# Nonequilibrium dynamics of fermions in a spatially homogeneous scalar background field

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We consider the time evolution of systems in which a spatially homogeneous scalar field is coupled to fermions. The quantum back reaction is taken into account in the one-loop approximation. We set up the basic equations and their renormalization in a form suitable for numerical computations. The initial singularities appearing in the renormalized equations are removed by a Bogoliubov transformation. The equations are then generalized to those in a spatially flat Friedmann-Robertson-Walker universe. We have implemented the Minkowski space equations numerically and present results for the time evolution with various parameter sets. We find that fermion fluctuations are not in general as ineffective as previously assumed, but show interesting features which should be studied further. In an especially interesting example we find that fermionic fluctuations can ''catalyze'' the evolution of bosonic fluctuations. [S0556-2821(98)04422-1]

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## I. INTRODUCTION

In recent years the study of nonequilibrium dynamics in quantum field theory has received much attention. Quantum fields out of equilibrium can play an essential role, e.g., in cosmology [1–16], in the quark-gluon plasma [17–25], or during phase transitions in solid state physics [26].

While the formalism of nonequilibrium dynamics in quantum field theory was established long ago [27,28], real time simulations for realistic systems have been developed only recently. Numerical simulations of the evolution equations have been studied by various authors. The general features are similar: The quantum back reaction cannot be described by Markovian friction terms. The relaxation of the classical field amplitude either shuts off or is powerlike; the quantum ensembles generated are characterized by parametric resonance bands, the full development of the resonance being suppressed by the quantum back reaction. The early- and late-time behavior has been analyzed analytically for the one-loop and large-$N$ approximations [29,30].

Most of the numerical simulations have been performed with a scalar classical field coupled to scalar quantum fields. The quantum back reaction of fermion fields on a classical scalar field has received little attention up to now. In [31] it has been stated, on the basis of some numerical evidence, that at least for large field amplitudes Pauli blocking would make fermions ineffective for dissipation and damping of the classical field. In another recent publication [32] the leading orders in perturbation theory have been evaluated, so far without numerical computations. The interaction between a classical electric field and fermionic fluctuations has been considered in [17] (see also [19]) as a model for $q\bar{q}$ production in the quark-gluon plasma. There the evolution of the system in quantum field theory was compared to the evolution using the Boltzmann-Vlasov equation.

In this paper we reconsider fermionic fluctuations coupled to a scalar field, with respect to both formal aspects and numerical simulations. We use a formalism developed by us recently [33,34] to formulate the renormalized equations of motion in a form which is suitable for numerical computation but satisfies at the same time the usual requirements of renormalized quantum field theory. The renormalization scheme is covariant and independent of the initial conditions. After renormalization we find the equations to be singular at $t=0$, a phenomenon known as Stueckelberg singularities [35,36]. As in the scalar case we have studied previously [37], these singularities can be removed by a Bogoliubov transformation of the initial fermionic quantum state. We then generalize the equations to those in a flat Friedmann-Robertson–Walker (FRW) universe. We finally formulate the linearized equations of motion, in order to be able to compare with the full quantum evolution. We have implemented numerically the formalism developed in this article. Numerical results for various parameter sets and various aspects of these results will be presented and discussed in Sec. VIII and in the conclusions.

The plan of this article is as follows. In Sec. II we formulate the basic relations and the equation of motion for a scalar field coupled to fermions, in Sec. III we present the energy momentum tensor and discuss the fermion number, the renormalization of the equation of motion and of the energy momentum tensor is developed in Sec. IV, in Sec. V we derive the Bogoliubov transformation which removes the initial singularities, the extension to a conformally flat Friedmann-Robertson-Walker is derived in Sec. VI, in Sec. VII we discuss the linearized equations of motion, the results of our numerical simulations are discussed in Sec. VIII, and conclusions are given in Sec. IX.

## II. BASIC RELATIONS AND EQUATION OF MOTION

We study a model consisting of a scalar ''inflaton'' field $\Phi$ coupled to a spin-1/2 field $\psi$ by a Yukawa interaction. We do not introduce a genuine mass term for the fermion field; it acquires a time-dependent mass via the Yukawa coupling. We introduce a $\lambda\Phi^4$ self-interaction, but do not consider the

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case of spontaneous symmetry breaking and do not consider the quantum fluctuations of the scalar field itself. This framework is sufficiently general for discussing renormalization and the typical effects introduced by the back reaction of the fermion fields. It can be easily generalized to include a fermion mass, bosonic fluctuations, and spontaneous symmetry breaking. The Lagrangian density is given by

$$\mathcal{L}=\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}M^2\Phi^2-\frac{\lambda}{4!}\Phi^4+\overline{\psi}(i\partial\!\!\!/-g\Phi)\psi \quad , \tag{2.1}$$

where $M$ is the mass of the scalar field, and $g$ is the Yukawa coupling.

We split the field $\Phi$ into its expectation value $\phi$ and the quantum fluctuations $\eta$:

$$\Phi(\mathbf{x},t)=\phi(t)+\eta(\mathbf{x},t), \tag{2.2}$$

with

$$\phi(t)=\langle\Phi(\mathbf{x},t)\rangle=\frac{\mathrm{Tr}\,\Phi\rho(t)}{\mathrm{Tr}\,\rho(t)}. \tag{2.3}$$

The scalar fluctuations have already been analyzed in [33,34]. The equations for the system we consider here, with the back reaction of the fermion field, have been derived in [31] using the Schwinger–Keldysh formalism [27,28] and the tadpole method [38]. We do not repeat it here. The equation of motion for the classical field is given by

$$\ddot{\phi}(t)+M^2\phi(t)+\frac{\lambda}{6}\phi^3(t)+\frac{\lambda}{2}\langle\eta^2\rangle+g\langle\overline{\psi}\psi\rangle=0 \quad . \tag{2.4}$$

Here $\langle\overline{\psi}\psi\rangle$ and $\langle\eta^2\rangle$ are the expectation values of the quantum fluctuations of the fermions and the scalar field, respectively. They are related to closed-time-path (CTP) Green functions. They can be expressed by mode functions which satisfy the linearized equations of motion in the background field and initial conditions at some time $t_0$. In the following we choose $t_0=0$. The scalar back reaction via $\langle\eta^2\rangle$ has been calculated previously by various groups within different approximation schemes, among them the large-$N$, Hartree, or one-loop approximation. As we have explained above, here we are merely interested in the fermionic back reaction and do not include the scalar one, except for some of the numerical examples in Sec. VIII.

The fermion field $\psi$ satisfies the Dirac equation

$$[i\partial_t-\mathcal{H}(t)]\psi(t,\mathbf{x})=0, \tag{2.5}$$

where the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H}(t)=-i\boldsymbol{\alpha}\nabla+m(t)\beta \quad . \tag{2.6}$$

The term $m(t)=g\phi(t)$ is the time-dependent fermion mass. We expand the fermion field in terms of the spinor solutions of the Dirac equation

$$\psi(t,\mathbf{x})=\sum_s\int\frac{d^3p}{(2\pi)^3 2E_0}[b_{\mathbf{p},s}U_{\mathbf{p},s}(t)+d^\dagger_{-\mathbf{p},s}V_{-\mathbf{p},s}(t)]e^{+i\mathbf{p}\cdot\mathbf{x}}, \tag{2.7}$$

with the time-independent creation and annhiliation operators whose mass is determined by the initial state. The creation and annihilation operators satisfy the usual anticommutation relations

$$\{b_{\mathbf{p},s},b^\dagger_{\mathbf{p}',s'}\}=2E_0(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{ss'}\,, \tag{2.8}$$

$$\{d_{\mathbf{p},s},d^\dagger_{\mathbf{p}',s'}\}=2E_0(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{ss'}\,. \tag{2.9}$$

For the positive and negative energy solutions we make the ansatz

$$U_{\mathbf{p},s}(t)=N_0[i\partial_t+\mathcal{H}_{\mathbf{p}}(t)]f_p(t)\begin{pmatrix}\chi_s\\0\end{pmatrix} \tag{2.10}$$

and

$$V_{\mathbf{p},s}(t)=N_0[i\partial_t+\mathcal{H}_{-\mathbf{p}}(t)]g_p(t)\begin{pmatrix}0\\\chi_s\end{pmatrix}, \tag{2.11}$$

with the Fourier-transformed Hamiltonian

$$\mathcal{H}_{\mathbf{p}}(t)=\boldsymbol{\alpha}\mathbf{p}+m(t)\beta \quad . \tag{2.12}$$

For the two-spinors $\chi_s$ we use helicity eigenstates, i.e.,

$$\hat{\boldsymbol{p}}\boldsymbol{\sigma}\chi_\pm=\pm\chi_\pm\,. \tag{2.13}$$

The mode functions $f_p$ and $g_p$ depend only on $p=|\mathbf{p}|$; they obey the second-order differential equations

$$\left[\frac{d^2}{dt^2}-i\dot{m}(t)+p^2+m^2(t)\right]f_p(t)=0 \quad , \tag{2.14}$$

$$\left[\frac{d^2}{dt^2}+i\dot{m}(t)+p^2+m^2(t)\right]g_p(t)=0 \quad . \tag{2.15}$$

The initial state for the fermion field is usually specified as a vacuum or thermal equilibrium state obtained by fixing the classical field $\phi$, and thereby the fermion mass, to some value $\phi_0$ for $t\leqslant 0$. The spinor solutions are then identical with the usual free field solutions of the Dirac equation with constant mass $m_0=m(0)=g\phi(0)$. Therefore, the mode functions, which would be plane waves for $t\leqslant 0$, satisfy the initial conditions

$$f_p(0)=1,\dot{f}_p(0)=-iE_0, \tag{2.16}$$

$$g_p(0)=1,\dot{g}_p(0)=iE_0 \quad . \tag{2.17}$$

For the spinors $U$ and $V$ we use the usual free field normalization conditions

$$\overline{U}_{\mathbf{p},s}(0)U_{\mathbf{p},s}(0)=-\overline{V}_{\mathbf{p},s}(0)V_{\mathbf{p},s}(0)=2m_0, \tag{2.18}$$

$$U^{\dagger}_{\mathbf{p},s}(0)U_{\mathbf{p},s}(0)=V^{\dagger}_{\mathbf{p},s}(0)V_{\mathbf{p},s}(0)=2E_0 \quad ; \tag{2.19}$$

we will also need the orthogonality relation

$$U^{\dagger}_{\mathbf{p},s}(0)V_{-\mathbf{p},s}(0)=0 \quad . \tag{2.20}$$

$E_0(p)$ denotes the mode energy in the initial state:

$$E_0^2=\mathbf{p}^2+m_0^2. \tag{2.21}$$

For the normalization constant we find

$$N_0=[E_0+m_0]^{-1/2} \quad . \tag{2.22}$$

For $t>0$ Eqs. (2.14), (2.15), and (2.16) imply that

$$f_p(t)=g_p^*(t). \tag{2.23}$$

Since the time evolution of the spinors $U_{\mathbf{p},s}(t)$ and $V_{-\mathbf{p},s}(t)$ is induced by the Hermitian operator $\mathcal{H}_{\mathbf{p}}$, their normalization and orthogonality relations (2.18) and (2.20) are conserved. This implies a useful relation for the mode functions [31]:

$$|\dot f_p(t)|^2-im(t)[f_p(t)\dot f_p^*(t)-\dot f_p(t)f_p^*(t)] +[p^2+m^2(t)]|f_p(t)|^2=2E_0(E_0+m_0), \tag{2.24}$$

which takes the role of the Wronskian. Using these mode functions $\langle\bar\psi\psi\rangle$ can be calculated once the initial state is specified.[1] If we use the Fock space vacuum defined by $b_{\mathbf{p},s}|0\rangle=0$ and $d_{\mathbf{p},s}|0\rangle=0$, we get

$$\langle\bar\psi\psi\rangle=\sum_s\int\frac{d^3p}{(2\pi)^3 2E_0}\bar V_{-\mathbf{p},s}(t)V_{-\mathbf{p},s}(t) =-2\int\frac{d^3p}{(2\pi)^3 2E_0}\left\{2E_0-\frac{2\mathbf{p}^2}{E_0+m_0}|f_p|^2\right\} \quad . \tag{2.25}$$

If we use a thermal density matrix defined in terms of the Fock space states, one obtains

$$\langle\bar\psi\psi\rangle=-2\int\frac{d^3p}{(2\pi)^3 2E_0} \times\tanh\left(\frac{E_0}{2T}\right)\left\{2E_0-\frac{2\mathbf{p}^2}{E_0+m_0}|f_p|^2\right\}; \tag{2.26}$$

the integration measure in the momentum integrals is modified accordingly in the expressions for the energy-momentum tensor.

We will denote $\langle\bar\psi\psi\rangle$ as the fluctuation integral

$$\mathcal{F}(t)=\langle\bar\psi(t)\psi(t)\rangle. \tag{2.27}$$

The fluctuation integral is divergent and has to be regularized and renormalized. This will be done in Sec. IV.

## III. ENERGY-MOMENTUM TENSOR AND PARTICLE NUMBER

The energy-momentum tensor of a Dirac field with mass $m=m(t)$ is given by

$$T_{\mu\nu}=\bar\psi\left(\frac{1}{2}i\gamma_\mu\overleftrightarrow{\partial}_\nu+mg_{\mu\nu}\right)\psi. \tag{3.1}$$

The expectation value of the energy-momentum tensor taken with the initial density matrix is spatially homogeneous and has therefore the form $T_{\mu\nu}=\mathrm{diag}(\mathcal{E},\mathcal{P})$. In addition to the contribution of the Dirac field it contains the classical contribution of the scalar field. The energy density $\mathcal{E}$ of the quantum fluctuations is then obtained as

$$\begin{aligned}\mathcal{E}_{\mathrm{fl}}(t)&=\langle\bar\psi(\beta\mathcal{H}_p)\psi\rangle\\ &=\sum_s\int\frac{d^3p}{(2\pi)^3 2E_0}\bar V_{-\mathbf{p},s}(t)(\beta\mathcal{H}_p)V_{-\mathbf{p},s}(t)\\ &=2\int\frac{d^3p}{(2\pi)^3 2E_0}\{i[E_0-m_0)]\\ &\quad\times(f_p\dot f_p^*-\dot f_pf_p^*)-2E_0m(t)\}.\end{aligned} \tag{3.2}$$

Using the equations of motion it is easy to see that the time derivative of the total energy density

$$\mathcal{E}=\mathcal{E}_{\mathrm{cl}}+\mathcal{E}_{\mathrm{fl}}=\frac{1}{2}\dot\phi^2(t)+\frac{1}{2}M^2\phi^2(t)+\frac{\lambda}{4!}\phi^4(t)+\mathcal{E}_{\mathrm{fl}}(t) \tag{3.3}$$

vanishes. The fluctuation pressure is given by

$$\begin{aligned}\mathcal{P}_{\mathrm{fl}}(t)&=\frac{1}{3}\langle\bar\psi\boldsymbol{\gamma p}\psi\rangle=\frac{1}{3}\sum_s\int\frac{d^3p}{(2\pi)^3 2E_0}\bar V_{-\mathbf{p},s}(t)\boldsymbol{\gamma p}V_{-\mathbf{p},s}(t)\\ &=\frac{2}{3}\int\frac{d^3p}{(2\pi)^3 2E_0}\{[E_0-m_0][i(f_p\dot f_p^*-\dot f_pf_p^*)\\ &\quad-2m(t)|f_p|^2]\},\end{aligned} \tag{3.4}$$

and the total pressure is

$$\mathcal{P}(t)=\dot\phi^2(t)-\mathcal{E}+\mathcal{P}_{\mathrm{fl}}. \tag{3.5}$$

Energy density and pressure are quartically divergent; their renormalization will be discussed in Sec. IV along with the renormalization of the fluctuation integral.

In contrast to the fluctuation integral and the energy-momentum tensor, the definition of the particle number relies on the creation and annihilation operators. The number of particles with momentum **p** and helicity $s$ is given generally via

[1] We use the Heisenberg picture; i.e., the field operators depend on time via the mode functions.

$$\mathcal{N}_{\mathbf{p},s}(t) \propto \langle b^\dagger_{\mathbf{p},s}(t) b_{\mathbf{p},s}(t)\rangle \quad . \tag{3.6}$$

The definition of time-dependent creation and annihilation operators implies an interpretation.[2] If we used the operators $b_{\mathbf{p},s}$ and $b^\dagger_{\mathbf{p},s}$, the particle number would remain equal to the initial one, zero for an initial vacuum state or $1/[\exp(E_0/T)+1]$ for a thermal one. While these operators refer to a decomposition of the field with respect to the exact mode functions $f_p(t)$, the concept of free particles implies plane-wave mode functions. If we define these modes to have mass $m_0$ and $E_0=\sqrt{p^2+m_0^2}$, i.e., if we use the modes corresponding to the initial state, we obtain

$$\begin{aligned} b^0_{\mathbf{p},s}(t) &= \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}+iE_0t} U^{0\dagger}_{\mathbf{p},s}\psi(\mathbf{x},t) \\ &= C^0_{\mathbf{p},s}(t) b_{\mathbf{p},s} + D^0_{\mathbf{p},s}(t) d^\dagger_{-\mathbf{p},s}, \end{aligned} \tag{3.7}$$

with

$$U^0_{\mathbf{p},s} = \frac{1}{\sqrt{E_0+m_0}} [E_0 + \mathcal{H}_{\mathbf{p}}(t_0)] \begin{pmatrix} \chi_s \\ 0 \end{pmatrix}. \tag{3.8}$$

We need only the coefficient $D^0_{\mathbf{p},s}$ which is given by

$$D^0_{\mathbf{p},s} = \frac{e^{iE_0t}}{2E_0} U^{0\dagger}_{\mathbf{p},s} V_{-\mathbf{p},s}(t), \tag{3.9}$$

and the particle number becomes

$$\mathcal{N}^0_{\mathbf{p},s}(t) = |D^0_{\mathbf{p},s}(t)|^2 \quad . \tag{3.10}$$

In terms of the mode functions we obtain, for the occupation number for one helicity eigenstate with momentum $\mathbf{p}$,

$$\begin{aligned} \mathcal{N}^0_{\mathbf{p}}(t) = \frac{E_0-m_0}{4E_0^2}\{&2E_0 + i[\dot f^*_p(t) f_p(t) - f^*_p(t)\dot f_p(t)] \\ &- 2(m_1-m_0)|f_p(t)|^2\}. \end{aligned} \tag{3.11}$$

This definition has been used in [31]. As it should be for fermions, the occupation number is strictly less or equal to 1; this is obvious from Eq. (3.9), since $D^0_{\mathbf{p},s}$ is the scalar product of two complex vectors of unit norm. Integrating the occupation number over momentum one obtains the total particle number density

$$\mathcal{N}_0(t) = \sum_s \int \frac{d^3p}{(2\pi)^3} \mathcal{N}^0_{\mathbf{p},s}(t) \quad . \tag{3.12}$$

If we imagine the time evolution being stopped at the time $t$, it seems more natural to use free quanta of mass $m_1=m(t)$ and

$$b^1_{\mathbf{p},s}(t) = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}+iE_0t} U^{1\dagger}_{\mathbf{p},s}\psi(\mathbf{x},t), \tag{3.13}$$

with

$$U^1_{\mathbf{p},s} = \frac{1}{\sqrt{E_1+m_1}} [E_1 + \mathcal{H}_{\mathbf{p}}(t_1)] \begin{pmatrix} \chi_s \\ 0 \end{pmatrix}. \tag{3.14}$$

The coefficient $D^1_{\mathbf{p},s}$ now reads

$$D^1_{\mathbf{p},s} = \frac{e^{iE_1t}}{\sqrt{4E_0E_1}} U^{1\dagger}_{\mathbf{p},s} V_{-\mathbf{p},s}(t_1), \tag{3.15}$$

and the occupation number for particles with mass $m_1=m(t)$ becomes

$$\begin{aligned} \mathcal{N}^1_{\mathbf{p},s}(t) = \frac{E_0-m_0}{4E_0E_1(E_0+m_0)(E_1+m_1)}\{&2E_0(E_0+m_0) \\ &+ i(E_1+m_1)[\dot f^*_p(t) f_p(t) - f^*_p(t)\dot f_p(t)]\}. \end{aligned} \tag{3.16}$$

The total particle number $\mathcal{N}_1(t)$ is again obtained by integrating $\mathcal{N}^1_{\mathbf{p},s}$ over momentum and summing over helicities. The particle number is divergent by power counting; the analysis of the divergent contributions of $\mathcal{N}_1(t)$ shows, however, that it is finite and does not need counterterms (see also Sec. IV). For $\mathcal{N}_0(t)$ we find a linearly divergent contribution that vanishes in dimensional regularization.

As we have mentioned above, the definition of the particle number relies on an interpretation which seems to be more straightforward if the particle number is computed for particles with the "final" mass $m_1=m(t)$. Of course, even if the classical scalar field relaxes to 0 as $t\to\infty$, the final state (taken in the Schrödinger picture) never becomes an ensemble of free particles. Such a state would be described by a density matrix which cannot arise in unitary evolution from a pure state $|0\rangle$. This is also true for the additional particles created in the case of a thermal initial state.

Equation (2.24) may be used to recast the expressions for energy density, pressure, and particle number into a different, sometimes advantageous form.

## IV. RENORMALIZATION

In order to develop the framework for renormalizing the one-loop equations, we write the equation of motion for the mode functions, Eq. (2.14), in the form

$$\left[\frac{d^2}{dt^2} + E_0^2\right] f_p(t) = -V(t) f_p(t), \tag{4.1}$$

with

$$V(t) = m^2(t) - m_0^2 - i\dot m(t). \tag{4.2}$$

Using the initial conditions (2.16) this equation can be recast into the form of an integral equation

[2] For an extensive discussion, in the context of general relativity, see [40].

$$f_p(t)=e^{-iE_0t}-\frac{1}{E_0}\int_0^t dt'\,\sin[E_0(t-t')]V(t')f_p(t'). \tag{4.3}$$

Using this integral equation, the mode functions may be expanded with respect to the potential $V(t)$. We split off the zeroth-order (plane-wave) contribution and an oscillating phase factor by writing

$$f_p(t)=e^{-iE_0t}[1+h_p(t)]. \tag{4.4}$$

The functions $h_p(t)$ satisfy differential and integral equations derived from Eqs. (4.1) and (4.3), respectively. These functions are discussed in Appendix A. They may be decomposed as

$$h_p(t)=\sum_{n=1}^{\infty} h_p^{(n)}, \tag{4.5}$$

where $h_p^{(n)}$ is of $n$th order in $V(t)$; we define further the inclusive sums

$$h_p^{\overline{(n)}}=\sum_{m=n}^{\infty} h_p^{(m)}. \tag{4.6}$$

In terms of these functions and their expansion discussed in Appendix A the integrand of the fluctuation integral can be written as

$$\begin{aligned}1-\left(1-\frac{m_0}{E_0}\right)|f_p(t)|^2 &=\frac{E_0}{m_0}-\left(1-\frac{E_0}{m_0}\right)[2\,\mathrm{Re}\,h_p(t)+|h_p(t)|^2]\\ &=\frac{m(t)}{E_0}-\frac{\ddot m(t)}{4(E_0)^3}-\frac{m^3(t)}{2(E_0)^3}+\frac{m(t)m^2(0)}{2(E_0)^3}\\ &\quad+\frac{\ddot m(0)}{4(E_0)^3}\cos(2E_0t)+K_{\rm F}(p,t).\end{aligned} \tag{4.7}$$

The first terms on the right hand side lead to divergent or singular momentum integrals. The function $K_{\rm F}(t)$ can be considered being defined by this equation. It behaves as $(E_0)^{-4}$ and its momentum integral is finite. While $K_{\rm F}(t)$ is defined here as the difference between the original, numerically computed integrand and its leading contributions, an alternative expression, avoiding such a subtraction, is given in Appendix A. We decompose the fluctuation integral as

$$\mathcal{F}(t)=\mathcal{F}_{\rm div}(t)+\mathcal{F}_{\rm sing}(t)+\mathcal{F}_{\rm fin}(t), \tag{4.8}$$

with

$$\mathcal{F}_{\rm div}=\int\frac{d^3p}{(2\pi)^3}\left\{\frac{m(t)}{E_0}-\frac{\ddot m(t)}{4(E_0)^3}-\frac{m^3(t)}{2(E_0)^3}+\frac{m(t)m^2(0)}{2(E_0)^3}\right\}, \tag{4.9}$$

$$\mathcal{F}_{\rm sing}=\int\frac{d^3p}{(2\pi)^3}\left\{\frac{\ddot m(0)}{4(E_0)^3}\cos(2E_0t)\right\}, \tag{4.10}$$

$$\mathcal{F}_{\rm fin}=\int\frac{d^3p}{(2\pi)^3}K_{\rm F}(p,t). \tag{4.11}$$

The divergent part $\mathcal{F}_{\rm div}$ is a local polynomial in $m(t)=g\phi(t)$. It will be absorbed by appropriate renormalization counterterms. The integral involving $\cos(2E_0t)$ is logarithmically singular at $t=0$ but finite otherwise. This contribution is obviously related to the initial conditions and will be discussed in Sec. V.

The fluctuation parts of the energy-momentum tensor can be analyzed in a similar way. The integrand of the energy density $\mathcal{E}_{\rm fl}$ can be expanded as

$$\begin{aligned}\frac{i}{2}\left(1-\frac{m_0}{E_0}\right)(f_p\dot f_p^*-\dot f_pf_p^*)-m(t) &=-(E_0-m_0)\left\{1+2\,\mathrm{Re}\,h_p+|h_p|^2+\frac{1}{E_0}\mathrm{Im}[\dot h_p(1+h_p^*)]\right\}-m(t)\\ &=-E_0-\frac{m^2(t)}{2E_0}+\frac{m_0^2}{2E_0}+\frac{\dot m^2(t)}{8(E_0)^3}+\frac{m^4(t)}{8(E_0)^3}\\ &\quad+\frac{m_0^4}{8(E_0)^3}-\frac{m^2(t)m_0^2}{4(E_0)^3}+K_{\rm E}(p,t).\end{aligned} \tag{4.12}$$

Again $K_{\rm E}(p,t)$ is defined by this equation and it behaves as $(E_0)^{-4}$ as $E_0\to\infty$. There is no cosine term here and, therefore, no singular contribution. So

$$\mathcal{E}(t)=\mathcal{E}_{\rm div}(t)+\mathcal{E}_{\rm fin}(t), \tag{4.13}$$

with

$$\mathcal{E}_{\rm div}=2\int\frac{d^3p}{(2\pi)^3}\left\{-E_0-\frac{m^2(t)}{2E_0}+\frac{m_0^2}{2E_0}+\frac{\dot m^2(t)}{8(E_0)^3}+\frac{m^4(t)}{8(E_0)^3}+\frac{m_0^4}{8(E_0)^3}-\frac{m^2(t)m_0^2}{4(E_0)^3}\right\}, \tag{4.14}$$

$$\mathcal{E}_{\rm fin}=\int\frac{d^3p}{(2\pi)^3}K_{\rm E}(p,t)\quad. \tag{4.15}$$

If the integrand of the fluctuation pressure is rewritten in terms of the functions $h_p$, it reads

$$-(E_0-m_0)\left\{\left(1+\frac{m(t)}{4E_0}\right)(1+2\,\mathrm{Re}\,h_p+|h_p|^2)-\frac{1}{E_0}\mathrm{Im}[\dot h_p(1+h_p^*)]\right\}-m(t). \tag{4.16}$$

We expand again in $V(t)$ in order to sort out the leading contributions. Finally, $\mathcal{P}(t)$ can be decomposed as

$$\mathcal{P}(t)=\mathcal{P}_{\rm div}(t)+\mathcal{P}_{\rm sing}(t)+\mathcal{P}_{\rm fin}(t), \tag{4.17}$$

with

$$\mathcal{P}_{\rm div}=\int\frac{d^3p}{(2\pi)^3}\left\{-\frac{4}{3}E_0+\frac{2m_0^2}{3E_0}+\frac{m_0^4}{6(E_0)^3}+\frac{\dot m^2(t)}{6(E_0)^3}-\frac{m(t)\ddot m(t)}{12(E_0)^3}-\frac{m^2(t)}{3E_0}-\frac{m^2(t)m_0^2}{6(E_0)^3}\right\}, \tag{4.18}$$

$$\mathcal{P}_{\rm sing}=\int\frac{d^3p}{(2\pi)^3}\frac{m(t)\ddot m(0)}{12(E_0)^3}\cos(2E_0t), \tag{4.19}$$

$$\mathcal{P}_{\rm fin}=\int\frac{d^3p}{(2\pi)^3}K_{\rm P}(p,t). \tag{4.20}$$

The integral over $K_{\rm P}$, which is defined by this decomposition, is finite. The divergent terms $\mathcal{F}_{\rm div}$, $\mathcal{E}_{\rm div}$, and $\mathcal{P}_{\rm div}$ are proportional to local terms in $\phi(t)$ and its derivatives. These can be absorbed in the usual way by introducing the appropriate counterterms into the Lagrangian and into the energy-momentum tensor.

The counterterms in the Lagrangian are introduced as

$$\mathcal{L}_{\rm c.t}=\frac{1}{2}\delta Z\dot\phi^2-\frac{1}{2}\delta M^2\phi^2-\frac{\delta\lambda}{24}\phi^4. \tag{4.21}$$

The divergent parts of the fluctuation integral can be evaluated, e.g., using dimensional regularization. One finds

$$\mathcal{F}_{\rm div}=\frac{\ddot m(t)}{8\pi^2}L_0+\frac{m^3(t)}{4\pi^2}L_0+\frac{m(t)m_0^2}{4\pi^2}, \tag{4.22}$$

with the abbreviation

$$L_0=\frac{2}{\epsilon}+\ln\frac{4\pi\mu^2}{m_0^2}-\gamma. \tag{4.23}$$

As already found for the scalar fluctuations [33], the dependence on the initial mass $m_0$ can be absorbed into finite terms, $\Delta Z$, $\Delta M^2$, and $\Delta\lambda$. Applying a modified minimal subtraction scheme (MS) prescription, the infinite renormalizations become

$$\delta Z=-\frac{g^2}{8\pi^2}L, \tag{4.24}$$

$$\delta\lambda=-6\frac{g^4}{4\pi^2}L, \tag{4.25}$$

with

$$L=\frac{2}{\epsilon}+\ln\frac{4\pi\mu^2}{M^2}-\gamma. \tag{4.26}$$

There is no infinite mass renormalization counterterm. Introducing the renormalization counterterms into the equation of motion, we obtain

$$(1+\Delta Z)\ddot\phi+(M^2+\Delta M^2)\phi+\frac{\lambda+\Delta\lambda}{6}\phi^3+g(\mathcal{F}_{\rm fin}+\mathcal{F}_{\rm sing})=0. \tag{4.27}$$

The coefficients of the finite terms left over after adding the renormalization counterterms to $\mathcal{F}_{\rm div}$ are given by

$$\Delta Z=\frac{g^2}{8\pi^2}\ln\frac{M^2}{m_0^2}, \tag{4.28}$$

$$\Delta\lambda M=6\frac{g^2}{4\pi^2}\ln\frac{M^2}{m_0^2}, \tag{4.29}$$

$$\Delta M^2=\frac{g^2m_0^2}{4\pi^2}. \tag{4.30}$$

Since the bare fermion mass vanishes, we have introduced the scalar mass $M$ as scale parameter. Obviously, the equation of motion is not yet acceptable in its present form, due to the singular term.

The divergent parts of the energy give, after dimensional regularization,

$$\mathcal{E}_{\rm div}=\frac{\dot m^2(t)}{16\pi^2}L_0+\frac{m^4(t)}{16\pi^2}L_0-\frac{m^4(0)}{32\pi^2}+\frac{m^2(t)m_0^2}{8\pi^2}. \tag{4.31}$$

The counterterms correspond to those in the Lagrangian, i.e.,

$$\mathcal{E}_{\rm c.t}=\frac{1}{2}\delta Z\dot\phi^2+\frac{1}{2}\delta M^2\phi^2+\frac{\delta\lambda}{24}\phi^4, \tag{4.32}$$

with the same coefficients as above. We need no infinite counterterm for the zero-point energy or cosmological constant. Adding the divergent part and the counterterms we are left with finite contributions

$$\mathcal{E}_{\rm div}+\mathcal{E}_{\rm c.t.}=\frac{1}{2}\Delta Z\dot\phi^2+\frac{1}{2}\Delta M^2\phi^2+\frac{\Delta\lambda}{24}\phi^4+\Delta\Lambda, \tag{4.33}$$

with

$$\Delta\Lambda=-\frac{m_0^4}{32\pi^2}\ . \tag{4.34}$$

The divergent part of the pressure is given by

$$\mathcal{P}_{\rm div}=\frac{\dot m^2(t)}{12\pi^2}L_0-\frac{m(t)\ddot m(t)}{24\pi^2}L_0-\frac{m_0^4}{24\pi^2}+\frac{m_0^2m^2(t)}{12\pi^2}. \tag{4.35}$$

In addition to the counterterms already introduced we have to add to the energy-momentum tensor the ''improvement'' counterterm [39]

$$\delta A(g_{\mu\nu}\partial^\alpha\partial_\alpha-\partial_\mu\partial_\nu)\phi^2(x); \tag{4.36}$$

since $\phi$ depends only on $t$, this term contributes only to the pressure, and we have

$$\mathcal{P}_{\rm c.t}=+\delta Z\dot\phi^2+\delta A\frac{d^2}{dt^2}\phi^2 \quad . \tag{4.37}$$

We choose

$$\delta A=\frac{g^2}{48\pi^2}L \quad ; \tag{4.38}$$

there is a finite remainder

$$\mathcal{P}_{\rm div}+\mathcal{P}_{\rm c.t.}=+\Delta Z\dot\phi^2+\Delta A\frac{d^2}{dt^2}\phi^2-\frac{m_0^4}{24\pi^2}+\frac{m_0^2m^2(t)}{12\pi^2}, \tag{4.39}$$

with

$$\Delta A=\frac{g^2}{48\pi^2}\ln\frac{M^2}{m_0^2}. \tag{4.40}$$

## V. REMOVING THE INITIAL SINGULARITY

We are now ready to discuss the terms proportional to $\cos(E_0t)$ which turn out to be singular as $t\to0$:

$$\mathcal{F}_{\rm sing}(t)=\int\frac{d^3p}{(2\pi)^3}\frac{\ddot m(0)}{4(E_0)^3}\cos(2E_0t)$$
$$\simeq-\frac{\ddot m(0)}{16\pi^2}\ln(2m_0t)\quad\text{as}\quad t\to0. \tag{5.1}$$

For the case of scalar fields in Minkowski space the fluctuation integral has only a logarithmic cusp $\propto t\ln t$ at $t=0$, as observed by Ringwald [1]; the energy is finite, and the pressure behaves as $\ln t$. In FRW cosmology the energy of scalar fluctuations is logarithmically singular whereas the pressure behaves as $1/t$. So the Friedmann equations become singular. Problems with the initial conditions in FRW cosmology have also been noted in [4,10] when using comoving time and the associated vacuum state; the problems disappear if conformal time is used. The two vacuum states are related by a Bogoliubov transformation.

For fermionic fluctuations we find that already the fluctuation integral is divergent, so that the numerical code cannot be started even in Minkowski space. We have shown recently, for the case of scalar fields, that such ''Stueckelberg singularities'' [35] can be removed by a Bogoliubov transformation of the initial state, which was constructed explicitly.

Within the Fock space based on the ''initial vacuum'' state $|0\rangle$ which is annihilated by the operators $b_{\mathbf{p},s}$ and $d_{\mathbf{p},s}$ we define a more general initial state by requiring that

$$[b_{\mathbf{p},s}-\rho_{\mathbf{p},s}d^\dagger_{-\mathbf{p},s}]|\overline{0}\rangle=0. \tag{5.2}$$

The Bogoliubov transformation from $|0\rangle$ to this state is given in Appendix B. If the fluctuation integral, the energy, and the pressure are computed by taking the trace with respect to this state, we just have to replace in the defining equations (2.25), (3.2), and (3.4) the functions $U_{\mathbf{p},s}(t)$ by

$$V_{-\mathbf{p},s}(t)\Rightarrow\cos(\beta_{\mathbf{p},s})V_{-\mathbf{p},s}(t)+\sin(\beta_{\mathbf{p},s})U_{\mathbf{p},s}(t) \quad . \tag{5.3}$$

For the particle number, the substitution is done in the Bogoliubov coefficients, Eqs. (3.9) and (3.15). The angle $\beta_{\mathbf{p},s}$ is related to $\rho_{\mathbf{p},s}$ via

$$\rho_{\mathbf{p},s}=\tan(\beta_{\mathbf{p},s}) \quad . \tag{5.4}$$

If the expectation value of $\bar\psi\psi$ is taken in the Bogoliubov-rotated initial state, the fluctuation integral becomes

$$\tilde{\mathcal{F}}(t)=\sum_s\int\frac{d^3p}{(2\pi)^3 2E_0}\{\overline{V}_{-\mathbf{p},s}(t)V_{-\mathbf{p},s}(t)\cos^2\beta_{\mathbf{p},s}+\overline{U}_{\mathbf{p},s}(t)U_{-\mathbf{p},s}(t)\sin^2\beta_{\mathbf{p},s}$$
$$+[\overline{U}_{\mathbf{p},s}(t)V_{-\mathbf{p},s}(t)+\overline{V}_{\mathbf{p},s}(t)U_{-\mathbf{p},s}(t)]\sin\beta_{\mathbf{p},s}\cos\beta_{\mathbf{p},s}\}. \tag{5.5}$$

Rewriting this expression in terms of the mode functions we find

$$\tilde{\mathcal{F}}(t)=-\sum_s\int\frac{d^3p}{(2\pi)^3 2E_0}\left\{\cos 2\beta_{\mathbf{p},s}\left[2E_0-\frac{2\mathbf{p}^2}{E_0+m_0}|f_p|^2\right]+\sin 2\beta_{\mathbf{p},s}\frac{sp}{E_0+m_0}[\mathrm{Im}\,\partial_t f_p^2-2m(t)\mathrm{Re}\,f_p^2]\right\}. \tag{5.6}$$

Using the perturbative expansion of the mode functions given in Appendix A this integral takes the form

$$\tilde{\mathcal{F}}(t) = -\sum_s \int \frac{d^3p}{(2\pi)^3} \Bigg\{ \cos 2\beta_{\mathbf{p},s} \left[ \frac{m(t)}{E_0} - \frac{\ddot{m}(t)}{4(E_0)^3} - \frac{m^3(t)}{2(E_0)^3} + \frac{m(t)m^2(0)}{2(E_0)^3} + \frac{\ddot{m}(0)}{4(E_0)^3}\cos(2E_0t) + K_{\rm F}(p,t) \right] + \sin 2\beta_{\mathbf{p},s} \frac{sp}{2E_0(E_0+m_0)} [-2E_0 \cos 2E_0 t + L_{\rm F}(p,t)] \Bigg\}. \tag{5.7}$$

The function $K_{\rm F}(p,t)$ has been defined above; $L_{\rm F}(p,t)$ is defined by

$$L_{\rm F}(p,t) = 2\,{\rm Im}\{e^{-2iE_0t}[(1+h_p)\dot{h}_p - iE_0(2h_p + h_p^2)]\} - 2m(t){\rm Re}\, f_p^2. \tag{5.8}$$

Obviously, one gets rid of the term proportional to $\cos(2E_0t)$ by requiring

$$\tan 2\beta_{\mathbf{p},s} = \frac{\ddot{m}(0)(E_0+m_0)}{8sp(E_0)^3}. \tag{5.9}$$

Thereby, the Bogoliubov transformation is explicitly specified. We notice that the helicity dependence of the Bogoliubov transformation cancels in Eq. (5.7). Using the asymptotic behavior

$$\beta_{\mathbf{p},s} \overset{p\to\infty}{\simeq} \frac{\ddot{m}(0)}{8sp(E_0)^2}, \tag{5.10}$$

and therefore

$$\cos(2\beta_{\mathbf{p},s}) - 1 = 2\sin^2(\beta_{\mathbf{p},s}) \overset{p\to\infty}{\simeq} \frac{\ddot{m}^2(0)}{64p^2(E_0)^4}, \tag{5.11}$$

$$\sin(2\beta_{\mathbf{p},s}) \overset{p\to\infty}{\simeq} \frac{\ddot{m}(0)}{4sp(E_0)^2}, \tag{5.12}$$

it is easy to convince oneself that this Bogoliubov transformation does not interfere with the analysis of the divergent parts and, therefore, with the renormalization discussed in the previous section. So $\tilde{\mathcal{F}}$ is rendered finite by adding the counterterms defined in the previous section. In the renormalized equation of motion (4.27) we just have to replace $\mathcal{F}_{\rm sing}(t) + \mathcal{F}_{\rm fin}(t)$ with $\tilde{\mathcal{F}}_{\rm fin}(t)$ which is given explicitly by

$$\tilde{\mathcal{F}}_{\rm fin}(t) = -2\int \frac{d^3p}{(2\pi)^3} \Bigg\{ -2\sin^2\beta_{\mathbf{p},s} \left[ \frac{m(t)}{E_0} - \frac{\ddot{m}(t)}{4(E_0)^3} - \frac{m^3(t)}{2(E_0)^3} + \frac{m(t)m^2(0)}{2(E_0)^3} \right] + \cos 2\beta_{\mathbf{p},s} K_{\rm F}(p,t) + \sin 2\beta_{\mathbf{p},s} \frac{sp}{2E_0(E_0+m_0)} L_{\rm F}(p,t) \Bigg\}. \tag{5.13}$$

The renormalization of the energy density proceeds in an analogous way. The fluctuation energy in the Bogoliubov transformed state is

$$\tilde{\mathcal{E}}_{\rm fl} = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_0} \{ \cos^2\beta_{\mathbf{p},s}\, \bar{V}_{-\mathbf{p},s}(t)(\beta\mathcal{H}_{\mathbf{p}})V_{-\mathbf{p},s}(t) + \sin^2\beta_{\mathbf{p},s}\, \bar{U}_{\mathbf{p},s}(t)(\beta\mathcal{H}_{\mathbf{p}})U_{-\mathbf{p},s}(t) + \sin\beta_{\mathbf{p},s}\cos\beta_{\mathbf{p},s}[\bar{U}_{\mathbf{p},s}(t)(\beta\mathcal{H}_p)V_{-\mathbf{p},s}(t) + \bar{V}_{\mathbf{p},s}(t)(\beta\mathcal{H}_p)U_{-\mathbf{p},s}(t)]\} \;. \tag{5.14}$$

We again insert the expansion of the mode functions to obtain

$$\tilde{\mathcal{E}}_{\rm fl} = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_0} \left( \cos 2\beta_{\mathbf{p},s}\{ i[E_0 - m(0)](f_p\dot{f}_p^* - \dot{f}_p f_p^*) - 2E_0 m(t)\} + \sin 2\beta_{\mathbf{p},s} \frac{sp}{E_0+m_0} {\rm Re}\,[\dot{f}_p^2 + E^2 f_p^2] \right). \tag{5.15}$$

After adding the counterterms defined in the previous section the finite part of the fluctuaion energy becomes

$$\tilde{\mathcal{E}}_{\rm fin}=2\int \frac{d^3p}{(2\pi)^3}\Biggl\{-2\sin^2\beta_{\mathbf{p},s}\left[-E_0-\frac{m^2(t)}{2E_0}+\frac{m_0^2}{2E_0}+\frac{\dot{m}^2(t)}{8(E_0)^3}+\frac{m^4(t)}{8(E_0)^3}+\frac{m_0^4}{8(E_0)^3}-\frac{m^2(t)m_0^2}{4(E_0)^3}\right]$$
$$+\cos 2\beta_{\mathbf{p},s}K_{\rm E}(p,t)+\sin 2\beta_{\mathbf{p},s}\frac{sp}{2E_0(E_0+m_0)}L_{\rm E}(p,t)\Biggr\}, \tag{5.16}$$

with

$$L_{\rm E}(p,t)={\rm Re}\ \{e^{-2iE_0t}[\dot{h}_p^2-2iE_0\dot{h}_p(1+h_p)+(m^2(t)-m_0^2)(1+h_p)^2]\}. \tag{5.17}$$

Finally, we consider the fluctuation pressure, taken in the new vacuum state. It reads

$$\tilde{\mathcal{P}}_{\rm fl}=-\tilde{\mathcal{E}}_{\rm fl}+\sum_s\int\frac{d^3p}{(2\pi)^3}\Biggl\{\cos 2\beta_{\mathbf{p},s}\left[\frac{4}{3}i(f_p\dot{f}_p^*-\dot{f}_pf_p^*)(E_0-m_0)-\frac{2}{3}m(t)(E_0-m_0)|f_p|^2-2m(t)E_0\right]$$
$$+\sin 2\beta_{\mathbf{p},s}\frac{sp}{E_0+m_0}{\rm Re}\left[\frac{4}{3}\dot{f}_p^2-\frac{1}{3}im(t)\partial_t f_p^2+\left(\frac{4}{3}\mathbf{p}^2+\frac{2}{3}m^2(t)\right)f_p^2\right]\Biggr\}. \tag{5.18}$$

The finite part becomes, after adding the counterterms,

$$\tilde{\mathcal{P}}_{\rm fin}=2\int\frac{d^3p}{(2\pi)^3}\Biggl\{-2\sin^2\beta_{\mathbf{p},s}\left[-\frac{4}{3}E_0+\frac{2m_0^2}{3E_0}+\frac{m_0^4}{6(E_0)^3}+\frac{\dot{m}^2(t)}{6(E_0)^3}-\frac{m(t)\ddot{m}(t)}{12(E_0)^3}-\frac{m^2(t)}{3E_0}-\frac{m^2(t)m_0^2}{6(E_0)^3}\right]$$
$$+\cos 2\beta_{\mathbf{p},s}K_{\rm P}(p,t)+\sin 2\beta_{\mathbf{p},s}\frac{sp}{2E_0(E_0+m_0)}L_{\rm P}(p,t)\Biggr\}, \tag{5.19}$$

with

$$L_{\rm P}(p,t)=\frac{1}{3}{\rm Re}\{4\dot{f}_p^2-2ime^{-2iE_0t}[\dot{h}_p(1+h_p)-iE_0(2h_p+h_p^2)]+[4\mathbf{p}^2+2m^2(t)]f_p^2\}. \tag{5.20}$$

The Bogoliubov transformation has removed the singular term in the pressure as well.

### VI. EXTENSION TO FRW SPACETIME

Now that we have set all basic equations and performed renormalization the extension to FRW spacetime is straightforward.

We consider the Friedmann–Robertson–Walker metric with curvature parameter $k=0$, i.e., a spatially isotropic and flat spacetime. We will treat the quantum fields and the cosmological background self-consistently. That is, the scale parameter $a(t)$ is obtained dynamically from the quantum fields.

The line element of a flat FRW universe is given by

$$ds^2=dt^2-a^2(t)d\vec{x}^2. \tag{6.1}$$

The time evolution of the $a(t)$ is governed by Einstein's field equation

$$G_{\mu\nu}+\alpha H^{(1)}_{\mu\nu}+\beta H^{(2)}_{\mu\nu}+\Lambda g_{\mu\nu}=-\kappa\langle T_{\mu\nu}\rangle, \tag{6.2}$$

with $\kappa=8\pi G$. The Einstein curvature tensor $G_{\mu\nu}$ is given by

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\quad. \tag{6.3}$$

The Ricci tensor and the Ricci scalar are defined as

$$R_{\mu\nu}=R^{\lambda}_{\mu\nu\lambda}, \tag{6.4}$$

$$R=g^{\mu\nu}R_{\mu\nu}, \tag{6.5}$$

where

$$R^{\lambda}_{\alpha\beta\gamma}=\partial_\gamma\Gamma^{\lambda}_{\alpha\beta}-\partial_\alpha\Gamma^{\lambda}_{\gamma\beta}+\Gamma^{\lambda}_{\gamma\sigma}\Gamma^{\sigma}_{\alpha\beta}-\Gamma^{\lambda}_{\alpha\sigma}\Gamma^{\sigma}_{\gamma\beta}\quad. \tag{6.6}$$

The terms $H^{(1)}_{\mu\nu}$ and $H^{(2)}_{\mu\nu}$ arise if terms proportional to $R^2$ and $R^{\mu\nu}R_{\mu\nu}$ are included into the Hilbert-Einstein action. If space-time is conformally flat, these terms are related by

$$H^{(2)}_{\mu\nu}=\frac{1}{3}H^{(1)}_{\mu\nu}\quad, \tag{6.7}$$

so that we can set $\beta=0$ in Eq. (6.2) without loss of generality [40]. We also replace $H^{(1)}_{\mu\nu}$ by $H_{\mu\nu}$ in the following.

These terms are usually not considered in standard cosmology. They are included here, as well as the cosmological constant term, only for the purpose of renormalization; they will absorb divergences of the energy-momentum tensor. So in principle they should appear on the right hand side as counterterms; they are related to the coefficients of these counterterms by $\Lambda=\kappa\delta\Lambda$ and $\alpha=\kappa\delta\alpha$.

As usual we can reduce the Einstein field equations to an equation for the time-time component and one for the trace of $G_{\mu\nu}$, the Friedmann equations

$$G_{tt}+\alpha H_{tt}+\Lambda=-\kappa T_{tt}, \tag{6.8}$$

$$G^{\mu}_{\mu}+\alpha H^{\mu}_{\mu}+4\Lambda=-\kappa T^{\mu}_{\mu}. \tag{6.9}$$

For the line element (6.1) the various terms take the form [40]

$$G_{tt}(t)=-3H^2(t),$$

$$G^{\mu}_{\mu}(t)=-R(t),$$

$$H_{tt}(t)=-6\left(H(t)\dot{R}(t)+H^2(t)R(t)-\frac{1}{12}R^2(t)\right),$$

$$H^{\mu}_{\mu}(t)=-6[\ddot{R}(t)+3H(t)\dot{R}(t)], \tag{6.10}$$

with the curvature scalar

$$R(t)=6[\dot{H}(t)+2H^2(t)] \tag{6.11}$$

and the Hubble expansion rate

$$H(t)=\frac{\dot{a}(t)}{a(t)} . \tag{6.12}$$

The Dirac equation in FRW spacetime (see, e.g., [40,41]) is given by

$$\left\{i\partial_t+i\frac{3}{2}\frac{\dot{a}(t)}{a(t)}+\frac{i}{a(t)}\boldsymbol{\alpha}\nabla-g\phi(t)\gamma_0\right\}\psi(t,\mathbf{x})=0 . \tag{6.13}$$

It proves convenient to introduce conformal time and scales. The conformal factors for the scalar field and the fermion field are

$$\psi(t,\mathbf{x})=a^{-3/2}(t)\tilde{\psi}(\tau,\tilde{\mathbf{x}}), \tag{6.14}$$

$$\phi(t,\mathbf{x})=a(t)^{-1}\tilde{\phi}(\tau,\tilde{\mathbf{x}}), \tag{6.15}$$

with $\mathbf{x}=a(t)\tilde{\mathbf{x}}$ and $dt=a(t)d\tau$. In conformal time, and using these redefinitions of the fields, the Dirac equation simplifies to

$$\{i\partial_\tau+i\boldsymbol{\alpha}\tilde{\nabla}-g\tilde{\phi}(\tau)\beta\}\tilde{\psi}(\tau,\tilde{\mathbf{x}})=0. \tag{6.16}$$

If we introduce, as in Sec. II, the Dirac Hamiltonian

$$\tilde{\mathcal{H}}(\tau)=-i\boldsymbol{\alpha}\tilde{\nabla}+g\tilde{\varphi}(\tau)\beta, \tag{6.17}$$

this equation takes the standard form

$$[i\partial_\tau-\tilde{\mathcal{H}}(\tau)]\tilde{\psi}(\tau,\tilde{\mathbf{x}})=0 . \tag{6.18}$$

So the formalism of quantization can be taken over from Sec. II, if the other functions and operators are understood as being rescaled quantities as well. The rescalings for the other quantities of interest are, omitting arguments, superscripts and subscripts:

$$x=a\tilde{x},\ p=a^{-1}\tilde{p},$$

$$b=a\tilde{b},\ d=a\tilde{d},$$

$$E_0=a^{-1}\tilde{E}_0,\ m=a^{-1}\tilde{m}=g\tilde{\phi},$$

$$U=a^{-1/2}22U,\ V=a^{-1/2}\tilde{V},$$

$$\mathcal{H}=a^{-1}\tilde{\mathcal{H}},\ N_0=a^{1/2}\tilde{N}_0. \tag{6.19}$$

The potential $V(t)$ , Eq. (4.2), becomes the analogous expression $\tilde{V}(\tau)$ with $m(t)$ replaced by $\tilde{m}(\tau)$. This means that the entire perturbative expansion and the analysis of divergences proceed in perfect analogy to the Minkowski space analysis. The metric does not appear in this formalism, and therefore there are no divergences related to the metric. So the Dirac field does not contribute to the wave function renormalization of the gravitational field $\delta Z_g$ (or, equivalently, the renormalization of Newton's constant) and to the terms $H^{(i)}_{\mu\nu}$.

In the equation of motion the fluctuation integral scales as $a^{-3}$ and so do the kinetic and $\lambda\phi^3$ terms. Therefore, the divergent parts of the fluctuation integral are absorbed by exactly the same counterterms $\delta Z\ddot{\phi}$ and $\delta\lambda\phi^3/6$ as in Minkowski space. The same holds true for the finite remainders proportional to $\Delta Z$ and $\Delta\Lambda$. However, the finite mass renormalization $\Delta M^2\phi$ is now replaced by $a^{-3}\Delta M^2\tilde{\phi}$ while the genuine mass term scales as $a^{-1}M^2\tilde{\phi}$.

The renormalized equation of motion for the scalar field $\tilde{\phi}$ takes therefore the form

$$(1+\Delta Z)\frac{d^2}{d\tau^2}\tilde{\phi}+a^2\left[M^2+\left(\xi-\frac{1}{6}\right)R\right]\tilde{\phi}+\Delta M^2\tilde{\phi}$$

$$+\frac{\lambda+\Delta\lambda}{6}\tilde{\phi}^3+g\tilde{\mathcal{F}}_{\rm fin}=0 . \tag{6.20}$$

The energy-momentum tensor generated by the fermionic fluctuations scales exactly as $a^{-4}$.

## VII. LINEARIZED EQUATIONS OF MOTION

A simple intuitive approach to the interplay between the classical Higgs field oscillating with a frequency of the order of the Higgs mass $M$ and the fermionic fluctuations is to treat the system as a Higgs field decaying at rest into fermions. For large amplitudes of the Higgs field this picture is certainly inadequate. For small amplitudes, e.g., at the end of inflation, the linearization of the equations of motion leads indeed to an approximation which supports the simple decay picture. We will compare the exact equations and the lowest approximation in our numerical examples. Here we analyze the behavior of the system analytically along the lines in [31].

If we retain only the $O(g^2)$ part of the fluctuation integral, we find a divergent part which is cancelled by the wave function renormalization counterterm, a finite part remains and the linearized equation of motion of the classical field takes the form

$$(1+\Delta Z)\ddot{\phi}(t)+M^2\phi(t)+\int_0^t dt'\Sigma_r(t-t')\dddot{\phi}(t') + \Sigma_r(t)\ddot{\phi}(0)=0. \tag{7.1}$$

The self-energy insertion is subtracted and is given explicitly by

$$\Sigma_r(t,t')=-2g^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_0^3}\cos[2E_0(t-t')]. \tag{7.2}$$

We define the Laplace transform of the condensate $\phi(t)$ via

$$\psi(s)=\int_0^\infty dt e^{-st}\phi(t) \quad , \tag{7.3}$$

the inverse transformation being given by

$$\phi(t)=\frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c}e^{st}\psi(s). \tag{7.4}$$

The ''Bromwich'' contour of the latter integral runs parallel to the imaginary axis. The constant $c$ has to be chosen in such a way that $\psi(s)$ is analytic for $\mathrm{Re}\, s>c$. In our application $\psi(s)$ will have cuts along the imaginary axis and poles in the half-plane Re $s<0$.

For the Laplace transform $\psi(s)$ the equation of motion reads

$$\begin{aligned}(1+\Delta Z)[s^2\psi(s)-s\phi(0)-\dot{\phi}(0)]+M^2\psi(s)\\ +\tilde{\Sigma}_r(s)[-\ddot{\phi}(0)-s\dot{\phi}(0)-s^2\phi(0)+s^3\psi(s)]\\ +\tilde{\Sigma}_r(s)\ddot{\phi}(0)=0,\end{aligned} \tag{7.5}$$

where $\tilde{\Sigma}(s)$ is the Laplace transform of the self-energy kernel. The equation can be solved readily with the result

$$\psi(s)=\frac{[\dot{\phi}(0)+s\phi(0)][1+\Delta Z+s\tilde{\Sigma}_r(s)]}{(1+\Delta Z)s^2+M^2+s^3\tilde{\Sigma}_r(s)} \quad . \tag{7.6}$$

The singularities of the right hand side in the complex $s$ plane are given, on the one hand, by the singularities of $\Sigma_r(s)$ and by possible zeros of the denominator, whose locations have to be determined. Since $\phi(t)$ cannot contain contributions that increase exponentially, these poles will have to lie on the left part of the complex plane. The dominant behavior at large $t$ will be governed, therefore, by the singularities of $\tilde{\Sigma}_r(s)$ on the imaginary axis.

The self-energy kernel has been defined in Eq. (7.2). Its Laplace transform is given by

$$\Sigma_r(s)=-2g^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{E_0^3}\frac{s}{s^2+4E_0^2} \quad . \tag{7.7}$$

It has cuts along the imaginary $s$ axis running from $s=4im_0$ to $s=i\infty$ and from $s=-4im_0$ to $s=-i\infty$. We introduce the frequency $\omega=-is$. Expressed in the variable $\omega$ the cuts run from $4m_0$ to $\infty$ and from $-\infty$ to $-4m_0$. The discontinuity across the cut along the positive imaginary axis is defined as

$$\rho(s)=\mathrm{disc}\,\tilde{\Sigma}_r(s)=\tilde{\Sigma}_r(i\omega+\epsilon)-\tilde{\Sigma}_r(i\omega-\epsilon). \tag{7.8}$$

One finds

$$\begin{aligned}\rho(s)&=-2g^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_0^3}\frac{2\pi\omega}{\omega+2E_0}\delta(\omega-2E_0)\\ &=-\frac{g^2}{4\pi}\frac{\sqrt{\omega^2-4m_0^2}}{\omega^2}.\end{aligned} \tag{7.9}$$

A more relevant quantity for $\psi(s)$ is the discontinuity of the denominator of Eq. (7.6):

$$i\gamma(s)=[\mathrm{disc}\, s^3\Sigma(s)]|_{s=i\omega}=i\frac{g^2}{4\pi}\omega\sqrt{\omega^2-4m_0^2}. \tag{7.10}$$

The discontinuity of the denominator is purely imaginary, as to be expected. If $\Sigma$ is small, we can expand the denominator around its zero at $s\simeq iM$. The contribution of this pole is related to the decay of the condensate particles of mass $M$ into fermions of mass $m_0$. We neglect the real part of $\Sigma$ as it just shifts the value of M. Indeed it should be zero at $s=\pm iM$ if the condensate field is renormalized on shell. We write

$$\begin{aligned}s^2+(M^-i\Gamma/2)^2&\simeq-(\omega^2-M^2+iM\Gamma)\\ &\simeq s^2+M^2-i\gamma(M)/2 \quad .\end{aligned} \tag{7.11}$$

It follows that

$$\Gamma=\frac{1}{2M}\gamma(M)=\frac{g^2}{8\pi}\sqrt{M^2-4m_0^2} \quad . \tag{7.12}$$

This is almost, but not quite, what one would expect for fermions. Evaluating the width of $\phi\to f\bar{f}$ in the standard way one finds

$$\Gamma_{\phi\to f\bar{f}}=\frac{g^2}{8\pi M}(M^2-4m_0^2)^{3/2} \quad . \tag{7.13}$$

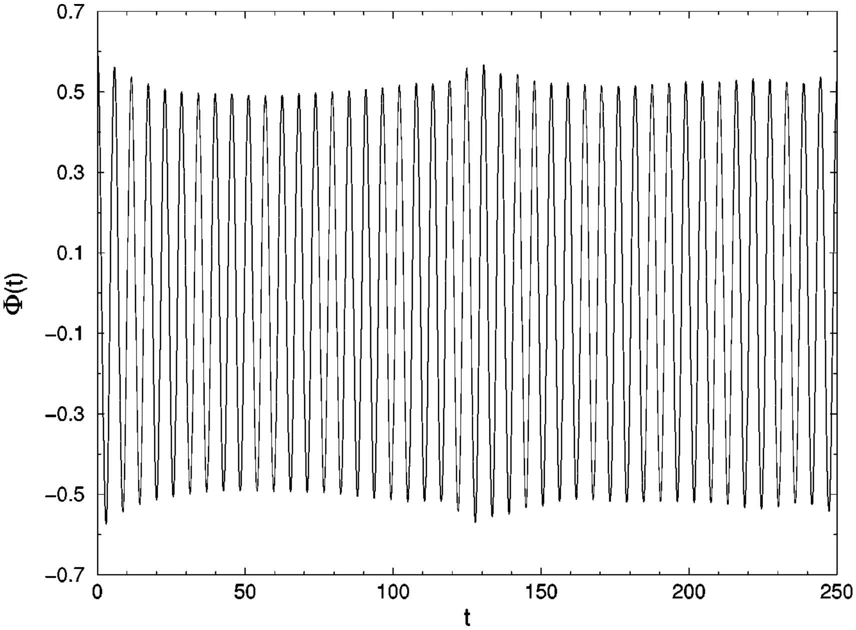


FIG. 1. $\phi(t)$ for $M=1$, $g=2$, $\lambda=1$, and $\phi(0)=0.6$, $\dot{\phi}(0)=0$.

A factor $p$ is typical for an $s$ wave; the factor $p^3$ in the correct decay formula arises from the Dirac traces. While at first glance the difference is surprising, one may recognize that the factor, by which the two expressions differ, is given by $1-4m_0^2/M^2$. The fermion mass is, however, of order $g^2$, so that the difference is higher order and our approximation to the full one-loop result is just lowest order in $g^2$ only.

The late-time behavior of $\phi(t)$ is determined by the strongest singularities of its Laplace transform. These are, on the one hand, the poles at $s=\pm iM-\Gamma/2$ and, on the other hand, the branch cuts on the imaginary axis. The contribution of the poles, which are actually in the second Riemann sheet, has been analyzed carefully in [31]. If the poles have residue $R$, they contribute

$$\begin{aligned}\phi(t) &\simeq \frac{1}{2\pi i}2\pi i R(e^{iM-\Gamma/2t}+e^{-iM-\Gamma/2t})\\ &= Re^{-\Gamma t/2}2\cos(Mt).\end{aligned} \tag{7.14}$$

Approximately, $R\simeq\phi(0)/2$. While this contribution decreases exponentially, the singularities on the imaginary axis yield a power behavior

$$\phi(t)=Rt^{-\alpha}\cos(\Omega t+\varphi) \quad , \tag{7.15}$$

where the powerlike decrease as $t^{-\alpha}$ is related to the order of the branch point $-1+\alpha$. Since the branch point is of the square root type, we have $\alpha=3/2$. Our treatment differs slightly, in terms of order $g^2$, from that of [31], where $\alpha=5/2$.

In our numerical computation we just find the exponential damping (7.14) with a value of $\Gamma$ which agrees with the theoretical expectation (7.12). The power behavior (7.15) is apparently suppressed due to a small coefficient.

## VIII. SOME NUMERICAL RESULTS

We have implemented numerically the formalism developed in the previous sections. We will discuss in this section some results of our numerical simulations. We will pay special attention to the phenomenon of Pauli blocking invoked in [31]. We have already seen that the occupation number cannot exceed unity on account of the unitary evolution of the mode functions $U_{\mathbf{p},s}(t),V_{\mathbf{p},s}(t)$. So an unlimited parametric resonance cannot develop. *A priori* this should not limit the production of particles as the available phase space is large. However, by a phenomenon similar to parametric resonance, the production of particles turns out to be concentrated within a very small band and it is only in this resonant region where Pauli blocking can be effective. One should keep in mind, however, that even for the bosonic case particle production shuts off in the one-loop approximation [29].

We expect Pauli blocking to be especially effective if the initial amplitude of the inflaton field is large. A typical case is displayed in Figs. 1–5. It corresponds to the parameters $M=1,\lambda=1,g=2$, and $\phi(0)=0.6$. Here $\dot{\phi}(0)$ is taken to be zero in all examples. We show the behavior of the inflaton amplitude in Fig. 1, the conserved total energy and its classical and fluctuation parts in Fig. 2, and the pressure in Fig. 3. The time dependence of the particle number is displayed in Fig. 4, using both definitions: $\mathcal{N}_0(t)$, referring to quanta of mass $m_0$ and $\mathcal{N}_1(t)$ referring to quanta of mass $m(t)$. The latter one is seen to behave more smoothly. The momentum spectrum of the occupation number $\mathcal{N}^1_{p,s}(t)$ varies strongly with time. We display therefore, in Fig. 5, the envelope obtained by selecting the maximal occcupation number reached at fixed momentum, as a function of $E_0$. The structure of this envelope shows resonance like enhancements at threshold $E_0=1.2$ and at $E_0=1.9$. The maximal occupation number is reached only at these two values. On the one hand, there is no Pauli blocking in the sense that all levels would be maximally occupied; on the other hand, the unitary evolution does

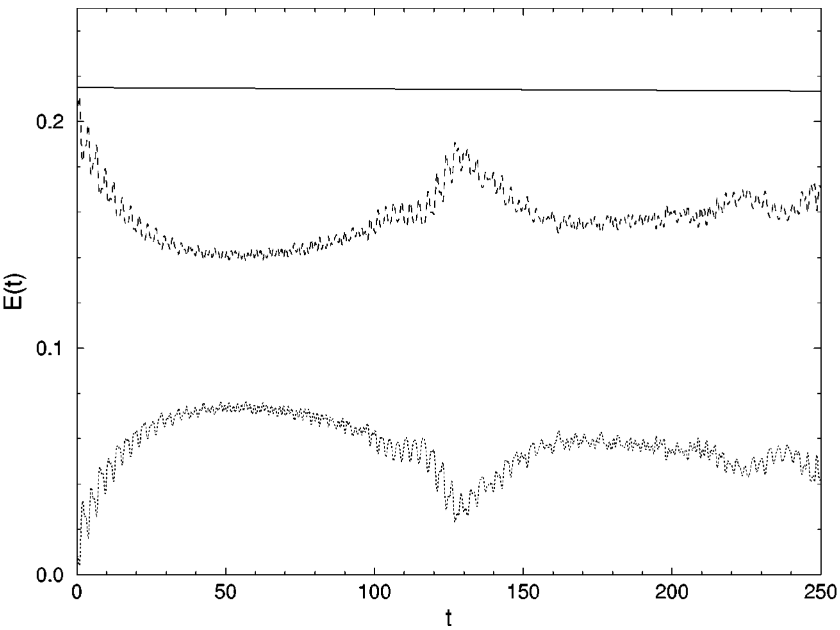


FIG. 2. Classical energy (dashed line), fluctuation energy (dotted line), and total energy (solid line) for the same parameters as in Fig. 1.

not allow for a parametric resonance with high occupation numbers. Clearly, for this parameter set the classical field is not able to efficiently transfer energy into the fermion fluctuations and its amplitude stays essentially constant. These results are similar to those obtained in [31].

The situation is simple in the case of very small excitations and moderate couplings. In this case the scalar field can decay into fermion-antifermion pairs. An exponential decrease is found for the exact quantum evolution and for the linearized equations of motion. This is displayed in Figs. 6–10. Figure 6 shows the exact evolution of $\phi(t)$ for the parameter set $M=1$, $g=2$, $\lambda=1$, and $\phi(0)=0.01$. The amplitude is seen to decrease exponentially, the decay rate being given approximately by $\Gamma\simeq g^2/8\pi$. In the same figure we also plot the solution of the linearized equation of motion for which $\Gamma$ is exactly equal to $g^2/8\pi$. The energy is transferred completely to the quantum fluctuations. The pressure, plotted in Fig. 7, becomes asymptotically equal to one-third of the total energy density $\mathcal{E}=7.5\times10^{-5}$. The quantum ensemble created is ultrarelativistic, as to be expected for massless quanta. The momentum distribution of the occupation number is shown in Fig. 8; it is characterized again by a resonancelike band. The total particle number is plotted in Fig. 9.

So far the results correspond to the expectations. The situation is, however, not as transparent. For intermediate initial amplitudes $\phi(0)$ the relaxation can shut off even for case $2m(0)<M$ where the scalar field can decay. An example is given in Fig. 10, with the parameters $M=1$, $g=1$, $\lambda=0$, and $\phi(0)=0.1$. The full evolution stagnates; the linearized equations of motion show the expected exponential decrease. On the other hand, even for large initial amplitudes the transfer of energy can be as efficient as for scalar fields. An example

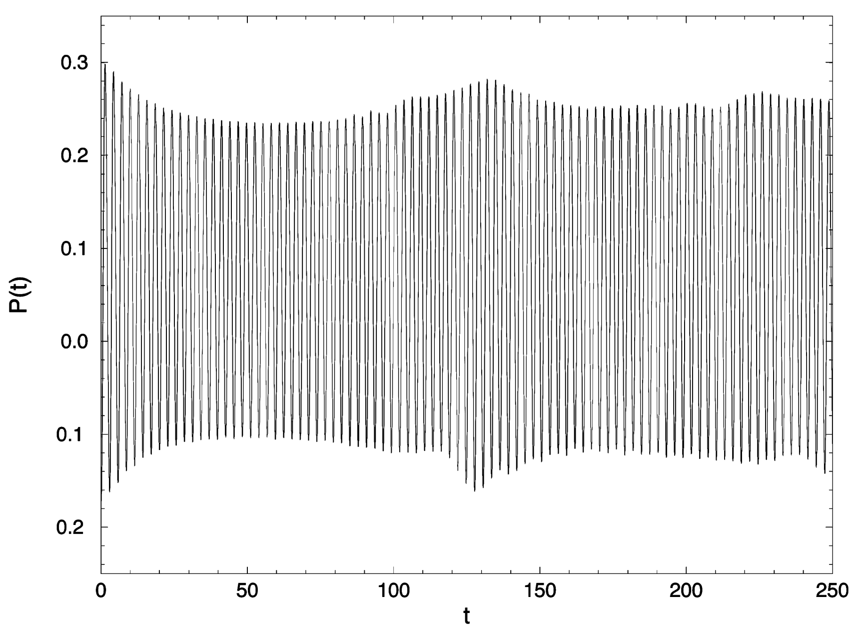


FIG. 3. Total pressure for the same parameters as in Fig. 1.

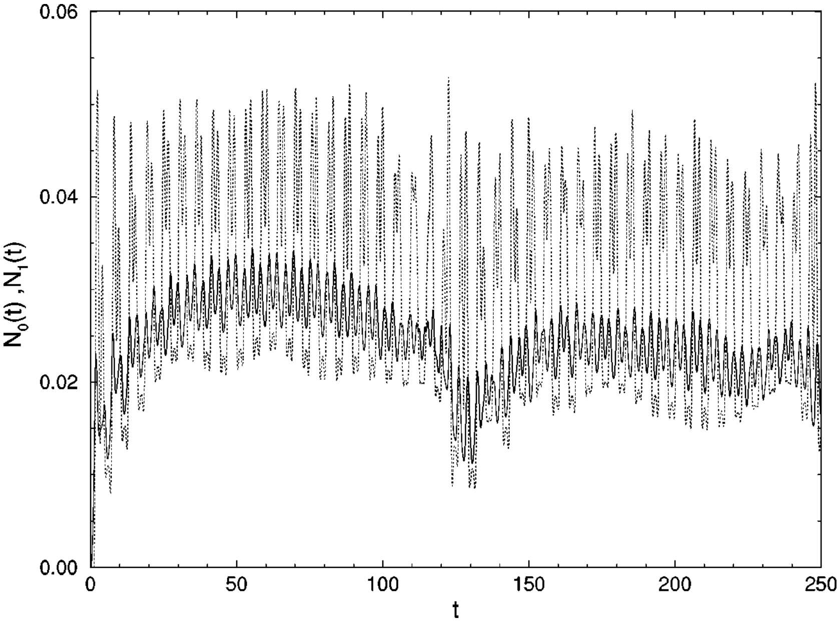


FIG. 4. Total particle number $\mathcal{N}_1(t)$ (solid line) and $\mathcal{N}_0(t)$ (dotted line) for the same parameters as in Fig. 1.

is displayed in Figs. 11–13. The amplitude $\phi(t)$, plotted in Fig. 11, decreases by roughly a factor of 3. The energy transfer is evident from Fig. 12. The spectrum of the occupation number, presented in Fig. 13, shows that the resonance-type band is occupied more strongly than in the first example, which had a smaller initial amplitude, and otherwise the same parameters. Obviously, the concept of Pauli blocking is too simple to describe the situation in an adequate way. If one aims at a better understanding of the quantum evolution, analytical methods should be developed. The simplest approach could be an analysis of the differential equation of the mode functions for a given oscillating classical field, analogous to the analysis of the Mathieu or Lamé equations. In order to illustrate the behavior for a given oscillating field we have solved numerically Eq. (2.14) with $m(t)=m_0\cos t$ for various values of $m_0$ and momentum $p$. We plot in Fig. 14 the envelope of the occupation number $\mathcal{N}_{\rm env}$, i.e., the maximal occupation reached at fixed $p$, as a function of $E_0(p)$. The structure resembles the envelopes plotted in Figs. 5 and 8.

Finally, we shall present an example where we include quantum fluctuations of both the fermion field and the scalar field itself. In Fig. 15 we display the behavior of the amplitude for this combined system; the parameters are $M=1$, $g=2$, $\lambda=4$, and $\phi(0)=1$. The field is seen to relax efficiently. On the contrary, if the scalar fluctuations are not included, the relaxation induced by the fermionic fluctuations is small, as seen in Fig. 16. Figure 17 shows the relaxation for the case that only the scalar fluctuations are included; it is seen to start later and to be less efficient than for the combined system. The growth of the fermionic and bosonic energy density is plotted in Fig. 18. The fermionic energy density is smaller but rises earlier. Its asymptotic value is only by roughly 20% higher than in the purely fermionic evolution. The fermion fluctuations seemingly act

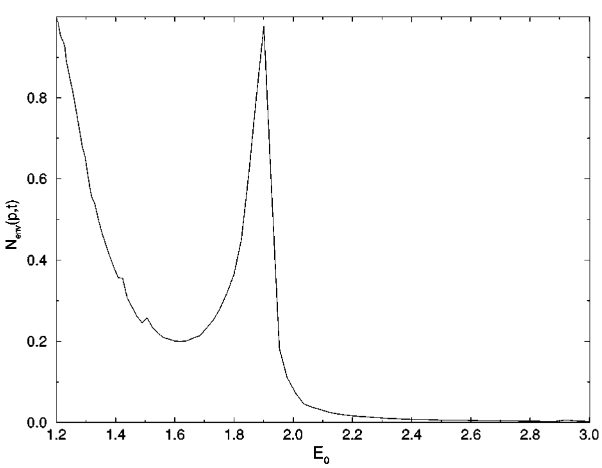


FIG. 5. Maximal occupation number $\mathcal{N}_{rmenv}$ as a function of $E_0$ for the same parameters as in Fig. 1.

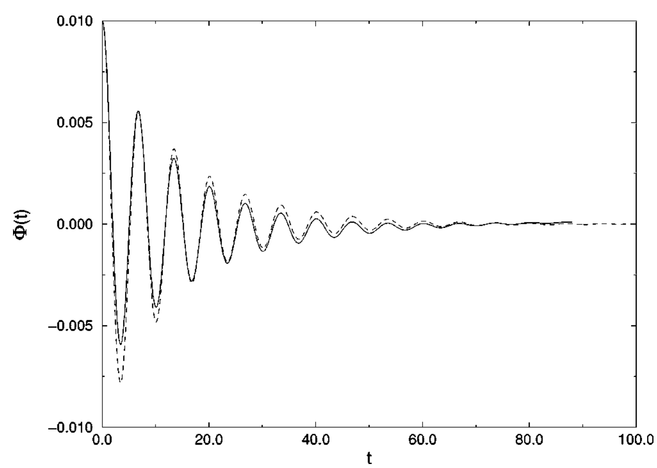


FIG. 6. Exact quantum evolution (dashed line) and linearized evolution (solid line) of $\phi(t)$ for $M=1$, $g=2$, $\lambda=1$, and $\phi(0)=0.01$, $\dot{\phi}(0)=0$.

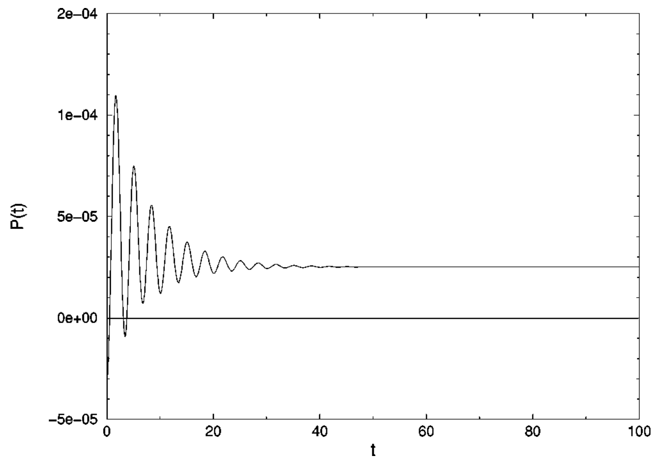


FIG. 7. Total pressure for the exact evolution for the same parameters as in Fig. 6.

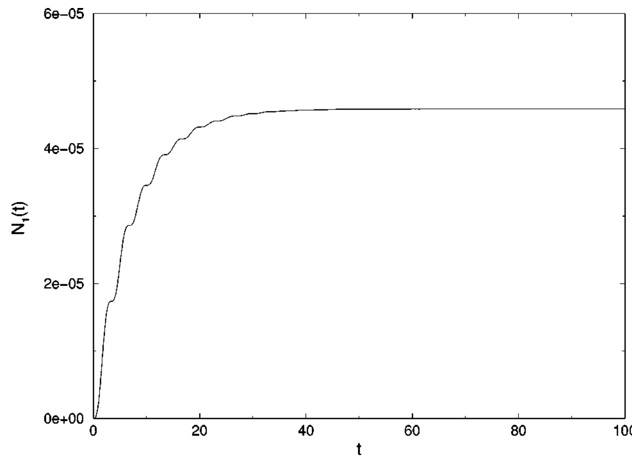


FIG. 9. Total particle number $\mathcal{N}_1(t)$ for the same case as in Fig. 6.

here as a kind of catalyzator, supporting the development of the bosonic quantum fluctuations.

## IX. CONCLUSION

We have developed the quantum field theory of the out-of-equilibrium evolution of fermionic quantum fluctuations driven by a scalar field. The quantum back reaction has been taken into account in the one-loop approximation. We have formulated the renormalization of the equations of motion and of the energy-momentum tensor in a covariant form and independent of the initial conditions. A restriction of suitable initial conditions for the fermionic quantum system, as required by the removal of initial singularities, has been obtained by selecting a Fock space built on a Bogoliubov-transformed vacuum state. Furthermore, we have formulated the renormalized equations for the case of a spatially flat FRW metric.

We have numerically implemented the evolution equations and we have presented some examples for the evolution of the quantum system. If the initial amplitude of the scalar field is very small, the system evolves as predicted for the linearized equations of motion, formulated in Sec. VII. It can be described as a decay of the scalar field into fermion-antifermion pairs. If the initial amplitude is larger, the evolution depends on the way in which a kind of resonance band at low momenta is situated kinematically and how it is filled. In some cases the fermions are indeed ineffective in damping the oscillation of the classical field; in others the relaxation develops as in the bosonic case and a considerable part of the energy is transferred to the quantum fluctuations. Analytical studies should help to clarify the features observed for large-amplitude oscillations. An example where bosonic as well as fermionic fluctuations are included shows an interesting interplay where the fermions catalyze the development of bosonic fluctuations.

We think that these results show that nonequilibrium systems with fermionic fluctuations show more interesting features and may play a more interesting role in cosmology than previously assumed.

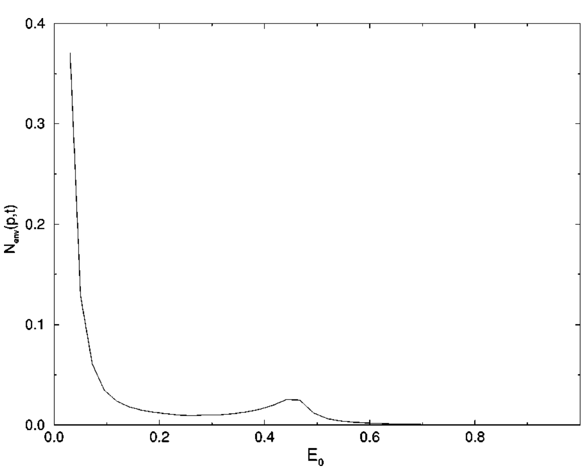


FIG. 8. Maximal occupation number $\mathcal{N}_{\rm env}$ as a function of $E_0$ for the same parameters as in Fig. 6.

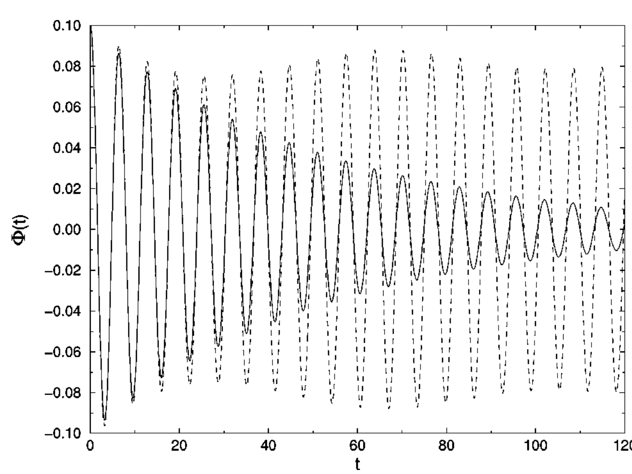


FIG. 10. Exact quantum evolution (dashed line) and linearized evolution (solid line) of $\phi(t)$ for $M=1$, $g=1$, $\lambda=0$, and $\phi(0)=0.1$, $\dot{\phi}(0)=0$.

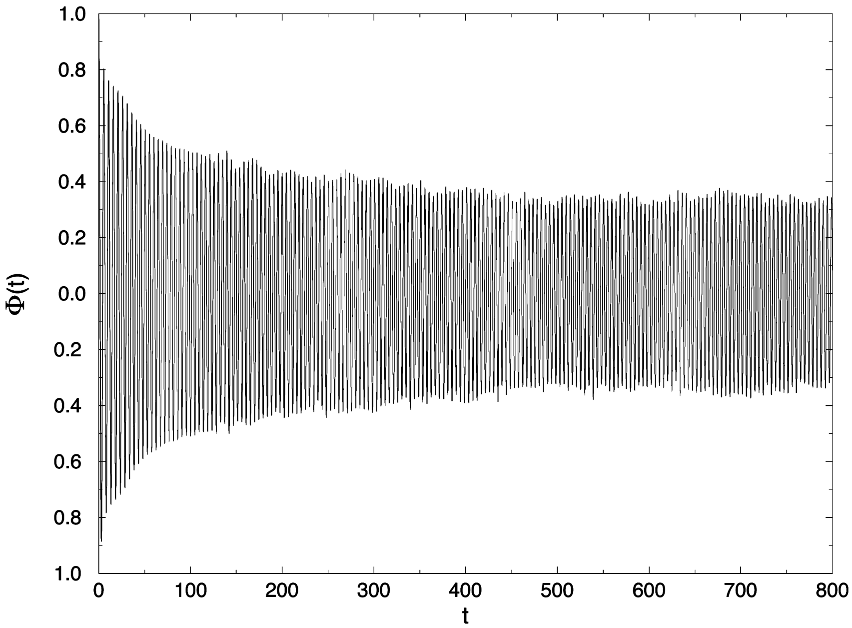


FIG. 11. $\phi(t)$ for $M=1$, $g=2$, $\lambda=1$, and $\phi(0)=2$, $\dot{\phi}(0)=0$.

## APPENDIX A: PERTURBATIVE EXPANSION OF THE MODE FUNCTIONS

We have introduced in Secs. II and III the mode functions $f_p(t)$ and $h_p(t)$ which are related via Eq. (4.4). In this appendix we will analyze the perturbation expansion and ultraviolet behavior of the functions $h_p(t)$. These mode functions satisfy the differential equation

$$\ddot{h}_p - 2iE_0\dot{h}_p = -V(t)[1+h_p], \tag{A1}$$

with the initial conditions $h_p(0)=\dot{h}_p(0)=0$. We expand $h_p$ with respect to orders in $V(t)$ by writing

$$h_p = h_p^{(1)} + h_p^{(2)} + h_p^{(3)} + \cdots, \tag{A2}$$

where $h_p^{(n)}(t)$ is of $n$th order in $V(t)$. Here $\overline{h_p^{(n)}}$ denotes the sum over all orders beginning with the $n$th one,

$$\overline{h_p^{(n)}} = \sum_{l=n}^{\infty} h_p^{(n)}, \tag{A3}$$

so that

$$h_p \equiv \overline{h_p^{(1)}} = h_p^{(1)} + \overline{h_p^{(2)}}. \tag{A4}$$

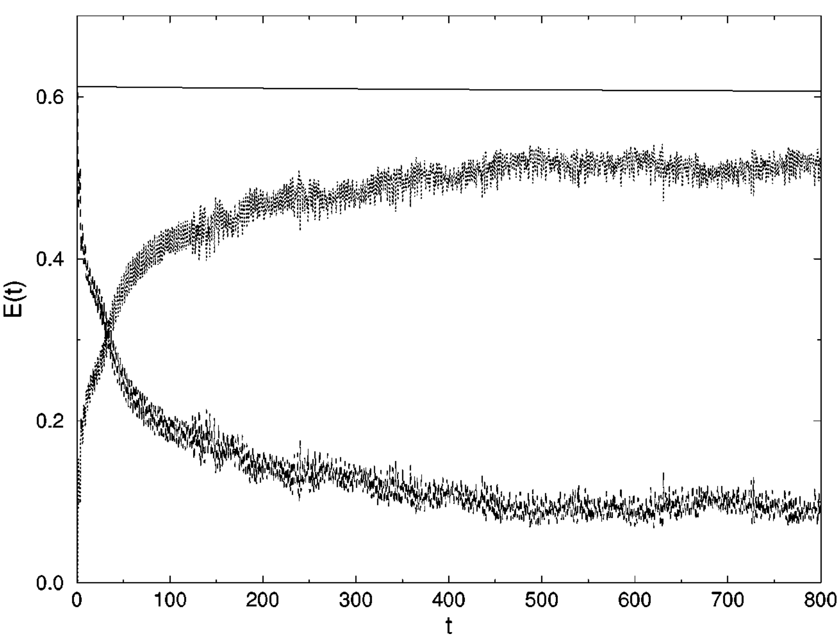


FIG. 12. Classical energy (dashed line), fluctuation energy (dotted line), and total energy (solid line) for the same parameters as in Fig. 11.

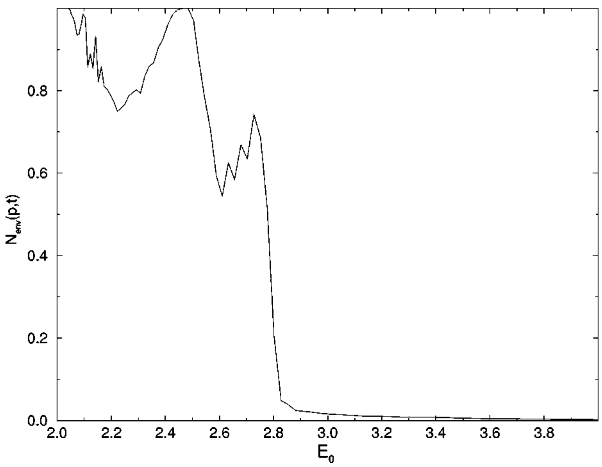


FIG. 13. Maximal occupation number $\mathcal{N}_{\rm env}$ as a function of $E_0$ for the same parameters as in Fig. 11.

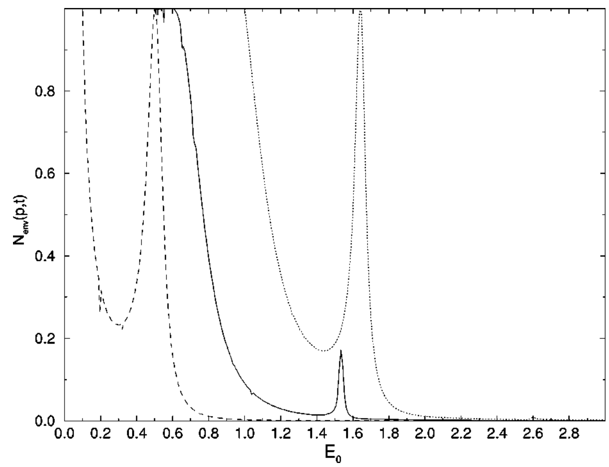


FIG. 14. Maximal occupation number $\mathcal{N}_{\rm env}$ for $m(t)=m_0\cos t$ with $m_0=0.1$ (dashed line), $m_0=0.5$ (solid line), and $m_0=1$ (dotted line), as a function of $E_0$.

The integral equation for the function $h_p$ can be derived in a straightforward way from the differential equation satisfied by the functions $f_p$; it reads

$$h_p=\frac{i}{2E_0}\int_0^t dt'(e^{2iE_0(t-t')}-1)V(t')[1+h_p(t')]. \tag{A5}$$

Using this integral equation we can obtain the functions $h_p^{(n)}(t)$ by iteration [33]. $h_p^{(1)}$ is given by

$$h_p^{(1)}=\frac{i}{2E_0}\int_0^t dt'(e^{2iE_0(t-t')}-1)V(t'). \tag{A6}$$

Using integrations by parts this function can be analyzed with respect to orders in $E_0$ via

$$h_p^{(1)}=\frac{-i}{2E_0}\int_0^t dt'V(t')+\sum_{l=0}^{n-1}\left(\frac{-i}{2E_0}\right)^{l+2}\times[V^{(l)}(t)-e^{2iE_0t}V^{(l)}(0)]+(h_p^{(1)})_{\overline{n}}, \tag{A7}$$

with

$$(h_p^{(1)})_{\overline{n}}=-\left(\frac{-i}{2E_0}\right)^{n+1}\int_0^t dt'e^{2iE_0(t-t')}V^{(n)}(t') \quad , \tag{A8}$$

Here $V^{(l)}(t)$ denotes the $l$th derivative of $V(t)$; the subscript $\overline{n}$ indicates that the expression in parentheses has been reduced to negative powers of $E_0$ equal or higher than $n$. For energy density and pressure we need the expansion of $\dot{h}_p^{(1)}\times(t)$ as well. From Eq. (A8) and the relation

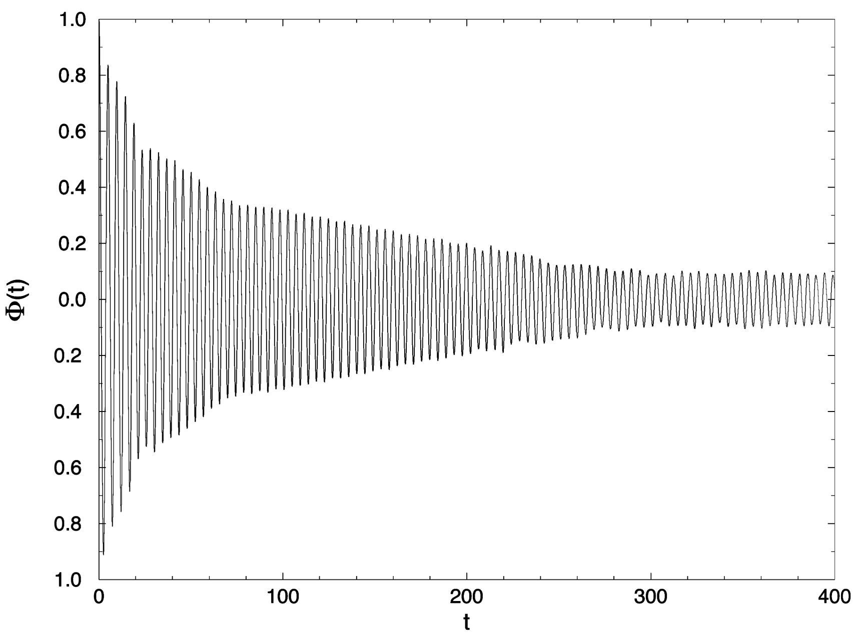


FIG. 15. $\phi(t)$ including back reaction of both fermionic and scalar fluctuations for $M=1$, $g=2$, $\lambda=4$, and $\phi(0)=1$, $\dot{\phi}(0)=0$.

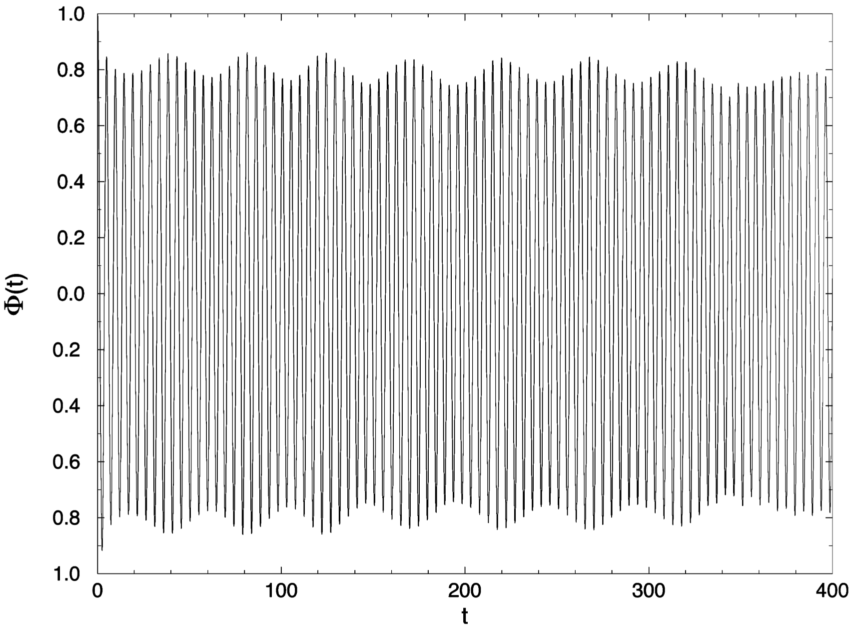


FIG. 16. $\phi(t)$ in the absence of scalar fluctuations for the same parameters as in Fig. 15.

$$\dot{h}_p^{(1)} = 2iE_0 h_p^{(1)} - \int_0^t dt' \, V(t'), \tag{A9}$$

we find

$$\dot{h}_p^{(1)} = \sum_{l=0}^{n} \left( \frac{-i}{2E_0} \right)^{l+1} [V^{(l)}(t) - e^{2iE_0 t} V^{(l)}(0)] - \left( \frac{-i}{2E_0} \right)^{n+1} \int_0^t dt' \, e^{2iE_0(t-t')} V^{(n+1)}(t') \quad . \tag{A10}$$

Similar expressions hold for the higher $h_p^{(n)}$ and $\dot{h}_p^{(n)}$.

In the numerical implementation the functions $(h_p^{(1)})_n^-$ can be obtained as the Fourier transform of the $n$th derivative of $V(t)$. Its computation needs just one update per time step. Alternatively one may construct differential equations satisfied by these functions,[3] e.g.,

$$(\ddot{h}_p^{(1)})_2^- - 2iE_0 (\dot{h}_p^{(1)})_2^- = \frac{i}{2E_0} \dot{V}. \tag{A11}$$

In order to isolate the divergent terms in the fluctuation integrals of the equation of motion and of the energy-momentum tensor we need an expansion of the mode function up to order $O[(E_0)^{-3}]$ and $O[(E_0)^{-4}]$, respectively. For this reason we will give relevant expansions of the $h_p$ in the following.

The expansion of $h_p^{(1)}$ up to $O[(E_0)^{-3}]$ gives

$$h_p^{(1)} = -\frac{i}{2E_0} \int_0^t dt' \, V(t') - \frac{V(t)}{4(E_0)^2} + \frac{i\dot{V}(t)}{8(E_0)^3} - \frac{i\dot{V}(0)}{8(E_0)^3} e^{2iE_0 t} - \frac{i}{8(E_0)^3} \int_0^t dt' \, e^{2iE_0(t-t')} \ddot{V}(t). \tag{A12}$$

For $h_p^{(2)}$ we obtain

$$h_p^{(2)} = -\frac{1}{8(E_0)^2} \left[ \int_0^t dt' \, V(t') \right]^2 + \frac{i}{8(E_0)^3} \int_0^t dt' \, V^2(t') + \frac{i}{8(E_0)^3} V(t) \int_0^t dt' \, V(t') + (h_p^{(2)})_4^-, \tag{A13}$$

where $(h_p^{(2)})_4^-$ includes all terms of $h_p^{(2)}$ that have at least four negative powers of $E_0$. It satisfies the differential equation

$$(\ddot{h}_p^{(2)})_4^- - 2iE_0 (\dot{h}_p^{(2)})_4^- = -V(t) (h_p^{(1)})_3^- - \frac{5i}{8(E_0)^3} \dot{V}(t) V(t) - \frac{i}{8(E_0)^3} \ddot{V}(t) \int_0^t dt' \, V(t') \quad . \tag{A14}$$

Finally, via integration by parts, $h_p^{(3)}$ takes the form

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[3]This had also been suggested by Boyanovsky [43].

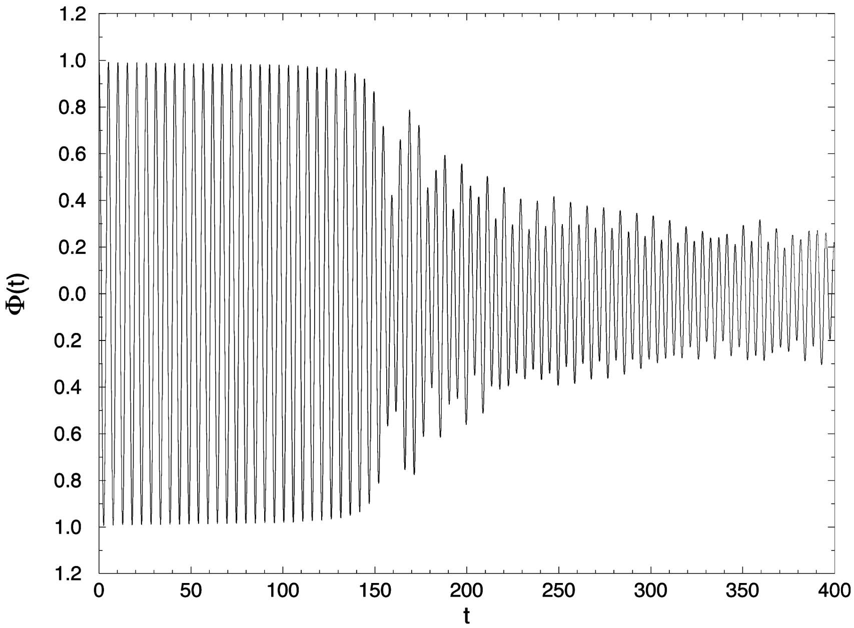


FIG. 17. $\phi(t)$ in the absence of fermion fluctuations, for the same parameters as in Fig. 15.

$$h_p^{(3)}=\frac{i}{48(E_0)^3}\left[\int_0^t dt'\,V(t')\right]^3+(h_p^{(3)})_{\overline{4}},\qquad\text{(A15)}$$

where the last term satifies the differential equation

$$(\ddot{h}_p^{(3)})_{\overline{4}}-2iE_0(\dot{h}_p^{(3)})_{\overline{4}}=-V(t)(h_p^{(2)})_{\overline{3}}-\frac{i\dot{V}(t)}{16(E_0)^3}\int_0^t dt'\,V(t')-\frac{iV^2(t)}{8(E_0)^3}\int_0^t dt'\,V(t').\qquad\text{(A16)}$$

For the integrands of the fluctuation integral we need repeatedly the two expressions

$$\mathcal{I}_1(p,t)=2\,\mathrm{Re}\;\overline{h_p^{(1)}}+\overline{|h_p^{(1)}|^2}\qquad\text{(A17)}$$

and

$$\mathcal{I}_2(p,t)=2\,\mathrm{Im}\;\overline{\dot{h}_p^{(1)}}+2\,\mathrm{Im}\;\overline{h_p^{(1)*}\dot{h}_p^{(1)}}.\qquad\text{(A18)}$$

Using the expansion up to $O[(E_0)^{-3}]$ we obtain

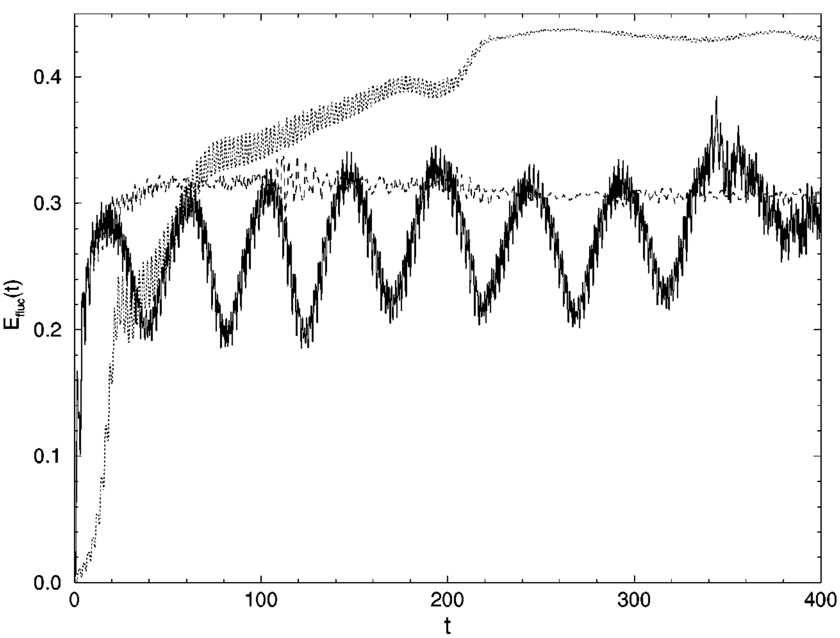


FIG. 18. Fluctuation energies of fermions for evolution without scalar fluctuations (solid line) and fluctuation energies of fermions (dashed line) and bosons (dotted line) for combined evolution.

$$\mathcal{I}_1(p,t)=\frac{1}{E_0}\int_0^t dt'\,\mathrm{Im}\,V(t')-\frac{\mathrm{Re}\,V(t)}{2(E_0)^2}+\frac{1}{2(E_0)^2}\left[\int_0^t dt'\,\mathrm{Im}\;V(t')\right]^2-\frac{\mathrm{Im}\,\dot V(t)}{4(E_0)^3}+\frac{\mathrm{Im}\;\dot V(0)}{4(E_0)^3}\cos(2E_0t)$$
$$+\frac{\mathrm{Re}\,\dot V(0)}{4(E_0)^3}\sin(2E_0t)-\frac{1}{4(E_0)^3}\int_0^t dt'\,\mathrm{Im}\;V^2(t')+\frac{1}{6(E_0)^3}\left[\int_0^t dt'\,\mathrm{Im}\;V(t')\right]^3$$
$$-\frac{1}{2(E_0)^3}\mathrm{Re}\;V(t)\int_0^t dt'\,\mathrm{Im}\,V(t')+2\mathrm{Re}\,(\overline{h_p^{(1)})_{\bar 4}}+(\overline{|h_p^{(1)}|^2})_{\bar 4},\tag{A19}$$

where we have introduced the terms of $O[(E_0)^{-4}]$ as

$$2\mathrm{Re}\;(\overline{h_p^{(1)})_{\bar 4}}=2\mathrm{Re}\;(h_p^{(1)}+h_p^{(2)}+h_p^{(3)})_{\bar 4}+2\mathrm{Re}\;\overline{h_p^{(4)}}\tag{A20}$$

and

$$(\overline{|h_p^{(1)}|^2})_{\bar 4}=(|h_p^{(1)}|^2)_{\bar 4}+2\mathrm{Re}\;(h_p^{(1)}h_p^{(2)*})_{\bar 4}+\overline{|h_p^{(2)}|^2}+2\mathrm{Re}\;(\overline{h_p^{(1)}h_p^{(3)*}}).\tag{A21}$$

Inserting the real and imaginary parts of the potential $V(t)$ we finally have the result

$$\mathcal{I}_1(p,t)=-\frac{1}{E_0}[m(t)-m(0)]-\frac{1}{(E_0)^2}[m(t)m(0)-m^2(0)]+\frac{1}{(E_0)^3}\Bigg[\frac{1}{2}m^3(t)+m^3(0)-\frac{3}{2}m(t)m^2(0)$$
$$-\frac{1}{4}\ddot m(0)\cos(2E_0t)+\frac{1}{2}\dot m(0)m_0\sin(2E_0t)\Bigg]+2\mathrm{Re}\;(\overline{h_p^{(1)})_{\bar 4}}+(\overline{|h_p^{(1)}|^2})_{\bar 4},\tag{A22}$$

$$\mathcal{I}_2(p,t)=-\frac{\mathrm{Re}\,V(t)}{E_0}-\frac{\mathrm{Im}\,\dot V(t)}{2(E_0)^2}+\frac{\mathrm{Im}\,\dot V(0)}{(2E_0)^2}\cos(2E_0t)+\frac{\mathrm{Re}\;\dot V(t)}{2(E_0)^2}\sin(2E_0t)+\frac{\mathrm{Re}\;\ddot V(t)}{4(E_0)^3}-\frac{\mathrm{Re}\;\ddot V(0)}{4(E_0)^3}\cos(2E_0t)$$
$$+\frac{\mathrm{Im}\;\ddot V(0)}{4(E_0)^3}\sin(2E_0t)+\frac{1}{(E_0)^2}\mathrm{Re}\;V(t)\int_0^t dt'\,\mathrm{Im}\;V(t')+\frac{3[\mathrm{Re}\;V(t)]^2}{4(E_0)^3}-\frac{[\mathrm{Im}\;V(t)]^2}{4(E_0)^3}-\frac{\mathrm{Im}\;\dot V(t)}{2(E_0)^3}\int_0^t dt'\,\mathrm{Im}\;V(t')$$
$$-\frac{\mathrm{Re}\;V(t)}{2(E_0)^3}\left[\int_0^t dt'\,\mathrm{Im}\;V(t')\right]^2+\frac{\mathrm{Re}\;\dot V(0)}{2(E_0)^3}\cos(2E_0t)\int_0^t dt'\,\mathrm{Re}\;V(t')-\frac{\mathrm{Im}\;\dot V(0)}{2(E_0)^3}\sin(2E_0t)\int_0^t dt'\,\mathrm{Re}\;V(t')$$
$$+2\mathrm{Im}\;(\overline{\dot h_p^{(1)})_{\bar 4}}+2\mathrm{Im}\;(\overline{h_p^{*(1)}\dot h_p^{(1)})_{\bar 4}}.\tag{A23}$$

For completeness we also give the expansion up to $O[(E_0)^{-4}]$:

$$\mathcal{I}_1(p,t)=\frac{1}{E_0}\int_0^t dt'\,\mathrm{Im}\;V(t')-\frac{\mathrm{Re}\;V(t)}{2(E_0)^2}+\frac{1}{2(E_0)^2}\left[\int_0^t dt'\,\mathrm{Im}\;V(t')\right]^2-\frac{\mathrm{Im}\;\dot V(t)}{4(E_0)^3}+\frac{\mathrm{Im}\;\dot V(0)}{4(E_0)^3}\cos(2E_0t)$$
$$+\frac{\mathrm{Re}\;\dot V(0)}{4(E_0)^3}\sin(2E_0t)-\frac{1}{4(E_0)^3}\int_0^t dt'\,\mathrm{Im}\;V^2(t')+\frac{1}{6(E_0)^3}\left[\int_0^t dt'\,\mathrm{Im}\;V(t')\right]^3-\frac{1}{2(E_0)^3}\mathrm{Re}\;V(t)\int_0^t dt'\,\mathrm{Im}\;V(t')$$
$$+\frac{3[\mathrm{Re}\;V(t)]^2}{8(E_0)^4}-\frac{[\mathrm{Im}\;V(t)]^2}{4(E_0)^4}-\frac{\mathrm{Im}\;\dot V(t')}{4(E_0)^4}\int_0^t dt'\,\mathrm{Im}\;V(t')-\frac{1}{4(E_0)^4}\int_0^t dt'\,\mathrm{Im}\;V(t')\int_0^t dt'\,\mathrm{Im}\;V^2(t')$$
$$-\frac{1}{4(E_0)^4}\mathrm{Re}\;V(t)\left[\int_0^t dt'\,\mathrm{Im}\;V(t')\right]^2+\frac{1}{24(E_0)^4}\left[\int_0^t dt'\,\mathrm{Im}\;V(t')\right]^4+\frac{\mathrm{Re}\;\dot V(0)}{4(E_0)^4}\cos(2E_0t)\int_0^t dt'\,\mathrm{Re}\;V(t')$$
$$+\frac{\mathrm{Re}\;\ddot V(t)}{8(E_0)^4}-\frac{1}{4(E_0)^4}\mathrm{Im}\;\dot V(0)\sin(2E_0t)\int_0^t dt\,\mathrm{Re}\;V(t')-\frac{\mathrm{Re}\;\ddot V(0)}{8(E_0)^4}\cos(2E_0t)+\frac{\mathrm{Im}\;\ddot V(0)}{8(E_0)^4}\sin(2E_0t)$$
$$+2\mathrm{Re}\;(\overline{h_p^{(1)})_{\bar 5}}+(\overline{|h_p^{(1)}|^2})_{\bar 5},\tag{A24}$$

where

$$2\,\mathrm{Re}\ \overline{(h_p^{(1)})_{\bar{5}}} = 2\,\mathrm{Re}\ \overline{(h_p^{(1)}+h_p^{(2)}+h_p^{(3)}+h_p^{(4)})_{\bar{5}}} + 2\,\mathrm{Re}\ \overline{h_p^{(5)}}. \tag{A25}$$

In terms of the notation introduced above the alternative representation for the functions $K_{\rm F}, K_{\rm E}$, and $K_{\rm P}$ become now

$$K_{\rm F}(p,t) = -[\mathcal{I}_1(p,t)]_{\bar{4}} + \frac{m_0}{E_0}[\mathcal{I}_1(p,t)]_{\bar{3}}, \tag{A26}$$

$$K_{\rm E}(p,t) = -E_0[\mathcal{I}_1(p,t)]_{\bar{5}} + [\mathcal{I}_2(p,t)]_{\bar{4}} + m_0[\mathcal{I}_1(p,t)]_{\bar{4}} - \frac{m_0}{E_0}[\mathcal{I}_2(p,t)]_{\bar{3}}, \tag{A27}$$

$$K_{\rm P}(p,t) = -\frac{4}{3}E_0[\mathcal{I}_1(p,t)]_{\bar{5}} + \left(\frac{4}{3}m_0 - \frac{1}{3}m(t)\right)[\mathcal{I}_1(p,t)]_{\bar{4}} + \frac{m(t)m_0}{3E_0}[\mathcal{I}_1(p,t)]_{\bar{3}} + \frac{4}{3}[\mathcal{I}_2(p,t)]_{\bar{4}} - \frac{4m_0}{3E_0}[\mathcal{I}_2(p,t)]_{\bar{3}}\ . \tag{A28}$$

Unlike the case of scalar fluctuations [33], the evaluation of these expressions has required extensive algebra, and the numerical implementation requires further efforts. This reflects the fact, that fermion loops are divergent up graphs with four external lines. The definition of $K_{\rm F}, K_{\rm E}$, and $K_{\rm P}$ given in Sec. IV is much easier to handle; it involves, however, potentially dangerous differences between expressions which are computed numerically and leading perturbative terms. We found in our numerical computations that the simple subtraction was tolerable; this is due, here, to the fact that most integrals are dominated by the low momentum region. The analysis presented in this appendix was necessary in any case, however, in order to find the divergent contributions.

## APPENDIX B: BOGOLIUBOV TRANSFORMATION

Here we recall the basic formulas for the Bogoliubov transformation of spin-1/2 fields (see e.g., [42]). A general canonical transformation compatible with the anticommutation relations is given, up to a further trivial phase transformation, by

$$b_{\mathbf{p},s} = \cos(\beta_{\mathbf{p},s})\tilde{b}_{\mathbf{p},s} + \sin(\beta_{\mathbf{p},s})e^{i\delta_{\mathbf{p},s}}\tilde{d}^\dagger_{-\mathbf{p},s} \tag{B1}$$

$$d^\dagger_{-\mathbf{p},s} = -\sin(\beta_{\mathbf{p},s})e^{-i\delta_{\mathbf{p},s}}\tilde{b}_{\mathbf{p},s} + \cos(\beta_{\mathbf{p},s})\tilde{d}^\dagger_{-\mathbf{p},s}\,. \tag{B2}$$

The state annihilated by the new operators $\tilde{b}_{\mathbf{p},s}$ and $\tilde{d}_{\mathbf{p},s}$ is given in discrete notation by

$$|\tilde{0}\rangle = \prod_{\mathbf{p},s}\left\{\cos(\beta_{\mathbf{p},s}) + \frac{1}{2E_0(p)V}\sin(\beta_{\mathbf{p},s})e^{-i\delta_{\mathbf{p},s}}d^\dagger_{-\mathbf{p},s}b^\dagger_{\mathbf{p},s}\right\}|0\rangle. \tag{B3}$$

For the expectation values of the various bilinear products of interest here one finds, in continuum notation,

$$\langle\tilde{0}|b^\dagger_{\mathbf{p},s}b_{\mathbf{p}',s'}|\tilde{0}\rangle = (2\pi)^3 2E_0\delta_{ss'}\delta^3(\mathbf{p}-\mathbf{p}')\sin^2\beta_{\mathbf{p},s}\,, \tag{B4}$$

$$\langle\tilde{0}|d_{-\mathbf{p},s}d^\dagger_{-\mathbf{p}',s'}|\tilde{0}\rangle = (2\pi)^3 2E_0\delta_{ss'}\delta^3(\mathbf{p}-\mathbf{p}')\cos^2\beta_{\mathbf{p},s}\,, \tag{B5}$$

$$\langle\tilde{0}|d_{-\mathbf{p},s}b_{\mathbf{p}',s'}|\tilde{0}\rangle = (2\pi)^3 2E_0\delta_{ss'}\delta^3(\mathbf{p}-\mathbf{p}') \times e^{i\delta_{\mathbf{p},s}}\cos\beta_{\mathbf{p},s}\sin\beta_{\mathbf{p},s}\,, \tag{B6}$$

$$\langle\tilde{0}|b^\dagger_{\mathbf{p},s}d^\dagger_{-\mathbf{p}',s'}|\tilde{0}\rangle = (2\pi)^3 2E_0\delta_{ss'}\delta^3(\mathbf{p}-\mathbf{p}') \times e^{-i\delta_{\mathbf{p},s}}\cos\beta_{\mathbf{p},s}\sin\beta_{\mathbf{p},s}\,, \tag{B7}$$

so that the fluctuation integral becomes

$$\tilde{\mathcal{F}}(t) = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_0}\{\bar{U}_{\mathbf{p},s}U_{\mathbf{p},s}\sin^2\beta_{\mathbf{p},s} + \bar{V}_{-\mathbf{p},s}V_{\mathbf{p},s}\cos^2\beta_{\mathbf{p},s} + \bar{V}_{-\mathbf{p},s}U_{\mathbf{p},s}e^{i\delta_{\mathbf{p},s}}\cos\beta_{\mathbf{p},s}\sin\beta_{\mathbf{p},s} + \bar{U}_{\mathbf{p},s}V_{-\mathbf{p},s}e^{-i\delta_{\mathbf{p},s}}\cos\beta_{\mathbf{p},s}\sin\beta_{\mathbf{p},s}\}. \tag{B8}$$

This corresponds to replacing

$$V_{-\mathbf{p},s}(t) \Rightarrow \cos(\beta_{\mathbf{p},s})V_{-\mathbf{p},s}(t) + e^{i\delta_{\mathbf{p},s}}\sin(\beta_{\mathbf{p},s})U_{\mathbf{p},s}(t) \tag{B9}$$

in the original expression (2.25). With the same substitution one obtains the expectation values for the energy and pressure. For the particle number the substitution is made in the Bogoliubov coefficients, Eqs. (3.9) and (3.15). As in the case of scalar fluctuations [37], we do not need the phase $\delta_{\mathbf{p},s}$ as long as we restrict the initial condition for the classical field to $\dot{\phi}(0)=0$.

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