Quasi-Bernoulli statistics of string-induced fluctuations of cosmic microwave background radiation

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A new kind of statistical distribution, a quasi-Bernoulli distribution, is suggested to describe the non-Gaussian string-induced perturbations of the cosmic microwave background radiation. Good agreement between predictions based on this statistics and data of numerical and laboratory simulations using some simplified models is established, and a possible relation of this statistic to a large-scale galaxy distribution is briefly discussed. [S0556-2821(98)02222-X]

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I. INTRODUCTION

Most inflationary models produce random Gaussian density fluctuations, and therefore one might hope to find a non-Gaussian signature to distinguish cosmic string models from inflationary models. The non-Gaussian spatial distribution of the cosmic string networks could be a basis for such expectations [1,2]. However, if many different strings contribute significantly to each resolution element of the temperature pattern, then the conditions for the central limit theorem are satisfied and the temperature pattern is close to Gaussian. Indeed, recent calculations (including all the relevant physical effects) show that realistic string models can produce Gaussian-like fluctuations (at least at low-order statistical analysis) [3,4,5]. Moreover, it is shown in Ref. [6] that a most favored cosmic string model is unlikely to produce a significant increase in the sheetlike nature of the matter distribution beyond that which occurs in Gaussian models (with the same power spectrum) due to the formation of Zeldovich pancakes. Thus the non-Gaussian features of the realistic string models are rather hidden for the observers. There are even indications of a serious problem for the defect theories reconciling the amplitude of large-scale cosmic microwave background (CMB) anisotropies with that in the matter distribution in a large flat universe [7,8].

On the other hand, it is suggested in some recent papers [9,10,11] that it is still possible to find a non-Gaussian signature of string-induced fluctuations even in Gaussian-like CMB radiation if one uses a *high*-order statistical analysis (see also [6] and Sec. IV). Interval of scales, chosen for the analysis, could be also crucial for the problem [7,10,12]. To understand this phenomenon it could be useful to study some string models (even if they are *less* realistic than those mentioned above) in which this phenomenon could be seen more clear (i.e., already at a low-order statistical analysis). One such model is suggested in [13,14] and is studied in a recent paper [15] by application of the multifractal analysis. This approach is suitable just for high-order statistical analysis, but as it is shown below the multifractal analysis of this specific string model gives good agreement between low- and high-order non-Gaussian statistics. A new type of non-Gaussian statistic, a quasi-Bernoulli distribution, which appears in this model as a general consequence of a morphological monofractality-multifractality phase transition, may turn out to be a statistical distribution relevant to the string-induced fluctuations of CMB radiation. To emphasize the general character of this non-Gaussian distribution, we compare this result with results of a laboratory simulation of the string-induced fluctuations of temperature gradients. This simulation is based on a two-dimensional temperature gradient map generated by a stochastic flow of mercury in a very strong external magnetic field [16]. It is known (see, for instance, [17] and references therein) that in such a flow the stochastic motion becomes two dimensional and mixing has been generated by a network of stochastically moving strings (vortex filaments), which are orthogonal to the plane of motion and are parallel to the external magnetic field. Our calculations show that the laboratory data are also in good agreement with the quasi-Bernoulli distribution and that parameters of the laboratory-obtained generalized dimension spectrum are close to those obtained from the numerical simulation performed in [15] (cf. Figs. 1 and 2).

Finally, we discuss briefly an idea of Ref. [6] that the string-dominated statistics could be also useful for an interpretation of the observed large-scale galaxy distributions.

II. QUASI-BERNOULLI DISTRIBUTION

Let us start from some standard definitions. Given a scalar signal $\Delta T/T$ along the scan line, the generalized box-counting partition function is defined as

$$Z_q(l)=\sum_{i=1}^{N}[\mu_i(l)]^q, \tag{1}$$

where N is the minimum number of one-dimensional segments with dimensionless length l which are needed to cover the set (the total length of the scan is $l_m=1$) and $\mu_i(l)$ is a measure on the line determined as follows:

$$\mu_i(l)=\sum_{j=i-Ml/2}^{i+Ml/2}\mu(j) \tag{2}$$

and

$$\mu(j)=\left[\frac{\Delta T}{T}(j)-\frac{\Delta T}{T}(j+1)\right]^2.$$

In this definition M is the total number of points in the data set, and $(i-ML/2)$ and $(i+Ml/2)$ are the lower and upper edges of the ith segment with Ml points, centered on the ith point of the scan. At the minimal resolution $l_{\min}=1/M$, one has $\mu_i(l_{\min})=\mu(i)$. The generalized dimensions D_q are then formally defined by

$$D_q=\frac{1}{(q-1)}\lim_{l\to 0}\frac{\log \Sigma_{i=1}^{N}[\mu_i(l)]^q}{\log l}. \tag{3}$$

For a continuous signal along line $N\sim l^{-1}$, i.e., $D_0=1$.

The above definition implies a scaling behavior of the partition function $Z_q(l)$ for small l, i.e.,

$$Z_q(l)\propto l^{\tau_q}, \tag{4}$$

where

$$\tau_q=D_q(q-1). \tag{5}$$

Then, if one uses standard averaging, one obtains

$$\langle \mu^q\rangle=\frac{\Sigma_{i=1}^{N}[\mu_i(l)]^q}{N}\propto l^{(\tau_q+1)}. \tag{6}$$

Let us define

$$\overline{\mu}_i=\mu_i/\max_i\{\mu_i\}. \tag{7}$$

Then

$$\langle \overline{\mu}^p\rangle=\frac{1}{N}\sum_i \overline{\mu}_i^{\,p}. \tag{8}$$

The simplest structure that can be used for fractal description is a system for which $\overline{\mu}_i$ can take only two values 0 and 1. It follows from Eqs. (7) and (8) that, for such a system (with $p>0$),

$$\langle \overline{\mu}^p\rangle=\langle \overline{\mu}\rangle, \tag{9}$$

and fluctuations in this system can be identified as Bernoulli fluctuations [18]. It is clear that the Bernoulli fluctuations can be *monofractal* only.

Generalization Eq. (9) in the form of a generalized scaling

$$\langle \overline{\mu}^p\rangle\sim\langle \overline{\mu}\rangle^{f(p)} \tag{10}$$

can be used to describe more complex (multifractal) systems. We use invariance of the generalized scaling (10) with dimension transform [19]

$$\overline{\mu}_i\to\overline{\mu}_i^{\,\lambda} \tag{11}$$

to find $f(p)$. This invariance means that

$$\langle(\overline{\mu}^{\lambda})^p\rangle\sim\langle(\overline{\mu}^{\lambda})\rangle^{f(p)} \tag{12}$$

for all positive λ. Then, it follows from Eqs. (10) and (12) that

$$\langle(\overline{\mu})^{\lambda p}\rangle\sim\langle\overline{\mu}\rangle^{f(\lambda p)}\sim\langle\overline{\mu}\rangle^{f(\lambda)f(p)}. \tag{13}$$

Hence

$$f(\lambda p)=f(\lambda)f(p). \tag{14}$$

A general solution of the functional equation (14) is

$$f(p)=p^{\gamma}, \tag{15}$$

where γ is a positive number. This relationship can be considered as a generalization of the Havlin-Bunde multifractal hypothesis [20]. It should be noted that the case $\gamma=1$ corresponds to Gauss fluctuations [21]. We, however, shall consider limit $\gamma\to 0$ (i.e., the transition to the Bernoulli fluctuations). This transition is nontrivial. Indeed, let us consider the generalized scaling

$$F_{qm}\sim F_{km}^{\alpha(q,k,m)}, \tag{16}$$

where

$$F_{qm}=\langle\overline{\mu^q}\rangle/\langle\overline{\mu^m}\rangle. \tag{17}$$

Substituting Eq. (10) into Eqs. (16), (17) and using Eq. (15), we obtain

$$\alpha(q,k,m)=\frac{q^{\gamma}-m^{\gamma}}{k^{\gamma}-m^{\gamma}}.$$

Hence

$$\lim_{\gamma\to 0}\alpha(q,k,m)=\frac{\ln(q/m)}{\ln(k/m)}. \tag{18}$$

If there is ordinary scaling

$$\langle\overline{\mu}^p\rangle\sim(l/L)^{\zeta_p}, \tag{19}$$

then

$$\alpha(q,k,m)=\frac{\zeta_q-\zeta_m}{\zeta_k-\zeta_m}. \tag{20}$$

From a comparison of Eqs. (18) and (20), we obtain, at the limit $\gamma\to 0$,

$$\frac{\zeta_q-\zeta_m}{\zeta_k-\zeta_m}=\frac{\ln(q/m)}{\ln(k/m)}. \tag{21}$$

A general solution of the functional equation (21) is

$$\zeta_q=a+c\ \ln q, \tag{22}$$

where a and c are some constants.

If we use the relationship

$$\max_i\{\mu_i\}\sim(l/L)^{D_\infty} \tag{23}$$

(see, for instance, [22]), then it follows from Eqs. (3), (7) and (19), (22), (23) that

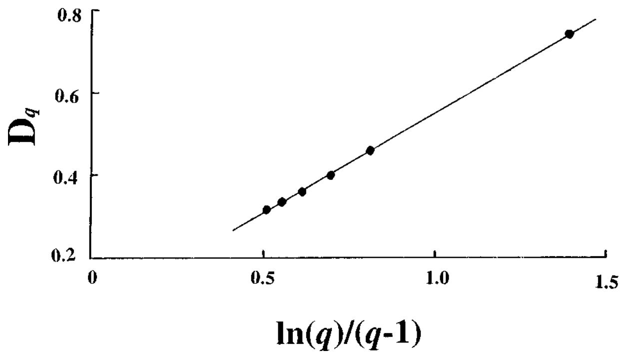

FIG. 1. Spectrum of generalized dimensions D_q (dots) obtained in a numerical simulation of the string-induced CMBR perturbations (adapted from [15]). The straight line is drawn for comparison with the quasi-Bernoulli representation (24).

$$D_q = D_\infty + c\,\frac{\ln q}{(q-1)} \tag{24}$$

for the multifractal Bernoulli fluctuations (i.e., for the fluctuations which appear at the limit $\gamma \to 0$).

From Eqs. (10), (19), and (22) we can find $f(p)$ corresponding to the multifractal Bernoulli fluctuations

$$f(p) = 1 + \frac{c}{a}\ln p, \tag{25}$$

where $a = d - D_\infty$. One can see that for finite c the dimension invariance is broken at the limit $\gamma \to 0$.

Let us find the characteristic function of the multifractal Bernoulli distribution. It is known that the characteristic function $\chi(\lambda)$ can be represented by the following series (see, for instance, [18]):

$$\chi(\lambda) = \sum_{p=0}^{\infty} \frac{(i\lambda)^p}{p!}\langle \overline{\mu^p} \rangle. \tag{26}$$

Then, using Eqs. (10) and (25), we obtain, from Eq. (26),

$$\chi(\lambda) = 1 + \langle \overline{\mu} \rangle \sum_{p=1}^{\infty} \frac{(i\lambda)^p}{p!} p^{\beta}, \tag{27}$$

where

$$\beta = \frac{c}{(d - D_\infty)} \ln \langle \overline{\mu} \rangle. \tag{28}$$

The characteristic function (27) gives a complete description of the multifractal Bernoulli distribution and when the $c = 0$ distribution (27) and (28) coincides with the simple Bernoulli distribution [18]. The multifractality-monofractality phase transition (with $\gamma \to 0$) corresponds to a gap from $c = 0$ to a finite nonzero value of c. If we use a thermodynamic interpretation of the multifractality represented in Ref. [23], then the constant c can be interpreted as the multifractal specific heat of the system. The gap of the multifractal specific heat at the multifractality-monofractality transition (i.e., with $\gamma \to 0$) allows us consider this transition as a thermodynamic phase transition [24].

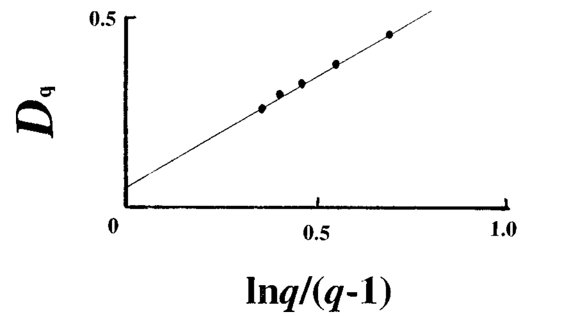

FIG. 2. Spectrum of generalized dimensions D_q (dots) obtained in a laboratory simulation of the string-induced perturbations of temperature gradients field [25]. The straight line is drawn for comparison with the quasi-Bernoulli representation (24).

III. SIMULATIONS

The authors of a recent paper [15] used the temperature maps produced by Bouchet, Bennet, and Stebbins [13,14] to simulate a CMBR anisotropy experiment in the presence of a network of cosmic strings. Figure 1 (adapted from [15]) shows a spectrum of generalized dimensions obtained at this simulation. The axes in this figure are chosen for comparison with the quasi-Bernoulli representation (24) [the straight line is drawn to indicate agreement between the data (dots) and representation (24)]. One can calculate the multifractal specific heat $c \simeq 0.5$ and $D_\infty \simeq 0.07$ from this figure.

We can also use data obtained in a laboratory simulation of string-induced temperature mixing in a stochastic flow of mercury (a well electricity conducting liquid) in a very strong ($B = 0.9$ T) transversal magnetic field [16]. It is known (see, for instance, [17]) that such mixing has been generated by a network of strings (vortex filaments) aligned along the magnetic field. The results of this laboratory simulation [25] are shown in Fig. 2. The straight line in this figure indicates agreement between the data (dots) and the quasi-

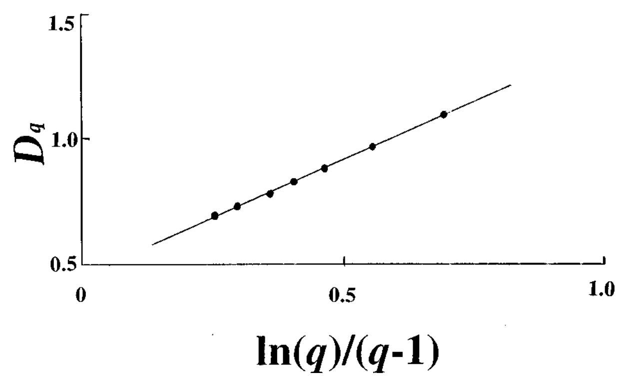

FIG. 3. Spectrum of generalized dimensions D_q (dots) obtained from a version of the CfA catalog of Huchra's compilation of redshifts [26]. The straight line is drawn for comparison with the quasi-Bernoulli representation (24).

Bernoulli representation (24). One can also calculate the multifractal specific heat $c \simeq 0.6$ and $D_\infty \simeq 0.05$ from this figure. These parameters are close to those calculated from Fig. 1.

Thus both the numerical and laboratory simulations give an indication that the quasi-Bernoulli statistics could be used to describe the string-induced perturbations of the temperature fields. An *a priori* reason for the applicability of the quasi-Bernoulli distribution to these perturbations could be related to the existence of the morphological phase transition at a generation stage.

IV. DISCUSSION

To understand the nature of the Gaussian-like behavior of the realistic CMB distribution (see the Introduction), it is useful to note that cosmic string models are unlikely to produce a significant increase in the sheetlike nature of the matter distribution beyond that which occurs in Gaussian models with the same power spectrum (see [6] for details), although it is suggested in Ref. [6] that string-dominated statistics could be useful for the interpretation of the observed galaxy distributions, so that it is interesting to compare the quasi-Bernoulli statistic with some observed galaxy distributions. In Fig. 3 we show a multifractal spectrum (dots) calculated in Ref. [26] using a version of the CfA catalog of Huchra's compilation of redshifts. The axes in this figure are chosen for comparison with the quasi-Bernoulli distribution (24) (straight line).

It is clear that examples given in this Brief Report have an illustrative character only. We hope, however, that the general nature of the quasi-Bernoulli distribution (related to the morphological phase transition) could be a reason to use this distribution for the cases where one can expect a transition from Gaussian to a new statistic just at high-order statistical analysis. Since analysis of realistic models indicates that in the case of the CMB radiation we are dealing with this situation (see the Introduction), then to check the applicability of the quasi-Bernoulli distribution to more realistic data sets (for high-order moments) seems to be an interesting problem for future investigations.

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