generalization for link variables. We decompose the general field A_μ in Eq. (36) into a long range part B_μ and the fluctuation Q_μ: $A_\mu(an)=Q_\mu(an)+aB_{\mu\nu}n_\nu\tau^af^a/2, f^2=1.$ Quasi-periodic boundary conditions for the fields Q can be generalized from Eq. (18) to all directions now,

$$Q_\mu(a(n+N))=e^{iw(n)}Q_\mu(n)e^{-iw(n)},$$

$$w(n)=a^2N_\alpha B_{\alpha\beta}n_\beta\tau^af^a/2. \tag{38}$$

Thus the following transformation of link variables is generated:

$$U_{n+N,\mu}=e^{iw(n)}U_{n,\mu}e^{-iw(n+\mu)}. \tag{39}$$

This has the structure of a gauge transformation. Thus gauge-invariant quantities such as the action are invariant. An integral in Eq. (35) includes an integration over all possible values of the strength tensor $B_{\mu\nu}$ and, hence, all values of $w(n)$; thus, the boundary conditions (38) and (39) are actually free, consistent with [10].

The most direct step next is to formulate the effective potential as the lattice functional integral

$$\int_{\mathcal{U}_Q} DU \exp\{-S[U\cdot V]\}$$

with V denoting a link variable generated by the background field,

$$V_{n,\mu}=\exp\{ia^2B_{\mu\nu}n_\nu\tau^b f^b/2\},$$

and where $\mathcal{U}_Q$ is now the space of quasiperiodic functions. Here $B_{\mu\nu}$ and f^a are external and particular directions in the color and Euclidean space can be fixed. The actual problem is to find an appropriate representation of the measure of the integral such that the exclusion of the given background field is manifest.

The consequence of the gauge function $w(n)$ in Eqs. (38),(39) being nonzero over the whole lattice is that all degrees of freedom are affected by the gauge transformation. Thus inclusion of covariantly constant field configurations in the space of integration $\mathcal{U}$ in Eq. (35) means actually that the space of allowed gauge functions cannot be restricted to the class of functions with local support. A significant consequence of this is that Elitzur's theorem which forbids spontaneous breakdown of local gauge symmetry [30] does not apply to this situation. This theorem concerns the integral

$$\lim_{J\to 0}\lim_{N\to\infty}\langle F(U)\rangle_{N,J}=\lim_{J\to 0}\lim_{N\to\infty} Z_{N,J}^{-1}\int_{\mathcal{U}} DU\int_{\mathcal{G}} DgF(U^g)\times\exp\{-S[U]+JU^g\}, \quad (40)$$

where $F(U)$ is gauge noninvariant, and J is an external source which breaks gauge invariance. The order of limits is important. The theorem states that if gauge transformations $\mathcal{G}$ are local—namely that they act on a finite (independent of N) number of degrees of freedom—then for sufficiently small sources $\|J\|<\epsilon$ the following inequality holds:

$$|\exp\{JU^g\}-1|\leq\eta(\epsilon), \quad (41)$$

with $\eta(\epsilon)$ being independent of N and vanishing as ϵ goes to zero. Periodicity of the functions in $\mathcal{U}$ is implicit.

As has been argued above, both conditions exclude covariantly constant field configurations which are long range modes that can produce symmetry breaking. Periodic boundary conditions and locality of gauge functions are in conflict with a self-consistent incorporation of these modes in the lattice functional integral. The choice of free boundary conditions for $\mathcal{U}$ and, in particular, the presence in $\mathcal{G}$ of gauge transformations which can act on all degrees of freedom results in nonuniformity in the function η in lattice size N, so that JU becomes an extensive quantity. In view of this, the drastic difference in the results of [10] (some evidence of nontrivial minimum with free boundary conditions and continuous field) and [6,7] (minimum at zero field strength with periodic boundary conditions and "quantized" field strength) seems unsurprising.

It would be instructive to give an example illustrating that inclusion of the homogeneous fields into the lattice integral allows the existence of an order parameter that is not gauge invariant. Let us consider the integral over $\mathcal{U}$ and $\mathcal{G}$ which includes now homogeneous fields and gauge functions of the form (38). If we put

$$F(U^g)=\mathrm{Im}\; U^g_{n,\mu\nu},$$

where $U_{n,\mu\nu}$ is a plaquette variable, and choose the source term in the form

$$\sum_{n,\mu\nu} J_{\mu\nu}\,\mathrm{Tr}\,\mathrm{Im}\,\tau^3 U^g_{n,\mu\nu}, \quad J_{\mu\nu}=\mathrm{const},$$

then the inequality (41) is not uniform in N for all field and gauge functions: if $U_{n,\mu\nu}$ contains the long range fields $a^2B_{\mu\nu}n_\nu f^b\tau^b/2$ and the gauge transformation corresponds to $\omega(n)=a^2N_\alpha B'_{\alpha\beta}n_\beta f'^b\tau^b/2$, then one gets, for the source term,

$$\sum_{n,\mu\nu} J_{\mu\nu}\,\mathrm{Tr}\,\mathrm{Im}\, e^{-i\omega(n)}\tau^3 e^{i\omega(n)}\exp\{-ia^2B_{\mu\nu}f^b\tau^b\}$$

$$=-2\sum_{n,\mu\nu} J_{\mu\nu}\sin(a^2B_{\mu\nu})\times[f^3-2(f'^3f^bf'^b-f^3)\sin^2(a^2N_\alpha B'_{\alpha\beta}n_\beta/2)].$$

Let for simplicity $B'_{13}=B'_{14}=B'_{23}=B'_{24}=0$, $B'_{12}=B'_{34}=B'$, and $N_1=N_2=N_3=N_4=N$. Using the summation formulas

$$\sum_{n=1}^{N}\sin^2(nx)=N/2-\cos(N+1)x\,\sin Nx/2\,\sin x,$$

$$\sum_{n=1}^{N}\cos^2(nx)=N/2+\cos(N+1)x\,\sin Nx/2\,\sin x,$$

$$\sum_{n=1}^{N}\sin(nx)=\sin\frac{N+1}{2}x\,\sin\frac{Nx}{2}\,\mathrm{cosec}\,\frac{x}{2},$$

one gets, in the limit $N\to\infty$,

$$\sum_{n_1,n_2,n_3,n_4=0}^{N}\sin^2[a^2B'N(n_1-n_2+n_3-n_4)/2]$$

$$=\frac{N}{2}\sum_{n_2,n_3,n_4=0}^{N}\{\sin^2[a^2B'N(n_3-n_2-n_4)/2]+\cos^2[a^2B'N(n_3-n_2-n_4)/2]\}+O(N^3)$$

$$=N^4/2+O(N^3).$$

Thus for the source term we arrive at the result

$$-N^4(4f^3-2\,f'^3f^bf'^b)\sum_{\mu\nu} J_{\mu\nu}\,\sin(a^2B_{\mu\nu})+O(N^3),$$

which shows that the gauge dependent part of the source term JU is an extensive quantity, and the order of limits $J\to 0$ and $N\to\infty$ cannot be interchanged.

It should be stressed that this example in no way violates Elitzur's theorem, but just underlines that its conditions are too restrictive for a self-consistent incorporation of homogeneous field configurations into the lattice functional integral (as mentioned also in the last reference of [30]).

We repeat that the picture of confinement with a self-dual homogeneous field can become reliable only with the inclusion of domain structures in the vacuum such that the symmetries broken by this field are restored at the hadronic level. The boundaries of the domains should be describable by some solitonic classical configurations. As far as we are aware, appropriate solutions are unknown. It thus remains a problem to verify our considerations of the Wilson criterion and Elitzur's theorem in the presence of domains, though we have given plausibility arguments why our results might be unaffected. The conclusion is thus that there are two interesting unsolved problems to be confronted which may be significant for understanding QCD vacuum structure: a calculation of the effective potential for the field, Eq. (1), in the strong coupling limit, and a search for topologically nontrivial classical configurations in the background of a homogeneous self-dual field.

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APPENDIX A: HEAVY MASS LIMIT

Below we derive Eq. (10) starting with Eq. (6). In the integrand of Eq. (6) let us integrate over ξ_4,

$$\int D\xi_4\,\exp\left\{-\int_0^\alpha d\tau\left[\frac{\dot\xi_4^2(\tau)}{2}+i\dot\xi_4(\tau)[\vec E\vec\xi(\tau)]\right]\right\}$$
$$=e^{-T^2/2\alpha}\,\exp\left\{-\int_0^\alpha d\tau\frac{1}{2}[\vec E\vec\xi(\tau)]^2\right\}$$

where $T=x_4-y_4$ and we have changed variables

$$\alpha=T/ms,\quad \tau=\beta/ms,\quad \vec\xi=\frac{\vec\eta}{\sqrt{ms}}.$$

Then the propagator takes the form

$$S(x,y|B)\propto\int_0^\infty ds e^{-mT/2(s+1/s)/2}\int D\vec\eta$$
$$\times\exp\left\{-\int_0^T d\beta\left[\frac{\dot{\vec\eta}^2}{2}-\frac{i}{2ms}\,\vec B[\dot{\vec\eta}\times\vec\eta]+\frac{1}{2(ms)^2}(\vec E\vec\eta)^2\right]\right\},$$

where we have omitted the phase factor and a constant in front of the integral. One can see that for $T\to\infty$ or more precisely for

$$\frac{|\vec x-\vec y|}{T}\ll 1$$

and $|\vec B|\sim|\vec E|\ll m^2$ the integral over s can be evaluated using a saddle-point approximation. The saddle-point is $s=1$; hence we arrive at

$$S(\vec x,\vec y,T|B)\propto e^{-mT}\int D\vec\eta\,\exp\left\{-\int_0^T d\beta\left[\frac{m}{2}\,\dot{\vec\eta}^2-\frac{i}{2}\,\vec B[\dot{\vec\eta}\times\vec\eta]+\frac{1}{2m}(\vec E\vec\eta)^2\right]\right\}. \tag{A1}$$

Inserting this result into Eq. (A1), we arrive at the representation Eq. (10).

At zero field $\vec E=\vec B=0$ we arrive at the correct nonrelativistic limit

$$S(\vec x,\vec y,T|B)\propto\exp\left\{-mT-\frac{m}{2}\frac{|\vec x-\vec y|^2}{T}\right\}$$
$$=\exp\left\{-\left(m+\frac{mv^2}{2}\right)T\right\}, \tag{A2}$$

$$v=\frac{|\vec x-\vec y|}{T}\ll 1, \tag{A3}$$

where the energy of the particle is

$$E=m+\frac{mv^2}{2}.$$

APPENDIX B: STATIC TEMPORAL GAUGE AND SELF-DUAL FIELDS

Consider a fixed direction in color space such that the phase of the gauge invariant Polyakov loop, $\mathrm{Tr\,P}\exp(ig\int_0^\beta dx_4A_4^a\tau^a/2)$, is in the τ^3 direction. The background self-dual field is also chosen to point in the same color direction. Gauge transformations now used to fix the gauge further may not change color axes.

An arbitrary gauge transformation on the gauge field $\tilde A_\mu=\tilde A_\mu^a\tau^a/2$ takes the form

$$\tilde{A}_\mu \to U\tilde{A}_\mu U^\dagger + \frac{i}{g} U\partial_\mu U^\dagger \equiv A_\mu . \quad \text{(B1)}$$

Under the decomposition of $\tilde{A}_\mu$ into background $\tilde{B}_\mu$ and fluctuating $\tilde{Q}_\mu$ parts we choose the separate pieces to transform under ''quantum'' gauge transformations:

$$B_\mu = U\tilde{B}_\mu U^\dagger$$

$$Q_\mu = U\tilde{Q}_\mu U^\dagger + \frac{i}{g} U\partial_\mu U^\dagger . \quad \text{(B2)}$$

Since we have specified the direction of the background, $B_\mu = \tilde{B}_\mu$.

We use the type (B2) to achieve a fixing of the gauge on $\tilde{Q}_\mu$. These fields satisfy quasiperiodic boundary conditions, Eq. (18),

$$(e^{\beta D_4})^{ab}\tilde{Q}^b_\mu(x_4,\vec{x}) = \tilde{Q}^a_\mu(x_4,\vec{x})$$

$$D_4^{ab} = \delta^{ab}\partial_4 - \epsilon^{3ab}B_4 \quad \text{(B3)}$$

which must not be changed under gauge-fixing. Thus, U must be quasiperiodic. The temporal (axial) gauge $Q_4 = 0$ cannot be achieved with such a group element. The static temporal gauge (14) followed by diagonalization of the zero mode is however possible. Explicitly the U bringing an arbitrary $\tilde{Q}_\mu$ into this gauge is

$$U[\tilde{Q}] = e^{-igx_4 q_4 \tau^3/2}\ \mathrm{P}\exp\left(ig\int_0^{x_4} dt\,\tilde{Q}_4(t,\vec{x})\right) \quad \text{(B4)}$$

with $q_4 = (1/\beta)\int_0^\beta dt Q_4$. Since $\tilde{Q}_\mu$ is quasiperiodic, so too is U. Thus Q_μ are also quasiperiodic: $e^{\beta D_4}Q_\mu(x_4,\vec{x}) = Q_\mu(x_4,\vec{x})$.

There are still two classes of gauge symmetry remaining: (1) Transformations $V(x) = \exp[ig\omega^3(x)\tau^3/2]$ with $\omega^3(x)$ strictly periodic in x_4 can be fixed by introducing an extra Lorentz-Coulomb gauge condition on the zero modes of the remaining neutral fields. As it is Abelian, this gauge fixing does not introduce Faddeev-Popov ghosts. The one-loop effective potential considered in the main body receives no contributions from these neutral fields. (2) Transformations $W(x) = \exp[(2in\pi x_4/\beta)\tau^3/2]$ cause a shift of $2n\pi/g\beta$ in the zero mode of the fluctuating field q_4. These are relevant for what follows.

We now implement these considerations in the quantum theory, using the Faddeev-Popov trick in the functional integral. The Faddeev-Popov determinant is defined by

$$\Delta_F^{-1}[Q] = \int \mathcal{D}g\,\delta[F[Q^g]] \quad \text{(B5)}$$

with Q^g all configurations related by gauge transformations, Eq. (B2), to a representative configuration Q which satisfies $F[Q]=0$. The functional F that selects this gauge is independent of the background field B_μ (unlike in the background field gauge). Inserting unity into the generating functional we obtain

$$\mathcal{Z}[B^2] = \mathcal{N}\int \mathcal{D}Q_\mu \mathcal{D}g\,\delta[F[Q^g]]\Delta_F[Q]\exp(-S[B+Q]). \quad \text{(B6)}$$

Now we perform a gauge transformation of type (B2) to bring $Q^g \to Q$. The measure and determinant are invariant, as stated. To recover the same action, a corresponding rotation of the background field must take place, as in Eq. (B2). Because $\mathcal{Z}$ is ultimately a functional only of the gauge invariant combination B^2, we recover again the same $\mathcal{Z}$. We may now absorb the integration $\int\mathcal{D}g$ into the normalization in the usual way and obtain

$$\mathcal{Z}[B^2] = \mathcal{N}\int \mathcal{D}Q_\mu\,\delta[F[Q]]\Delta_F[Q]\exp(-S[B+Q]). \quad \text{(B7)}$$

The form of the determinant for the static temporal gauget is well known. Using a lattice regularization for space $\vec{x}$, it can be written as

$$\Delta_F[Q] = \prod_{\vec{x}} \sin^2[g\beta q_4(\vec{x})/2]. \quad \text{(B8)}$$

The Jacobian is independent of the background component B_4. The zeros of the Jacobian indicate the appropriate range of integration for q_4, which in turn is seen in the symmetry under transformations W at the classical level. The appropriate functional integral after implementing the delta functional is then

$$\mathcal{Z}[B^2] = \mathcal{N}\int \mathcal{D}Q_i(x)\int_0^{\pi/g\beta}\mathcal{D}q_4(\vec{x})\sin^2[g\beta q_4(\vec{x})/2] \times\exp(-S[B+Q]_{F[Q]=0}). \quad \text{(B9)}$$

This is still symmetric under the V transformation. Performing the Faddeev-Popov trick again with the Lorentz-Coulomb gauge condition on the neutral zero mode fields enables factoring out of this redundant gauge volume. With the normalization $\mathcal{N}$ being done at $B=0$, the neutral field contributions to this functional integral will anyway cancel out. The W symmetry is however fixed by restriction of the range of integration of q_4.

We now show how the integral over q_4 can be performed. We consider

$$\int \mathcal{D}Q_i \mathcal{D}q_4\,\sin^2(g\beta q_4/2)\exp\left\{-S + \int d^4x JQ\right\}. \quad \text{(B10)}$$

We integrate over q_4 in a diagrammatic expansion in order to derive an effective theory for Q_i. Thus the fields Q_i and B_μ appear only in external lines of the diagrams. This is a strong restriction on the allowed diagrams. The zero mode q_4 couples only to charged gluons via the three- and four-point vertices, and never to itself. The three-gluon vertex leads to a

$q_4 \to$two-charged-gluon vertex, while the four-point vertex gives $(q_4)^2 \to$charged-anticharged spatial gluons. This means that the perturbation series in this functional integral stops at one loop. Only three topologically distinct classes of diagrams are present and of these only one specific diagram gives a non-vanishing contribution as the lattice spacing is taken zero after subtraction out of a pure infinite constant. This leaves a mass term in the off-diagonal fields. Its form is determined by the propagator for two fluctuating fields $q_4(\vec{x})$, namely,

$$\langle 0|T(q_4(\vec{x})q_4(\vec{y}))|0\rangle = \frac{\int \mathcal{D}Q_i \int \mathcal{D}q_4(\vec{z})\sin^2[g\beta q_4(\vec{z})/2]e^{-S_0[B+Q]}q_4(\vec{x})q_4(\vec{y})}{\int \mathcal{D}Q_i \int \mathcal{D}q_4(\vec{z})\sin^2[g\beta q_4(\vec{z})/2]e^{-S_0[B+Q]}} \quad \text{(B11)}$$

with S_0 representing the action with the couplings between q_4 and the remaining fields dropped. The unregularized form for this is

$$S_0[B+Q] = \frac{L}{2}\int d^3x[B_4(\vec{x})+q_4(\vec{x})]\vec{\nabla}^2[B_4(\vec{x})+q_4(\vec{x})]$$
$$= \frac{\beta}{2}\int d^3x q_4(\vec{x})\vec{\nabla}^2 q_4(\vec{x})$$
$$= S_0[Q] \quad \text{(B12)}$$

because $\vec{\nabla}^2 B_4 = \partial_i\partial_i B_{4j}x_j = \partial_i B_{4i} = 0$ and $q_4(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$. We see that the arguments of [16] go through unchanged: we shift by half the fundamental domain $q_4(\vec{x}) \to q_4(\vec{x}) - \pi/g\beta = q_4'(\vec{x})$ so that the Jacobian becomes a cosine-squared and the boundary conditions in x_4 of the charged gluons acquire an extra term: they go from being quasi-periodic to quasi-*anti*periodic. Next we discretize $\vec{x} = l\vec{n}$, with directional unit vectors $\hat{e}$, and dimensionless field $\varphi_{\vec{n}} \equiv g\beta q_4'(\vec{x})$. We obtain, for the action,

$$S_0^{(l)}[Q] = \frac{l}{2g^2\beta}\sum_{\vec{n}}\sum_{\hat{e}} \varphi_{\vec{n}}[\varphi_{\vec{n}+2\hat{e}} - 2\varphi_{\vec{n}+\hat{e}} + \varphi_{\vec{n}}]. \quad \text{(B13)}$$

Thus the weight factor appearing in the functional integral is

$$e^{-S_0^{(l)}[Q]} = \sum_{r=0}^{\infty} \mathcal{C}_r\left(\frac{l}{g^2\beta}\right)^r. \quad \text{(B14)}$$

The functional integration in Eq. (B11) can be done explicitly:

$$\langle 0|T(q_4(\vec{x})q_4(\vec{y}))|0\rangle = \frac{1}{4g^2\beta^2}\left(\frac{\pi^2}{3}-2\right)\delta_{\vec{m}^x,\vec{m}^y} + \mathcal{O}\left(\frac{l}{g^4\beta^3}\right). \quad \text{(B15)}$$

So also in the presence of the self-dual background field, the correlator of the fluctuating part of Polyakov loops is ultralocal, being proportional to $\delta^{(3)}(\vec{x}-\vec{y})$ in the continuum limit. The result, Eq. (B15), guarantees that the mass term for the charged gluons is as derived by [16], namely Eq. (15).

We are thus led to an effective action after integration out of q_4 which contains charged gluon fields $Q_i^{1,2}$ with a mass diverging with increasing temperature but, in the presence of the self-dual field, quasi-antiperiodic boundary conditions. Moreover, the background field component $B_4(x)$ is still present in the action in the usual terms where the original A_4 was located, but the field q_4 has been successfully integrated out.

APPENDIX C: ORTHOGONALITY AND COMPLETENESS RELATIONS

In this section we derive the orthogonality and completeness relations for the eigenfunctions of the Laplace operator in the presence of the self-dual homogeneous field and the formulas for the propagator subject to quasi-antiperiodic boundary conditions. Let us recall first the solution to the eigenvalue problem at zero temperature [2,5]:

$$-\nabla^2(x)\psi(x) = \lambda\psi(x),$$

$$\nabla^2(x) = [\partial_\mu - iB_\mu(x)]^2, \quad B_\mu = \frac{1}{2}B_{\mu\nu}x_\nu,$$

in the space of functions vanishing at infinity. The operator $-\nabla^2$ can be represented in the form

$$-\nabla^2 = 2B(a_\mu^\dagger Q_{\mu\nu}^- a_\nu + 1)$$

$$a_\mu = \frac{1}{\sqrt{B}}\left(\frac{1}{2}Bx_\mu + \partial_\mu\right), \quad a_\mu^\dagger = \frac{1}{\sqrt{B}}\left(\frac{1}{2}Bx_\mu - \partial_\mu\right),$$
$$[a_\mu, a_\nu^\dagger] = \delta_{\mu\nu},$$

$$Q_{\mu\nu}^\pm = (\delta_{\mu\nu} \pm ib_{\mu\nu})/2,$$
$$Q^\pm Q^\pm = Q^\pm, \quad Q^\mp Q^\pm = 0, \quad b_{\mu\nu} = B_{\mu\nu}/B.$$

The matrix $(ib_{\mu\nu})$ can be diagonalized by means of an appropriate unitary transformation U:

$$U^\dagger ibU = \text{diag}(1,-1,1,-1), \quad U^\dagger a = \alpha, \quad [\alpha_\mu, \alpha_\nu^\dagger] = \delta_{\mu\nu},$$

$$\alpha_1 = (a_1 + ia_2)/\sqrt{2}, \quad \alpha_2 = (a_1 - ia_2)/\sqrt{2},$$
$$\alpha_3 = (a_3 + ia_4)/\sqrt{2}, \quad \alpha_4 = (a_3 - ia_4)/\sqrt{2}.$$

The eigenvalue problem then takes the form

$$2B(\alpha_2^\dagger\alpha_2 + \alpha_4^\dagger\alpha_4 + 1)\psi(x) = \lambda\psi(x),$$

with the solution

$$\lambda_{k_1k_2} = 2B(k_1 + k_2 + 1),$$

$$\psi_{k_1k_2k_3k_4} = \frac{1}{\sqrt{k_1!k_2!}}(\alpha_2^\dagger)^{k_1}(\alpha_4^\dagger)^{k_2}\psi_{00k_3k_4}(x)$$

$$\psi_{00k_3k_4}=\frac{B}{2\pi}\frac{1}{\sqrt{k_3!k_4!}}(\alpha_1^\dagger)^{k_3}(\alpha_3^\dagger)^{k_4}\exp\left(-\frac{1}{4}Bx^2\right). \tag{C1}$$

The orthogonality and completeness relations have the following form:

$$\int_{-\infty}^{\infty}d^4x\,\psi^\dagger_{k_1k_2k_3k_4}(x)\,\psi_{k_1'k_2'k_3'k_4'}(x)=\delta_{k_1k_1'}\delta_{k_2k_2'}\delta_{k_3k_3'}\delta_{k_4k_4'},$$

$$\sum_{k_1k_2k_3k_4}\psi^\dagger_{k_1k_2k_3k_4}(x)\,\psi_{k_1k_2k_3k_4}(y)=\delta(x-y). \tag{C2}$$

One sees from Eqs. (C1) that the spectrum of ∇^2 is infinitely degenerate which is a consequence of the homogeneity of the background field. To proceed further, it is advantageous to introduce the eigenfunctions $\phi_{k_1k_2}(x,y)$:

$$\begin{aligned}\phi_{k_1k_2}(x,y)=&\frac{B^2}{4\pi^2}\sum_{k_3k_4}\sqrt{\frac{(B/2)^{k_3+k_4}}{k_3!k_4!}}\\&\times(y_1-iy_2)^{k_3}(y_3-iy_4)^{k_4}\\&\times\exp\left(-\frac{1}{4}By^2\right)\psi_{k_1k_2k_3k_4}(x)\\=&\frac{B^2}{4\pi^2}\frac{1}{\sqrt{k_1!k_2!}}(\alpha_2^\dagger)^{k_1}(\alpha_4^\dagger)^{k_2}\\&\times\exp\left(-\frac{1}{4}B(x-y)^2+\frac{i}{2}x_\mu B_{\mu\nu}y_\nu\right).\end{aligned}$$

Now the degeneracy is parametrized by the continuous variable y. The function $\phi_{00}(x,y)$ can be seen as a matrix element of the projector onto the subspace spanned by the lowest mode $(k_1=k_2=0)$.

Making use of Eqs. (C2), we arrive at the following equations for the case of infinite β (zero temperature):

$$-\nabla^2(x)\Delta(x,y)=\delta(x-y),$$

$$-\nabla^2(x)\,\phi_{k_1k_2}(x,y)=\lambda_{k_1k_2}\phi_{k_1k_2}(x,y),$$

$$\phi_{k_1k_2}(x,y)=\phi_{k_1k_2}(x-y)\exp\left\{\frac{i}{2}x_\mu B_{\mu\nu}y_\nu\right\},$$

$$\sum_{k_1k_2}\int_{-\infty}^{\infty}d^4y\,\phi^\dagger_{k_1k_2}(x,y)\,\phi_{k_1k_2}(z,y)=\delta(z-x),$$

$$\int_{-\infty}^{\infty}d^4x\,\phi^\dagger_{k_1k_2}(x,y)\,\phi_{k_1'k_2'}(x,z)=\delta_{k_1k_1'}\delta_{k_2k_2'}\phi_{00}(y,z).$$

Together with Eq. (C3), these define respectively the Green's function and eigenfunctions for the operator ∇^2 as well as completeness and orthogonality relations for the eigenfunctions. The propagator can be decomposed into a sum over projectors onto the subspaces corresponding to the different eigen-numbers

$$\Delta(z,x)=\sum_{k_1k_2}\frac{\mathcal{P}_{k_1k_2}(z,x)}{\lambda^2_{k_1k_2}},$$

$$\begin{aligned}\mathcal{P}_{k_1k_2}(z,x)&=\int_{-\infty}^{\infty}d^4y\,\phi_{k_1k_2}(z,y)\,\phi^\dagger_{k_1k_2}(x,y)\\&=\mathcal{P}_{k_1k_2}(z-x)e^{iz_\mu B_{\mu\nu}x_\nu/2}.\end{aligned} \tag{C3}$$

The completeness, for instance, is derived in the following way:

$$\begin{aligned}&\int_{-\infty}^{\infty}d^4y\sum_{k_1k_2}\phi^\dagger_{k_1k_2}(x,y)\,\phi_{k_1k_2}(z,y)\\&=\frac{B^2}{4\pi^2}\sum_{k_1k_2k_3k_4}\psi^\dagger_{k_1k_2k_3k_4}(x)\,\psi_{k_1k_2k_3k_4}(z)\\&\quad\times\int_{-\infty}^{\infty}d^4y\,\frac{(B/2)^{k_3+k_4}}{k_3!k_4!}(y_1^2+y_2^2)^{k_3}(y_3^2+y_4^2)^{k_4}e^{-By^2/2}\\&=\sum_{k_1k_2k_3k_4}\psi^\dagger_{k_1k_2k_3k_4}(x)\,\psi_{k_1k_2k_3k_4}(z)=\delta(z-x).\end{aligned} \tag{C4}$$

Then at finite β the function

$$\begin{aligned}D^\beta(x,y)=&\sum_{n=-\infty}^{\infty}(-1)^n\Delta(x_4-y_4+n\beta;\vec{x}-\vec{y})\\&\times\exp\left\{\frac{i}{2}x_\mu B_{\mu\nu}y_\nu+in\beta B_4(\vec{x}+\vec{y})\right\}\end{aligned} \tag{C5}$$

is a solution to the equation

$$-\nabla^2(x)D^\beta(x,y)=\delta_\beta(x,y),$$

$$\begin{aligned}\delta_\beta(x,y)=&\sum_{n=-\infty}^{\infty}(-1)^n\delta(x_4-y_4+n\beta)\,\delta(\vec{x}-\vec{y})\\&\times\exp\left\{\frac{i}{2}x_\mu B_{\mu\nu}y_\nu+in\beta B_4(\vec{x}+\vec{y})\right\},\end{aligned}$$

satisfying the boundary conditions

$$D^\beta(x_4+\beta,\vec{x};y)=-D^\beta(x_4,\vec{x};y)\exp\{-i\beta B_4(\vec{x})\},$$

$$D^\beta(x;y_4+\beta,\vec{y})=-D^\beta(x:y_4,\vec{y})\exp\{i\beta B_4(\vec{y})\}.$$

Here $\delta_\beta(x,y)$ is the δ-function on the linear space Φ_β of functions $f_\beta(x)$ obeying the boundary condition

$$f_\beta(x_4+\beta,\vec{x})=-f_\beta(x_4,\vec{x})\exp\{-i\beta B_4(\vec{x})\}.$$

One can check that completeness is satisfied:

$$\int_{-\infty}^{\infty}d^3y\int_0^\beta dy_4\,\delta_\beta(x,y)f_\beta(y)=f_\beta(x).$$

Moreover, the functions

$$\phi^{\beta}_{k_1k_2}(x,y)=\sum_{n=-\infty}^{\infty}(-1)^n\phi_{k_1k_2}(x_4-y_4+n\beta,\vec{x}-\vec{y})\times\exp\left\{\frac{i}{2}x_\mu B_{\mu\nu}y_\nu+in\beta B_4(\vec{x}+\vec{y})\right\}$$

being the eigenfunctions of $\nabla^2(x)$,

$$-\nabla^2(x)\phi^{\beta}_{k_1k_2}(x,y)=\lambda_{k_1k_2}\phi^{\beta}_{k_1k_2}(x,y),$$

and satisfying the boundary condition

$$\phi^{\beta}_{k_1k_2}(x_4+\beta,\vec{x};y)=-\phi^{\beta}_{k_1k_2}(x_4,\vec{x})\exp\{-i\beta B_4(\vec{x})\},$$

then give

$$\sum_{k_1k_2=0}^{\infty}\int_{-\infty}^{+\infty}d^3y\int_0^{\beta}dy_4\phi^{\beta\dagger}_{k_1k_2}(x,y)\phi^{\beta}_{k_1k_2}(z,y)=\delta_\beta(z,x),$$

$$\int_{-\infty}^{+\infty}d^3x\int_0^{\beta}dx_4\phi^{\beta\dagger}_{k_1k_2}(x,y)\phi^{\beta}_{l_1l_2}(x,z)=\delta_{k_1l_1}\delta_{k_2l_2}\phi^{\beta}_{00}(y,z)$$

so that they define an orthogonal complete basis for the space Φ_β. The propagator D^β can be decomposed over projectors

$$D^{\beta}(z,x)=\sum_{k_1k_2}\frac{\mathcal{P}^{\beta}_{k_1k_2}(z,x)}{\lambda^2_{k_1k_2}},$$

$$P^{\beta}_{k_1k_2}(z,x)=\int_{-\infty}^{\infty}d^3y\int_0^{\beta}dy_4\phi_{k_1k_2}(z,y)\phi^{\dagger}_{k_1k_2}(x,y)$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n\mathcal{P}_{k_1k_2}(z_4-x_4+n\beta,\vec{z}-\vec{x})\exp\left\{\frac{i}{2}z_\mu B_{\mu\nu}x_\nu+im\beta B_4(\vec{z}+\vec{x})\right\}. \tag{C6}$$

Taking into account the representation of the zero temperature propagator in terms of the projector operators we get Eq. (C5).

Let us show how the completeness of the set can be derived. We have to evaluate the integral

$$\sum_{k_1k_2=0}^{\infty}\int_{-\infty}^{+\infty}d^3y\int_0^{\beta}dy_4\phi^{\beta\dagger}_{k_1k_2}(x,y)\phi^{\beta}_{k_1k_2}(z,y)=\sum_{k_1k_2=0}^{\infty}\int_{-\infty}^{+\infty}d^3y\int_0^{\beta}dy_4$$
$$\times\sum_{n,m=-\infty}^{\infty}(-1)^{n+m}\phi^{\dagger}_{k_1k_2}(x_4-y_4+n\beta,\vec{x}-\vec{y})\phi_{k_1k_2}(z_4-y_4+m\beta,\vec{z}-\vec{y})$$
$$\times\exp\left\{-\frac{i}{2}x_\mu B_{\mu\nu}y_\nu+\frac{i}{2}z_\mu B_{\mu\nu}y_\nu-in\beta B_4(\vec{x}+\vec{y})+im\beta B_4(\vec{z}+\vec{y})\right\}.$$

After the change of integration variable $y'_4=y_4-n\beta$ we get, for the right hand side of this equation,

$$\sum_{k_1k_2=0}^{\infty}\int_{-\infty}^{+\infty}d^3y\sum_{n,m=-\infty}^{\infty}(-1)^{n+m}\int_{-n\beta}^{(1-n)\beta}dy'_4\phi^{\dagger}_{k_1k_2}(x_4-y'_4,\vec{x}-\vec{y})\phi_{k_1k_2}(z_4-y'_4+(m-n)\beta,\vec{z}-\vec{y})$$
$$\times\exp\left\{-\frac{i}{2}x_\mu B_{\mu\nu}y'_\nu+\frac{i}{2}z_\mu B_{\mu\nu}y'_\nu+i(m-n)\beta B_4(\vec{z}+\vec{y})\right\}.$$

Finally, shifting the variable $m'=m-n$ in the sum and denoting $z'_4=z_4+m'\beta$ we arrive at

$$\sum_{k_1k_2=0}^{\infty}\int_{-\infty}^{+\infty}d^3y\int_0^{\beta}dy_4\phi^{\beta\dagger}_{k_1k_2}(x,y)\phi^{\beta}_{k_1k_2}(z,y)=\sum_{m'=-\infty}^{\infty}(-1)^{m'}\sum_{k_1k_2=0}^{\infty}$$
$$\times\int_{-\infty}^{+\infty}d^3y\int_{-\infty}^{\infty}dy_4\phi^{\dagger}_{k_1k_2}(x,y)\phi_{k_1k_2}(z',y)\exp\{im'\beta B_4(\vec{z})\}$$
$$=\sum_{m=-\infty}^{\infty}(-1)^m\delta(z_4-x_4+m\beta)\delta(\vec{z}-\vec{x})\exp\left\{\frac{i}{2}z_\mu B_{\mu\nu}x_\nu+im\beta B_4(\vec{z}+\vec{x})\right\}$$
$$=\delta_\beta(z,x).$$

The orthogonality and decomposition of the propagator over the projectors can be obtained in a similar way.

APPENDIX D: GLUON PROPAGATOR IN A SELF-DUAL BACKGROUND FIELD IN THE TEMPORAL GAUGE

We start with the function

$$D_{ij}(x,y|B,m)=\Delta_{ij}(x_4-y_4;\vec{x}-\vec{y}|B,m)\exp\left(\frac{i}{2}x_\mu B_{\mu\nu}y_\nu\right)$$

which is a solution of the equation

$$[(-\nabla^2+M^2)\delta_{ij}+D_iD_j-2iB_{ij}]D_{jk}(x,y|B,m)=\delta_{ik}\delta(x-y)$$

in the limit of infinite β. The function Δ is then a solution to the equation

$$[(-\nabla^2+M^2)\delta_{ij}+D_iD_j-2iB_{ij}]\Delta_{jk}(x|B,m)=\delta_{ik}\delta(x).$$

The matrix iB_{ij} can be diagonalized by an appropriate unitary transformation U, so that we arrive at

$$[(-\nabla^2+m^2)\delta_{rs}+\tilde{D}_r\tilde{D}'_s-2B\delta_{rs}\xi_s]\tilde{\Delta}_{st}(x|B,m)=\delta_{rt}\delta(x) \tag{D1}$$

with $r,s,t\in\{0,1,-1\}$ and $\xi_s=s$ the gluon spin projections onto the third spatial axis. Moreover,

$$\nabla^2=D_4^2+\tilde{D}'_r\tilde{D}_r,\quad \tilde{D}_s=U^\dagger_{sj}D_j,\quad \tilde{D}'_s=D_jU_{js}.$$

We next decompose the propagator as

$$\tilde{\Delta}_{rs}=\delta_{rs}F_s+\tilde{D}_r\tilde{D}'_sH_s+i\delta_{r0}\tilde{D}'_sL+i\delta_{s0}\tilde{D}_rN+\delta_{r0}\delta_{s0}P. \tag{D2}$$

Using this and the relations

$$[D_4^2,\tilde{D}_s]=2iB\delta_{s0}D_4,\quad [\tilde{D}_r,\tilde{D}'_s\tilde{D}_s]=2B\delta_{rt}\xi_t\tilde{D}_t,$$

$$[D_4,\tilde{D}_s]=[D_4,\tilde{D}'_s]=iB\delta_{s0},\quad [\tilde{D}_0,\tilde{D}_s]=0,$$

$$\sum_j[-\tilde{D}^2\delta_{rs}+\tilde{D}_r\tilde{D}'_s-2B\delta_{rs}\xi_s]\tilde{D}_s\equiv 0,$$

we can rewrite Eq. (D1) as a system of differential equations for the functions F, H, L, N, P:

$$(-\nabla^2-2B\xi_s+m^2)F_s(x)=\delta(x),$$

$$i\tilde{D}_0L_s(x)+(-D_4^2+m^2)H_s(x)+F_s(x)=0,$$

$$-2iBD_4H_s(x)+(-\nabla^2-2B\xi_s+m^2)L_s(x)=0,$$

$$-i\tilde{D}'_0P_s(x)-2iBD_4H_s(x)-(-D_4^2+m^2)N_s(x)=0,$$

$$2iBD_4[L_s(x)-N_s(x)]+2B^2H_s(x)+(-\nabla^2+m^2)P_s(x)=0. \tag{D3}$$

We now show that, for the calculation of the space-time trace of the propagator, we only need to know the functions H, L, N and P in the neighborhood of $x=0$. Consider the integral

$$\mathcal{T}_{jk}=\int_V d^3x\int_0^\beta dx_4D^\beta_{jk}(x,x|B,m),$$

which is contained in Eq. (24). With L_3 the length of the third space direction and using Eqs. (27),(20) we get

$$\mathcal{T}_{jk}=\int_V d^3x\int_0^\beta dx_4\sum_{n=-\infty}^{\infty}(-1)^n\Delta_{jk}(n\beta;0|B,m)\times\exp\left(\mp\frac{i}{2}n\beta Bx_3\right)$$

$$=V\beta\left[\Delta_{jk}(0;0|B,m)+4\lim_{L_3\to\infty}\times\sum_{n=1}^{\infty}(-1)^n\Delta_{jk}(n\beta;0|B,m)\times\frac{\sin\left(\frac{1}{2}nBL_3\beta\right)}{nBL_3\beta}\right]$$

$$=V\beta[\Delta_{jk}(0;0|B,m)+O(L_3^{-1}\beta^{-1})]. \tag{D4}$$

Equation (D4) leads to the result that the terms with $n\neq 0$ do not contribute to the effective potential. Further calculations can be simplified due to this property of the external field (1). The solution of Eqs. (D3) depends on x^2, and the first order derivatives are proportional to $x_0=U^\dagger_{0j}x_j$ or x_4. If we need the propagator only for $x\to 0$, we can omit all terms which contain the first order derivatives. Thus, in the limit $x\to 0$, we have to solve the equations

$$(-\nabla^2-2B\xi_s+m^2)F_s(x)=\delta(x),$$

$$(-D_4^2+m^2)H_s(x)+F_s(x)=0,$$

$$(-\nabla^2-2B\xi_s+m^2)L_s(x)=0,$$

$$(-D_4^2+m^2)N_s(x)=0,$$

$$(-\nabla^2+m^2)P_s(x)+2B^2H_s(x)=0.$$

One can check the positive-definiteness of the spectrum of the operators $(-D_4^2+m^2)$ and $(-\nabla^2-2B\xi_s+m^2)$ in the space of functions vanishing at infinity. This means that $L_s(x)\to 0$ and $N_s(x)\to 0$ for $x\to 0$. Finally one gets, for small x^2,

$$F_s(x)=(-\nabla^2-2B\xi_s+m^2)^{-1}\delta(x),$$

$$H_s(x)=-(-D_4^2+m^2)^{-1}F_s(x),$$

$$P_s(x) = -2B^2(-\nabla^2 + m^2)^{-1} H_s(x),$$
$$L_s(x) = 0, \quad N_s(x) = 0. \tag{D5}$$

Using Eqs. (D5) and (D2), we arrive at the relation (29) given in the main body of the paper. We have only to insert the delta-function into the equations for H_s and P_s and to represent these functions as convolutions of the propagators F_k, F_0 and Δ_4, where the latter is the Green's function corresponding to $(-D_4^2 + m^2)$. These lead to the expressions in Eq. (30).

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