Real-time relaxation and kinetics in hot scalar QED: Landau damping

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(Received 23 February 1998; revised manuscript received 6 August 1998; published 12 November 1998)

The real time evolution of non-equilibrium expectation values with soft length scales $\sim k^{-1} > (eT)^{-1}$ is solved in hot scalar electrodynamics, with a view towards understanding relaxational phenomena in the QGP and the electroweak plasma. We find that the gauge invariant non-equilibrium expectation values relax via *power laws* to asymptotic amplitudes that are determined by the quasiparticle poles. The long time relaxational dynamics and relevant time scales are determined by the behavior of the retarded self-energy not at the small frequencies, but at the Landau damping thresholds. This explains the presence of power laws and not of exponential decay. In the process we rederive the HTL effective action using *non-equilibrium* field theory. Furthermore we obtain the influence functional, the Langevin equation and the fluctuation-dissipation theorem for the soft modes, identifying the correlators that emerge in the classical limit. We show that a Markovian approximation fails to describe the dynamics *both* at short and long times. We find that the distribution function for soft quasiparticles relaxes with a power law through Landau damping. We also introduce a novel kinetic approach that goes beyond the standard Boltzmann equation by incorporating off-shell processes and find that the distribution function for soft quasiparticles relaxes with a power law through Landau damping. We find an unusual dressing dynamics of bare particles and anomalous (logarithmic) relaxation of hard quasiparticles. [S0556-2821(98)03722-9]

PACS number(s): 11.10.Wx, 12.38.Mh

I. INTRODUCTION

There is currently a great deal of interest in understanding non-perturbative real time dynamics in gauge theories at high temperature, both within the realm of heavy ion collisions and the study of the quark gluon plasma [1–8], as well as the possibility for anomalous baryon number violation in the electroweak theory [9,10]. In both situations the dynamics of soft gauge fields with typical length scales $>(gT)^{-1}$ is non-perturbative.

Their treatment requires a resummation scheme where one can consistently integrate out the hard scales associated with momenta $\approx T$ to obtain an effective theory for the soft scales. This is the program of resummation of hard thermal loops [11–15]. Physically, the hard scale represents the typical energy of a particle in the plasma while the soft scale is associated with collective excitations [16].

The recognition of the non-perturbative physics associated with soft degrees of freedom has led to an effort to describe the dynamics by implementing numerical simulations of *classical* gauge theories [17–24] since soft degrees of freedom have very large occupation numbers and could in principle be treated classically [20]. Effective classical descriptions for the infrared bosonic modes have been obtained consistently in scalar field theory by integrating out the hard modes [25]. However, it was recognized that the dynamics of the soft modes in gauge theories is sensitive to the hard modes [26–29] and that the Rayleigh-Jeans divergences associated with the hard modes provide non-trivial contributions to the soft dynamics.

For example, the one-loop correction to the gauge boson self-energy contains a leading contribution from the hard momenta of order g^2T^2 which gives the hard thermal loop (HTL) contribution, and a subleading contribution from the soft scales. The T^2 dependence is a reflection of the UV quadratic divergence of the zero temperature theory which is cutoff at momentum scales $>T$ by the Bose-Einstein factor. Restoring the appropriate $\hbar's$ we see that this contribution is of order $\hbar(g^2T^2/\hbar^2)$ where the first $\hbar$ is associated with the loop and the denominator follows from the usual manner in which $\hbar$ enters with temperature. This term is therefore $\mathcal{O}(T^2/\hbar)$ and reveals the usual Rayleigh-Jeans divergence. For hard external momenta ($K_{ext}\sim T$) this one-loop correction to the propagator is obviously subleading and bare perturbation theory is valid. But when the external momenta are soft ($K_{ext}\sim gT$), clearly the one-loop correction is of the same order as the tree level term. This is in fact at the heart of the breakdown of the perturbative expansion. The problem is resolved by using HTL-resummed propagators and vertices for the soft external lines while hard scales may always be treated within the usual perturbation theory. This procedure is akin to obtaining a Wilsonian effective action for the soft modes by integrating out all the momenta above a certain soft scale which in this case is gT. For a detailed discussion of the relevant issues we refer the interested reader to the original works of Braaten and Pisarski [11,12].

All of the HTL contributions may be divided into two distinct categories: (i) the contributions from tadpole diagrams and (ii) those from diagrams with discontinuities. While the tadpole contributions are independent of the external momenta, the diagrams with discontinuities lead to momentum dependent terms and it is these that lead to the non-local effective HTL Lagrangian. The non-locality of the HTL effective Lagrangian for the soft modes originates in the process of Landau damping which results in discontinuities below the light cone [11,13,30–32]. A very interesting method to deal with the non-locality in the HTL effective action in numerical simulations of classical gauge fields has been recently proposed [33] and is based on the particle method akin to that used in transport theory. This method has been used to study the diffusion of Chern-Simons number in a lattice approach [34]. A local Hamiltonian approach that is intrinsically gauge invariant has also been recently proposed [35] and has the potential for numerical implementation. A proposal to study the classical dynamics of soft gauge fields in terms of an effective Langevin equation has been put forth in [29]. However in our view such a proposal does not seem to incorporate consistently the non-Markovian nature of the noise that is a result of the non-localities associated with Landau damping.

The focus of this article is precisely to study in detail the real-time dynamics of the evolution of gauge fields, given an arbitrary field condensate in the initial state. This study will reveal that *Landau damping processes* dominate the most relevant aspects of the dynamics and as argued above, determine the non-local aspects of the HTL effective action. The main goal of this investigation is: (i) to provide a deeper understanding of the time scales associated with dissipative *off-shell* processes, (ii) a consistent microscopic description that can be used as a yardstick to test lattice results on real-time correlation functions, and (iii) a detailed real-time description of relaxation and kinetics of soft collective excitations in gauge theories.

Landau damping [7,11–13] occurs when a hard quasiparticle from the thermal bath (with momentum $\sim T$) scatters off a soft collective mode (momentum $\sim gT$), borrowing energy from (and damping) the soft excitations in the process. Simple kinematics dictates that these processes can occur only *off shell* and below the light-cone i.e. for spacelike four-momentum. Furthermore, Landau damping gives a non-zero contribution only in the presence of a heat bath and when the external momentum is non-zero. Phrased differently, the Landau discontinuities are *purely thermal cuts* arising only at non-zero temperature and lead to damping of *spatially inhomogeneous* field configurations only. Whereas the real-time dynamics of similar processes has been studied in a scalar theory [36], such a study is lacking for the case of gauge fields. The gauge boson self-energy in the presence of these processes has been known for a long time [30–32] and can be computed in the imaginary time formalism of finite temperature field theory in the HTL limit [37].

The main focus and goals of these article are:

To compute explicitly the *real time evolution* of inhomogeneous field configurations (non-equilibrium field expectation values) in the ultrarelativistic plasma as an *initial value problem*. We linearize the field equations of motion in the condensate amplitude. In this weak field regime, the evolution equations for the condensate can be solved in closed form through Laplace transform. The analytic structure of the propagator in momentum space (s-plane) determines the real time behavior of the solution. We obtain the propagator to one-loop order in the HTL approximation. In this approximation the long-time behavior of the condensate turns out to be governed by the Landau discontinuities resulting in Landau damping processes. Although the HTL corrections to the gauge boson self-energy are well-known, we believe that the calculation of the *real time* dependence of the damping of soft excitations is new.

While an understanding of the real-time relaxation of non-equilibrium, inhomogeneous field configurations is of fundamental importance in the physics of relaxation in the QGP [7], such a calculation also has phenomenological implications for sphaleron induced B-violating processes. It has been recently pointed out [27–29] that standard estimates of the topological transition rate at finite temperature in the electroweak theory ignore the effects of damping in the thermal bath and these authors have argued that Landau damping plays a very important role. Since the sphaleron is an inhomogeneous excitation associated with a soft length scale ($\sim 1/g^2T$) Landau damping effects and the full HTL-resummed propagators must necessarily be taken into account when studying the sphaleron damping rates [27–29].

One of our main goals is to assess in detail the real time non-equilibrium dynamics of soft excitations in the plasma with particular attention to a critical analysis of the long-standing belief that the small frequency region of the spectral function dominates the long time relaxational dynamics. We find, to the contrary that the Landau damping *thresholds* at $\omega = \pm k$ determine the long- time dynamics and that the early time dynamics is sensitive to several moments of the total spectral density. This is an important point that bears on recent arguments that seek to clarify the damping effects on the sphaleron rate [27–29]. We analyze this novel result in detail both analytically and numerically, thus proving that the long time behavior is dominated by the Landau damping *thresholds* and that the small frequency region gives rise to sub-leading corrections to the long-time dynamics in the leading order HTL approximation.

In this article we concentrate on the case of scalar electrodynamics (SQED) since this theory has the same HTL structure (to lowest order) as the non-Abelian case [37,38]. Most of our results can therefore, be taken over to the non-Abelian case with little or no changes at least in the lowest order HTL approximation. Scalar electrodynamics has already been used as an example to study the HTL resummation for the infrared modes [38], but the scope of this article is different in that we study explicitly the real time dynamics of the damping processes.

After resummation of the one-loop HTL contributions, at long times the relaxation of either transverse or longitudinal field expectation values is given by two contributions. The first is from the quasiparticle modes, and is standard—an oscillatory function in time. The second contribution arises from branch point singularities in the HTL self energies at nonzero frequency. These produce correlations in time which are oscillatory times power law tails; these power law tails are a new feature of HTL's. Long time power law tails in current-current correlators have been recently reported in Ref. [28].

Furthermore we obtain consistently the effective Langevin equation for soft modes by integrating out the scalar fields and obtaining the influence functional for the gauge invariant observables. This allows us to extract the noise correlation function that displays all of the non-localities associated with HTLs and Landau damping. By deriving the relevant fluctuation-dissipation relation we identify the proper correlation function that emerges in the classical limit. This analysis reveals that the range of both the dissipation kernel as well as the noise correlation function are determined by the *soft* scale. This results in the kernels being long-ranged, typically falling off with a power of time and with no Markovian limit. Again this result is deeply related to the non-localities of the HTL effective action and prevents a local description of relaxational dynamics associated with Landau damping.

Our detailed analysis and the consistent formulation of the influence functional starting from the microscopic Lagrangian, unequivocally leads us to the conclusion that an effective stochastic Langevin description of gauge field relaxation to leading order in the HTL limit is *non-Markovian*. Because of the long-range kernels associated with Landau damping, a Markovian limit cannot be consistently extracted. This result points out the limited utility of a Langevin equation for describing relaxation via Landau damping.

Having established the relaxation of inhomogenous gauge field configurations via off-shell processes associated with Landau damping, we ask how these processes contribute to the relaxation of the distribution function of transverse degrees of freedom. Clearly the evolution of the distribution function cannot be described to this order by a Boltzmann equation, since this kinetic approach only includes on-shell processes. Thus we provide one of the novel results of this work: we incorporate the non-equilibrium relaxation effects from Landau damping into a kinetic equation that describes the relaxation of the occupation number of transverse gauge fields. This kinetic equation incorporates *off-shell* effects and therefore constitutes an advance over the usual Boltzmann kinetic description in terms of completed collisions. We argue that since Landau damping results in an exchange of energy (and momentum) between the quasiparticles in the bath and the out-of-equilibrium field configuration, this in turn will naturally lead to a depletion of the particle number from the field configuration and must necessarily be included in any accurate kinetic description which aims to probe relaxational phenomena on the relevant time scales. A framework to study these transient, off-shell relaxational phenomena as initial value problems is given in Refs. [40–42]. The analysis presented in this article implies a Dyson-like resummation that goes far beyond Boltzmann kinetics.

We compare the results from these new kinetic equations to several different approximations to the kinetics. In comparing the relaxation of dressed soft quasiparticles and that of bare particles and hard quasiparticles, we find a remarkable dressing dynamics of the degrees of freedom in the medium, and anomalous relaxation for the hard quasiparticles. We also argue that a proper kinetic description must incorporate consistently the HTL effects and the quasiparticle nature of the excitations.

In Sec. II we introduce the model under study and focus on a gauge invariant description of the dynamics. Section III is devoted to a derivation of the non-equilibrium equation of motion for the inhomogeneous condensate to one-loop order in the hard thermal approximation. The real time relaxational dynamics of the inhomogeneous configuration via Landau damping is investigated in detail in Sec. IV. Contact is established with the fluctuation-dissipation theorem and stochastic dynamics in Sec. V, where we derive a Langevin description for the soft gauge invariant degrees of freedom in the thermal bath and recognize the relevant correlation functions that emerge in a (semi) classical stochastic description. We introduce a new kinetic description of transport phenomena induced by the non-collisional Landau damping process in Sec. VI. In this section we study the relaxation of the distribution function for soft quasiparticles, bare particles and mention some interesting features of the relaxation for hard quasiparticles. Finally, we summarize our analysis, discuss the modifications that will arise when higher order corrections leading to collisional lifetimes are included and present our conclusions and possible future directions of study.

II. PRELIMINARIES

As mentioned in the Introduction, our ultimate goal is to understand relaxational processes associated with off-shell effects, such as Landau damping, in a non-Abelian gauge theory [7]. What we will do here is to treat the same problem in the context of scalar quantum electrodynamics (SQED) model. To leading order, we expect that this should be a good analogue of what happens in the non-Abelian case; it also has the advantage that it is simpler to deal with, and as we will see below, it can be cast from the outset in terms of *gauge invariant* variables. This will eliminate any ambiguities associated with the usual problem of gauge dependence of off-shell quantities.

We will start with some inhomogeneous field configuration which is excited in the SQED plasma at $t=0$. What we want to do is to follow the time development of this configuration as it interacts with the hard modes in the plasma, and in particular, we want to know whether the relaxation is of the usually assumed exponential sort or something different.

Let us first reformulate SQED in terms of gauge invariant variables. The SQED Lagrangian is given by

$$\mathcal{L}=D_\mu\Phi^\dagger D^\mu\Phi-m^2|\Phi|^2-\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$D_\mu\Phi=(\partial_\mu-ieA_\mu)\Phi. \tag{2.1}$$

A description of the dynamics in terms of gauge invariant observables begins with the identification of the constraints associated with gauge invariance. The Abelian gauge theory has two first class constraints, namely Gauss's law and vanishing canonical momentum for A_0. What we will do is project the theory directly onto the physical Hilbert space, defined as usual as the set of states annihilated by the constraints. The procedure is simple. First, we obtain gauge invariant observables that commute with the first class constraints and write the Hamiltonian in terms of these. All of the matrix elements between gauge invariant states (annihilated by first class constraints) are the same as those that would be obtained by fixing Coulomb gauge $\vec{\nabla}\cdot\vec{A}=0$. The Hamiltonian, when defined in the physical subspace, can be written solely in terms of transverse components and includes the instantaneous Coulomb interaction as would be obtained in Coulomb gauge. This instantaneous Coulomb interaction can then be traded for a *gauge invariant* Lagrange multiplier field $A_0(\vec{x},t)$ (a non-propagating field whose canonical momentum is absent from the Hamiltonian) linearly coupled to the charge density $\rho(\vec{x},t)$ and obeying the algebraic equation of motion $\nabla^2 A_0(\vec{x},t)=\rho(\vec{x},t)$. Alternatively, one can use a phase-space path integral representation of the generating functionals, trade the Coulomb interaction with a Lagrange multiplier linearly coupled to the charge density and perform the path integral over the canonical momenta as usual. Both methods lead to the following Lagrangian density:

$$
\begin{aligned}
\mathcal{L}=&\partial_\mu\Phi^\dagger\,\partial^\mu\Phi+\frac{1}{2}\partial_\mu\vec{A}_T\cdot\partial^\mu\vec{A}_T\\
&-e\vec{A}_T\cdot\vec{j}_T-e^2\vec{A}_T\cdot\vec{A}_T\Phi^\dagger\Phi+\frac{1}{2}(\nabla A_0)^2+e^2A_0^2\Phi^\dagger\Phi\\
&-ieA_0(\Phi\dot{\Phi}^\dagger-\Phi^\dagger\dot{\Phi}),\\
\vec{j}_T=&i(\Phi^\dagger\vec{\nabla}_T\Phi-\vec{\nabla}_T\Phi^\dagger\;\;\Phi).
\end{aligned}
\tag{2.2}
$$

where A_T is the transverse component of the gauge field.

In order to provide an initial value problem for studying the relaxational dynamics of charge density fluctuations we introduce an external source $\mathcal{J}_L(\vec{x},t)$ linearly coupled to A_0 and study the linear response to this perturbation. Furthermore, it is convenient to introduce external sources coupled to the transverse gauge fields to study the linear response of *transverse* gauge field configurations. These external fields could in principle play the role of a semiclassical configuration coupled to small perturbations in a linearized approximation. Therefore we include external source terms in the Lagrangian density:

$$
\mathcal{L}\rightarrow\mathcal{L}-\mathcal{J}_L(\vec{x},t)A_0(\vec{x},t)-\vec{\mathcal{J}}_T(\vec{x},t)\cdot\vec{A}_T(\vec{x},t). \tag{2.3}
$$

The relaxational dynamics of our initial inhomogeneous configurations is clearly an out of equilibrium process, and needs to be treated by an appropriate formalism [43–45].

In the Schrödinger picture the dynamics is completely described by a time-evolved density matrix $\hat{\rho}$ that obeys the quantum Liouville equation:

$$
i\hbar\frac{\partial\hat{\rho}}{\partial t}=[\hat{\rho},H], \tag{2.4}
$$

where H is the Hamiltonian of the system. The expectation value of any operator $\mathcal{O}$ is given by

$$
\langle\mathcal{O}\rangle=\mathrm{Tr}[\hat{\rho}(t)\mathcal{O}]. \tag{2.5}
$$

For thermal initial conditions with an initial temperature given by $1/\beta$, the density matrix at $t=0$ is $\rho_i=e^{-\beta H_i}$ and the above expectation value can be rewritten easily as a functional integral defined on a complex-time contour. Notice that H_i *is not* the Hamiltonian of the system for $t>0$. The system thus evolves out of equilibrium. The contour has two branches running forward and backward in time and a third leg along the imaginary axis stretching to $t=-i\beta$. This is the standard Schwinger-Keldysh closed time path formulation of non-equilibrium field theory (see Refs. [43–45] for details). Fields defined on the forward and backward time contours are accompanied with $(+)$ and $(-)$ superscripts respectively and are to be treated independently. The expectation value of any string of field operators may be obtained by introducing independent sources on the forward and backward time contours and taking functional derivatives of the generating functional with respect to these sources. The imaginary time leg of the complex time contour does not contribute to the dynamics. Since the path integral represents a trace, the initial and final states must be identified and therefore all the local bosonic fields $\mathcal{O}(\vec{x},t)$ satisfy the Kubo-Martin-Schwinger (KMS) periodicity condition

$$
\mathcal{O}^{(+)}(\vec{x},t_0)=\mathcal{O}^{(\beta)}(\vec{x},t_0-i\beta). \tag{2.6}
$$

The non-equilibrium SQED Lagrangian is given by

$$
\mathcal{L}_{noneq}=\mathcal{L}[\vec{A}_T^+,\Phi^+,\Phi^{\dagger+},A_0^+]-\mathcal{L}[\vec{A}_T^-,\Phi^-,\Phi^{\dagger-},A_0^-]. \tag{2.7}
$$

Perturbative calculations are carried out with the following non-equilibrium Green's functions: Scalar propagators:

$$
\langle\Phi^{(a)\dagger}(\vec{x},t)\Phi^{(b)}(\vec{x},t')\rangle=-i\int\frac{d^3k}{(2\pi)^3}G_k^{ab}(t,t')e^{-i\vec{k}\cdot(\vec{x}-\vec{x'})},
$$

where $(a,b)\in\{+,-\}$.

$$
G_k^{++}(t,t')=G_k^{>}(t,t')\Theta(t-t')+G_k^{<}(t,t')\Theta(t'-t), \tag{2.8}
$$

$$
G_k^{--}(t,t')=G_k^{>}(t,t')\Theta(t'-t)+G_k^{<}(t,t')\Theta(t-t'),
$$

$$
G_k^{\pm\mp}(t,t')=-G_k^{<(>)}(t,t'), \tag{2.9}
$$

$$
G_k^{>}(t,t')=\frac{i}{2\omega_k}[(1+n_k)e^{-i\omega_k(t-t')}+n_ke^{i\omega_k(t-t')}], \tag{2.10}
$$

$$
G_k^{<}(t,t')=\frac{i}{2\omega_k}[n_ke^{-i\omega_k(t-t')}+(1+n_k)e^{i\omega_k(t-t')}], \tag{2.11}
$$

$$
\omega_k=\sqrt{\vec{k}^2+m^2};\quad n_k=\frac{1}{e^{\beta\omega_k}-1}.
$$

Photon propagators:

$$\langle A^{(a)}_{Ti}(\vec{x},t)A^{(b)}_{Tj}(\vec{x},t')\rangle = -i\int \frac{d^3k}{(2\pi)^3}\mathcal{G}^{ab}_{ij}(k;t,t')e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')},$$

$$\mathcal{G}^{++}_{ij}(k;t,t') = \mathcal{P}_{ij}(\vec{k})[\mathcal{G}^{>}_{k}(t,t')\Theta(t-t') + \mathcal{G}^{<}_{k}(t,t')\Theta(t'-t)],$$

$$\mathcal{G}^{--}_{ij}(k;t,t') = \mathcal{P}_{ij}(\vec{k})[\mathcal{G}^{>}_{k}(t,t')\Theta(t'-t) + \mathcal{G}^{<}_{k}(t,t')\Theta(t-t')],$$

$$\mathcal{G}^{\pm\mp}_{ij}(k;t,t') = -\mathcal{P}_{ij}(\vec{k})\mathcal{G}^{<(>)}_{k}(t,t'),$$

$$\mathcal{G}^{>}_{k}(t,t') = \frac{i}{2k}[(1+N_k)e^{-ik(t-t')} + N_k e^{ik(t-t')}], \tag{2.12}$$

$$\mathcal{G}^{<}_{k}(t,t') = \frac{i}{2k}[N_k e^{-ik(t-t')} + (1+N_k)e^{ik(t-t')}], \tag{2.13}$$

$$N_k = \frac{1}{e^{\beta k}-1}.$$

Here $\mathcal{P}_{ij}(\vec{k})$ is the transverse projection operator:

$$\mathcal{P}_{ij}(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}. \tag{2.14}$$

With these tools we are ready to begin our analysis of non-equilibrium SQED.

III. LINEAR RELAXATION

We now introduce the inhomogeneous non-equilibrium expectation values $\vec{\mathcal{A}}_T(\vec{x},t);\mathcal{A}_0(\vec{x},t)$ which are excited at time $t=0$ in the plasma. The non-equilibrium expectation values of the transverse components represent electric and magnetic fields, whereas the expectation of the Lagrange multiplier field A_0 corresponds to an initial charge density in the system.

The dynamics of these non-equilibrium expectation value will be analyzed by treating $\vec{\mathcal{A}}_T(\vec{x},t);\mathcal{A}_0(\vec{x},t)$ as background fields, i.e. the expectation values of the corresponding fields in the non-equilibrium density matrix, and expanding the Lagrangian about this configuration. Therefore we split the full quantum fields into c-number expectation values (which are the non-equilibrium expectation value) and quantum fluctuations about these expectation values:

$$\vec{A}^{(\pm)}_T(\vec{x},t) = \vec{\mathcal{A}}_T(\vec{x},t) + \delta\vec{A}^{(\pm)}_T(\vec{x},t);$$

$$A^{(\pm)}_0(\vec{x},t) = \mathcal{A}_0(\vec{x},t) + \delta A^{(\pm)}_0(\vec{x},t) \tag{3.1}$$

$$\vec{\mathcal{A}}_T(\vec{x},t) = \langle \vec{A}^{(\pm)}_T(\vec{x},t)\rangle; \quad \mathcal{A}_0(\vec{x},t) = \langle A^{(\pm)}_0(\vec{x},t)\rangle \tag{3.2}$$

where the expectation values of the field operators are taken in the time-evolved density matrix.

The equations of motion for the background field can be obtained to any order in the perturbative expansion by imposing the requirement that the expectation value of the quantum fluctuations in the time evolved density matrix vanishes identically. This is referred to as the tadpole equation [45] which follows from Eq. (3.1) and Eq. (3.2):

$$\langle \delta\vec{A}^{(\pm)}_T\rangle = 0; \quad \langle \delta A^{(\pm)}_0\rangle = 0. \tag{3.3}$$

The equations obtained via this procedure are the equations of motion obtained by variations of the non-equilibrium effective action. The perturbative expansion needed to compute the relevant expectation values is obtained by treating *all* the *linear* terms [45] in the fluctuations as interactions along with the usual interaction vertices.

Although in principle the tadpole method could be used to study arbitrary background configurations including non-perturbative ones (for e.g. sphalerons in the non-Abelian case) we restrict our discussions to the small amplitude regime in close analogy to the work of Ref. [36] for scalar field theory. In other words, the effective action equation of motion will be studied in the linear approximation for the condensate amplitude so that $\mathcal{O}(\mathcal{A}^2_{Ti})$ and $\mathcal{O}(\mathcal{A}^2_0)$ and higher orders will be neglected. In addition, the evolution kernel will be approximated to the *one-loop* order only.

Defining the Fourier components of the electromagnetic condensate as

$$\mathcal{A}_{Ti}(\vec{k},t) = \int d^3x e^{i\vec{k}\cdot\vec{x}}\mathcal{A}_{Ti}(\vec{x},t), \tag{3.4}$$

$$\mathcal{A}_0(\vec{k},t) = \int d^3x e^{i\vec{k}\cdot\vec{x}}\mathcal{A}_0(\vec{x},t) \tag{3.5}$$

we obtain the following equation of motion to one-loop order for the transverse part:

$$\frac{d^2}{dt^2}\mathcal{A}_{Ti}(\vec{k},t) + k^2\mathcal{A}_{Ti}(\vec{k},t) + 2e^2\langle\Phi^\dagger\Phi\rangle\mathcal{A}_{Ti}(\vec{k},t) - 2e^2\int_0^t d\tau\int\frac{d^3p}{(2\pi)^3\omega_p\omega_{p+k}}p_{Ti}p_{Tj}[(1+n_p+n_{p+k})\sin\{(\omega_{k+p}+\omega_p) \times(t-\tau)\} + (n_p-n_{k+p})\sin\{(\omega_{k+p}-\omega_p)(t-\tau)\}]\mathcal{A}_{Tj}(\vec{k},\tau) = \mathcal{J}_{Ti}(\vec{k},t). \tag{3.6}$$

Here p_{Ti} refers to the component of the spatial momentum $\vec{p}$ which is transverse to the wave-vector $\vec{k}$ and $\mathcal{A}_{Tj}(\vec{k},\tau)$ stands for the Fourier transform of the external source $\vec{A}_{Tj}(\vec{x},t)$ in Eq. (2.3).

The tadpole term which appears due to the 4-point ''seagull'' vertex can be evaluated easily in the limit where $T \gg m$ and yields the following hard thermal loop contribution

$$2e^2\langle\Phi^\dagger\Phi\rangle = 2e^2\int \frac{1+2n_k}{2\omega_k}\frac{d^3k}{(2\pi)^3} \simeq \frac{e^2T^2}{6}. \tag{3.7}$$

We remark that in evaluating the tadpole diagram above and in all subsequent calculations it is always *assumed* that the zero temperature divergences have already been absorbed in the proper mass and wavefunction renormalizations.

The longitudinal part obeys a similar equation:

$$k^2\mathcal{A}_0(\vec{k},t) + e^2\int_0^t d\tau \int \frac{d^3p}{(2\pi)^3}\Bigg[\left(\frac{\omega_{k+p}}{\omega_p}-1\right)(1+n_p+n_{p+k})\sin\{(\omega_{k+p}+\omega_p)(t-\tau)\}$$
$$+\left(\frac{\omega_{k+p}}{\omega_p}+1\right)(n_p-n_{k+p})\sin\{(\omega_{k+p}-\omega_p)(t-\tau)\}\Bigg]\mathcal{A}_0(\vec{k},\tau) = \mathcal{J}_L(\vec{k},t). \tag{3.8}$$

It should be noted that the equation of motion for the longitudinal component has no time derivatives, indicating the non-dynamical nature of the field. The source term for the longitudinal component is interpreted as an external disturbance that induces a charge density fluctuation in the SQED plasma. The response to a general disturbance can be obtained by convolution from the result of linear response to the impulsive perturbation.

We also note that unlike the transverse case, there is no contribution from the tadpole term to the effective action equations of motion. This fact will be important in understanding the origin of a non-zero Debye mass.

The nonlocal terms in Eqs. (3.6) and (3.8) are the one-loop self-energies for the transverse and longitudinal components respectively, resulting from the photon-Higgs trilinear coupling. The first piece in the nonlocal terms proportional to $1+n_p+n_{k+p}$ is the difference of the following creation and annihilation processes in the medium [46] : $\gamma \rightarrow \bar{\Phi}\Phi$ with Bose factor $(1+n_p)(1+n_{p+k})$ and $\bar{\Phi}\Phi \rightarrow \gamma$ with a statistical factor $n_p n_{p+k}$. The piece proportional to $(n_p - n_{k+p})$ is the Landau damping contribution [13,46]. When $k=0$ this contribution vanishes indicating that it affects inhomogeneous excitations only. Furthermore, it has no zero temperature counterpart since the Bose factor $n_k - n_{k+p}$ vanishes identically at $T=0$. This term arises from the difference of the processes $\gamma\Phi \rightarrow \Phi$ with statistical factor $(1+n_{k+p})n_p$ and $\Phi \rightarrow \gamma\Phi$ with the factor $(1+n_p)n_{k+p}$. Although it does not give rise to an imaginary part for the *on-shell* self-energy, it will nevertheless have an effect on the physical processes associated with the relaxation of the inhomogeneous condensate as described in detail below.

We can solve Eqs. (3.6), (3.8) via the Laplace transform. Introducing the Laplace transformed fields

$$\tilde{\mathcal{A}}_{Ti}(\vec{k},s) = \int_0^\infty dt e^{-st}\mathcal{A}_{Ti}(\vec{k},t), \tag{3.9}$$

$$\tilde{\mathcal{A}}_0(\vec{k},s) = \int_0^\infty dt e^{-st}\mathcal{A}_0(\vec{k},t), \tag{3.10}$$

and performing the transform on the above equations of motion we get following the same methods as in Ref. [36],

$$(s^2+k^2+e^2T^2/6)\tilde{\mathcal{A}}_{Ti}(\vec{k},s) - 2e^2\int \frac{d^3p}{(2\pi)^3\omega_p\omega_{k+p}}\Bigg[(1+n_p+n_{p+k})\frac{\omega_{k+p}+\omega_p}{s^2+(\omega_{k+p}+\omega_p)^2}$$
$$+(n_p-n_{k+p})\frac{\omega_{k+p}-\omega_p}{s^2+(\omega_{k+p}-\omega_p)^2}\Bigg]p_{Ti}p_{Tj}\tilde{\mathcal{A}}_{Tj}(\vec{k},s)$$
$$= s\mathcal{A}_{Ti}(\vec{k},0) + \dot{\mathcal{A}}_{Ti}(\vec{k},0) + \tilde{\mathcal{J}}_{Ti}(\vec{k},s) \tag{3.11}$$

for the transverse part, and

$$\left\{k^2 + e^2\int \frac{d^3p}{(2\pi)^3}\Bigg[\left(\frac{\omega_{k+p}}{\omega_p}-1\right)(1+n_p+n_{p+k})\frac{\omega_{k+p}+\omega_p}{s^2+(\omega_{k+p}+\omega_p)^2} + \left(\frac{\omega_{k+p}}{\omega_p}+1\right)(n_p-n_{k+p})\frac{\omega_{k+p}-\omega_p}{s^2+(\omega_{k+p}-\omega_p)^2}\Bigg]\right\}\tilde{\mathcal{A}}_0(\vec{k},s)$$
$$= \tilde{\mathcal{J}}_L(\vec{k},s) \tag{3.12}$$

for the longitudinal component. The Laplace transform variable s plays the role of an (imaginary) time component of the photon four-momentum.

IV. REAL TIME LANDAU DAMPING

With the equations of motion for the non-equilibrium expectation value in hand, we turn to the analysis of its time evolution, paying particular attention to those parts of the spectral density which contribute to the long time dynamics.

We do this by solving Eqs. (3.11), (3.12) via the inverse Laplace transform. Recall that this requires an integration along a contour parallel to the imaginary s axis placed in such a way so as to have all the singularities of the integrand to the left of the contour. To do this, we will need a clear understanding of the analytic structure of the photon self-energy in the s-plane. We will explicitly outline the details of the calculation for the transverse part only, since those for the longitudinal part are similar.

A. The transverse part

It is illuminating to write the equation of motion (3.11) in terms of a spectral density function $\rho_{ij}(\omega',\vec{k})$ [36]:

$$(s^2+k^2+e^2T^2/6)\tilde{\mathcal{A}}_{Ti}(\vec{k},s) + \int_{-\infty}^{+\infty} d\omega' \frac{\omega'}{s^2+\omega'^2}\rho_{ij}(\omega',\vec{k})\tilde{\mathcal{A}}_{Tj}(\vec{k},s) = s\mathcal{A}_{Ti}(\vec{k},0)+\dot{\mathcal{A}}_{Ti}(\vec{k},0)+\tilde{\mathcal{J}}_{Ti}(\vec{k},s) \tag{4.1}$$

where

$$\rho_{ij}(\omega',\vec{k}) = -2e^2\int \frac{d^3p}{(2\pi)^3\omega_p\omega_{k+p}} p_{Ti}p_{Tj} \times[(1+n_p+n_{p+k})\delta(\omega'-\omega_{k+p}-\omega_p) +(n_p-n_{k+p})\delta(\omega_{k+p}-\omega_p-\omega')]. \tag{4.2}$$

The one-loop transverse self-energy in the s-plane is given by

$$\mathcal{P}_{ij}(\vec{k})\Sigma^t(s) = \frac{e^2T^2}{6}\mathcal{P}_{ij}(\vec{k}) + \int_{-\infty}^{\infty} d\omega' \frac{\omega'}{s^2+\omega'^2}\rho_{ij}(\omega',\vec{k},s), \tag{4.3}$$

where we have recognized the fact that $\rho_{ij}\propto\mathcal{P}_{ij}(\vec{k})$.

Using $\mathcal{P}_{ij}(\vec{k})\mathcal{A}_{Tj}(\vec{k},s)=\mathcal{A}_{Ti}(\vec{k},s)$ the Laplace transform of the transverse part of the condensate is given by

$$\tilde{\mathcal{A}}_{Ti}(\vec{k},s) = \frac{s\,\mathcal{A}_{Ti}(\vec{k},0)+\dot{\mathcal{A}}_{Ti}(\vec{k},0)+\tilde{\mathcal{J}}_{Ti}(\vec{k},s)}{s^2+k^2+\Sigma^t(s)}. \tag{4.4}$$

The real time dependence of the inhomogeneous background is given by the inverse Laplace transform of the above expression which is in fact the retarded propagator defined in the s-plane. The inverse transform is calculated by performing the following integral along the Bromwich contour.

$$\mathcal{A}_{Ti}(\vec{k},t) = \int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{st}\tilde{\mathcal{A}}_{Ti}(\vec{k},s) = \int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{st}\frac{s\,\mathcal{A}_{Ti}(\vec{k},0)+\dot{\mathcal{A}}_{Ti}(\vec{k},0)+\tilde{\mathcal{J}}_{Ti}(\vec{k},s)}{s^2+k^2+\Sigma^t(s)}. \tag{4.5}$$

Here we choose $c\geqslant 0$, such that the contour is to the right of all the singularities in the s-plane.

The time dependence of this integral crucially depends on the analytic properties of the propagator and hence a clear understanding of the poles and cuts of the retarded propagator is essential. At this stage we will set the external current to zero, and analyze the contribution of the sources at the end of this subsection.

In all of the above and in what follows it is implicitly assumed that the zero temperature divergences have been dealt with already by renormalizing the amplitude of the field, i.e. wave-function renormalization and that we are working with the subtracted spectral function. Recall that the divergences are determined solely by zero temperature fluctuations.

We define the real and imaginary parts of the self-energy near the imaginary axis through

$$\Sigma^t(i\omega\pm 0^+) = \Sigma^t_R(i\omega\pm 0^+) + i\Sigma^t_I(i\omega\pm 0^+). \tag{4.6}$$

In fact it is easy to see from Eq. (4.3) that

$$\Sigma^t_I(i\omega\pm 0^+) = \mp\,\mathrm{sgn}(\omega)\frac{\pi}{2}(\rho_{ij}(|\omega|)-\rho_{ij}(-|\omega|)) = \pm\frac{e^2}{8\pi^2}\mathrm{sgn}(\omega)\int\frac{d^3p\,p^2\sin^2\theta}{\omega_p\omega_{k+p}}[(1+n_p+n_{k+p})\delta(\omega_{k+p}+\omega_p-|\omega|) +(n_p-n_{k+p})\{\delta(\omega_{k+p}-\omega_p-|\omega|)-\delta(\omega_{k+p}-\omega_p+|\omega|)\}]. \tag{4.7}$$

This shows that $\Sigma^t_I(i\omega-0^+)=-\Sigma^t_I(i\omega+0^+)$ and that the cuts will appear whenever the delta functions are satisfied. Analysis of the arguments of the delta functions reveals that the two-particle cuts arising from the first term in Eq. (4.7) stretch from $s=\pm i(m+\omega_k)$ to $s=\pm i\infty$. The second and third terms in Eq. (4.7) contain the hard thermal contributions. They have support when $\omega_{k+p}-\omega_p\simeq\vec{k}\cdot\hat{p}=\pm\omega$ which corresponds to the four-vector $(\omega,\vec{k})$ being spacelike. The resulting branch cut runs from $s=-ik$ to $s=+ik$. Hence these processes are induced only by off-shell space-like photons. These HTL contributions lead to *Landau damping*. In summary, the physical region singularities and the Landau discontinuities show up as discontinuities in the imaginary part of the self-energy $\Sigma^t(s)$ when approaching the imaginary axis of the s-plane i.e. $s=i\omega$.

In addition, as argued in the previous section the contribution of the physical region cuts from $s=\pm i(m+\omega_k)$ to $s=\pm i\infty$ to the self-energy are $\propto \ln T$ only, and therefore subleading compared with the $O(T^2)$ terms.

Furthermore the long time dynamics due to the cuts will be dominated by the thresholds (or the end-points). Since the end-points of the Landau damping discontinuities are at $s\approx\pm ik$, they will be the dominant contribution and the two particle cuts from $s=\pm i(m+\omega_k)$ to $s=\pm\infty$ will be subleading at long times. Thus we can simply focus on Landau damping both as the leading high temperature and long time contributions. This argument is necessary because it is not *a priori* obvious that the high temperature and long time limits are described by the same processes.

The leading term in the self-energy $\propto T^2$ is given by

$$\Sigma^t_I(i\omega\pm0^+)\simeq\pm\frac{e^2}{8\pi}\mathrm{sgn}(\omega)\int dp\, p^2\left(-\frac{dn_p}{dp}\right)\times\int_{-1}^{1}dx(1-x^2)kx\times[\delta(kx-|\omega|)-\delta(kx+|\omega|)]=\pm\frac{e^2T^2\pi}{12}\frac{\omega}{k}\left(1-\frac{\omega^2}{k^2}\right)\Theta(k^2-\omega^2). \tag{4.8}$$

The real part can be obtained either by using dispersion relations or by explicitly solving for the hard thermal self-energy by calculating the relevant integrals and we find

$$\Sigma^t(s)=-\frac{e^2T^2}{12}\left[2\frac{s^2}{k^2}+i\frac{s}{k}\left(1+\frac{s^2}{k^2}\right)\ln\left(\frac{is-k}{is+k}\right)\right]+\mathcal{O}(\ln T). \tag{4.9}$$

From this expression the transverse self-energy along the imaginary axis when approaching from the right, can be read off easily:

$$\Sigma^t(i\omega+0^+)=\frac{e^2T^2}{12}\left[2\frac{\omega^2}{k^2}+\frac{\omega}{k}\left(1-\frac{\omega^2}{k^2}\right)\ln\left|\frac{k+\omega}{k-\omega}\right|\right]+i\frac{e^2T^2\pi}{12}\frac{\omega}{k}\left(1-\frac{\omega^2}{k^2}\right)\Theta(k^2-\omega^2) \tag{4.10}$$

and agrees with known results [31,32,37,38,46].

In addition to the branch cut singularities, the retarded propagator will also have isolated poles corresponding to the quasiparticle excitations which can propagate in the plasma. The poles for the transverse excitations will be given by the solutions to

$$\omega_P^2=k^2+\frac{e^2T^2}{12}\left[2\frac{\omega_P^2}{k^2}+\frac{\omega_P}{k}\left(1-\frac{\omega_P^2}{k^2}\right)\ln\left|\frac{k+\omega_P}{k-\omega_P}\right|\right] \tag{4.11}$$

$$+i\frac{e^2T^2\pi}{12}\frac{\omega_P}{k}\left(1-\frac{\omega_P^2}{k^2}\right)\Theta(k^2-\omega_P^2). \tag{4.12}$$

They will be in the physical sheet provided the imaginary part vanishes at the pole.

Though the equation cannot be solved analytically, in the case of interest when the external momenta are extremely soft (for e.g. $k\sim e^2T\ll eT$) representing a small amplitude long wavelength field configuration, the approximate location of the poles is found to be $s=\pm i\omega_P\simeq\pm i(eT/3)$ [31,32,37,38].

The two-particle cuts were shown to run from $s=\pm i(m+\omega_k)$ to $\pm i\infty$ where m is the mass of the scalar. A consistent HTL resummation should also include the shift in the scalar masses, thus ensuring that to this order the quasiparticle pole is in the physical sheet. Higher order contributions will provide a collisional broadening to the pole. It is a noteworthy point that the quasiparticle poles are located beyond the Landau discontinuities which stretch from $-ik$ to $+ik$ in the s-plane, and below the two-particle threshold.

In summary, to this order in the HTL approximation, the analytic structure of the retarded propagator in the high temperature limit features a Landau discontinuity running from $-ik$ to $+ik$ and quasiparticle poles at $s=\pm i\omega_p(k)$. The two-particle cut contributions have been shown to give a subleading contribution both in temperature and in the long time dynamics.

Using Eq. (4.5) we can now invert the transform by deforming the contour and wrapping it around the poles and the cuts to pick up the corresponding residues and discontinuities repectively so that

$$\mathcal{A}_{Ti}(\vec{k},t)=\mathcal{A}^{pole}_{Ti}(\vec{k},t)+\mathcal{A}^{cut}_{Ti}(\vec{k},t). \tag{4.13}$$

The contributions from the quasiparticle poles add up to give a purely oscillatory behavior in time. The residues at the poles give rise to a wave function renormalization with both T-dependent and T-independent contributions. The T-independent contribution contains the typical logarithmic divergence and has been absorbed in the usual zero temperature wave-function renormalization. We thus obtain

$$\mathcal{A}_{Ti}^{pole}(\vec{k},t)=Z^t[T]\left[\mathcal{A}_{Ti}(\vec{k},0)\cos(\omega_P t)+\dot{\mathcal{A}}_{Ti}(\vec{k},0)\frac{\sin(\omega_P t)}{\omega_P}\right], \tag{4.14}$$

$$Z^t[T]=\left[1-\frac{\partial\Sigma^t(i\omega)}{\partial\omega^2}\right]^{-1}_{\omega=\omega_P\simeq eT/3}. \tag{4.15}$$

Here $Z^t(T)$ is the temperature dependent wave function renormalization defined on-shell at the quasiparticle pole whose leading HTL contribution is obtained from the self-energy (4.10). The continuum contribution is given by

$$\mathcal{A}_{Ti}^{cut}(\vec{k},t)=\frac{2}{\pi}\int_0^k d\omega\frac{\Sigma_I^t(i\omega+0^+)[\mathcal{A}_{Ti}(\vec{k},0)\omega\cos(\omega t)+\dot{\mathcal{A}}_{Ti}(\vec{k},0)\sin(\omega t)]}{[\omega^2-k^2-\Sigma_R^t(i\omega)]^2+[\Sigma_I^t(i\omega+0^+)]^2}. \tag{4.16}$$

Evaluating Eqs. (4.13), (4.14) and (4.16) at $t=0$ we obtain an important sum rule,

$$Z^t[T]+\frac{2}{\pi}\int_0^k d\omega\frac{\omega\Sigma_I^t(i\omega+0^+)}{[\omega^2-k^2-\Sigma_R^t(i\omega)]^2+[\Sigma_I^t(i\omega+0^+)]^2}=1. \tag{4.17}$$

This sum rule is a consequence of the canonical commutation relations but its content in the HTL approximation is that in the high temperature limit the wave function renormalisation which is evaluated *on-shell* is completely determined by the Landau discontinuities which originate from *strongly off-shell* processes. A similar sum rule was also obtained in [11] using different methods.

The integral over the cut (4.16) cannot be evaluated in closed form but its long time asymptotics is dominated by the end-point contributions as can be understood from the following argument. The integral along the real ω axis from $\omega=0$ to $\omega=k$ can be obtained by deforming the integral into the upper complex ω plane so that it runs along $\omega=iz;0<z<\infty$, then around an arc at infinity and back to the real axis along the line $\omega=k+iz;0<z<\infty$ for the term $\propto e^{i\omega t}$ and similarly into the lower complex plane for the term $\propto e^{-i\omega t}$. A detailed analysis of the asymptotic behavior of the integral reveals that only the $\omega=k$ end-point contributes because the contributions from $\omega=0$ vanish for large t faster than any negative power of t. This is a consequence of the regular behavior of the spectral density in the vicinity of $\omega=0$.

We find the contribution from the $\omega=k$ end-point in the long time limit $t\gg 1/k$ to be given by

$$\mathcal{A}_{Ti}^{cut}(\vec{k},t)\overset{t\to\infty}{=}-\frac{12}{e^2T^2}\left\{\mathcal{A}_{Ti}(\vec{k},0)\frac{\cos(kt)}{t^2}+\dot{\mathcal{A}}_{Ti}(\vec{k},0)\frac{\sin(kt)}{kt^2}\right\}\times\left[1+\mathcal{O}\left(\frac{1}{t}\right)\right]. \tag{4.18}$$

From the sum rule (4.16) and (4.17) we find that the *early* time behavior is approximately given by

$$\mathcal{A}_{Ti}(\vec{k},t)=\mathcal{A}_{Ti}(\vec{k},0)[1-e^2T^2t^2\Delta_e+\mathcal{O}(t^4)]+\dot{\mathcal{A}}_{Ti}(\vec{k},0)\Delta_o\frac{e^2T^2}{k^2}t[1+\mathcal{O}(t^2)], \tag{4.19}$$

where $\Delta_{e,o}$ are constants depending on the wave-function renormalization and moments of the spectral density.

At long times, the dominant contributions are from the nearest singularities in the complex ω plane. This includes the usual contributions from the quasiparticle modes at $\omega=\pm\omega_P$, given by Eq. (4.14). In addition, there is also the contributions from the branch points at $\omega=\pm k$, Eq. (4.18). There is no contribution from $\omega=0$, because the HTL self-energy is regular about zero frequency.

Figure 1 shows the cut contribution $\mathcal{A}_T^{cut}(k,t)/\mathcal{A}_T(k,0)$ vs time for $e^2T^2/k^2=2$ and Fig. 2 shows $t^2\mathcal{A}_T^{cut}(k,t)/$

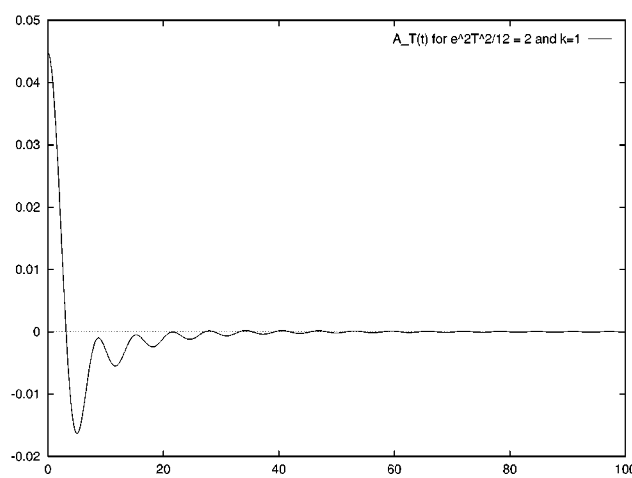

FIG. 1. Cut contribution $\mathcal{A}_T^{cut}(k,t)/\mathcal{A}_T(k,0)$ for $e^2T^2/12=2$ and $k=1$ vs t.

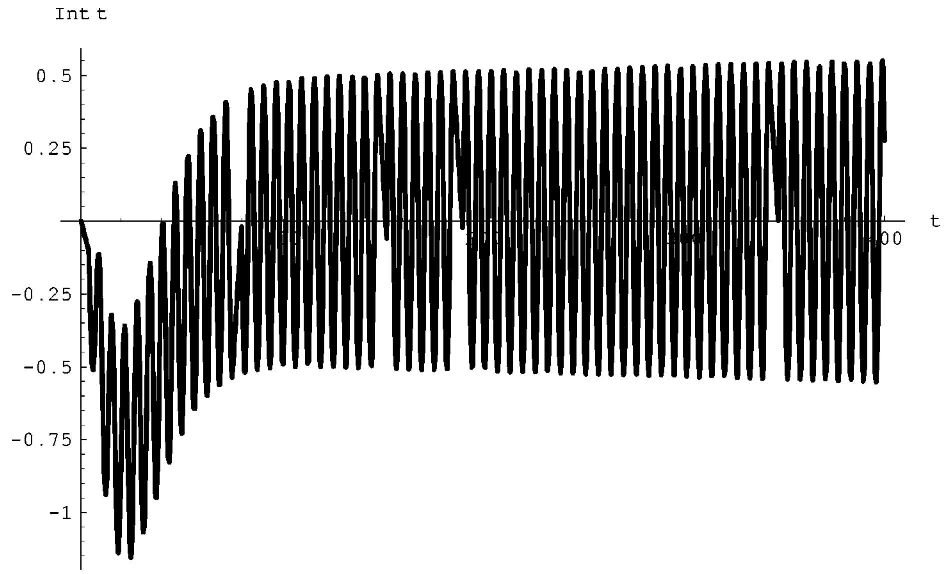

FIG. 2. $t^2 \times \mathcal{A}_T^{cut}(k,t)/\mathcal{A}_T(k,0)$ vs t (in units of 1/k) for $m_D^2/k^2 = 12; m_D^2 = e^2T^2/3$.

$\mathcal{A}_T(k,0)$ vs t (in units of $1/k$) for $m_D^2/k^2 = 12$ for the case $\dot{\mathcal{A}}_{Ti}(\vec{k},0) = 0$.

The point being made is that the real time dynamics of the condensate is completely determined by the analytic structure of the retarded propagator and the *global structure* of the spectral density in the s-plane.

The second important point to note is that the long time behavior is a *power law* $\sim t^{-2}$ (times oscillations) and *not an exponential decay*. This means that Landau damping effects *cannot* be reproduced by phenomenological ''viscous'' terms of the type $\sim \Gamma(d/dt)$ neither at long nor at short times. The failure of such a phenomenologically motivated ansatz was already noticed at zero temperature in different contexts in Ref. [44]. We stress that such a description not only fails to reproduce the power law behavior but in fact ignores *all the non-local* physics of Landau damping which is so clearly encoded in the hard thermal loop kernels.

One might argue that higher order processes, both Landau damping and collisional, could lead to an exponential relaxation. However, the point of the above analysis is to argue that the full relaxational physics will be described by a *competition* between the power laws from the lowest order Landau damping contributions and the higher order exponential damping. The time scale of interest will determine which process dominates.

For non-zero external sources we can obtain the general real time evolution by inserting the Laplace transform of the source in Eq. (4.5)

$$\tilde{\mathcal{J}}_{Ti}(\vec{k},s) = \int_0^\infty dt e^{-st} \mathcal{J}_{Ti}(\vec{k},t).$$

This is an analytic function of s for $\mathrm{Re}(s) > 0$ provided $\mathcal{J}_{Ti}(\vec{k},t)$ is a non-singular function of time. Let us assume that $\tilde{\mathcal{J}}_{Ti}(\vec{k},s)$ is analytic for $\mathrm{Re}(s) > -\alpha$, where $\alpha > 0$ is a positive number. [Since $\tilde{\mathcal{J}}_{Ti}(\vec{k},s)$ vanishes for $\mathrm{Re}(s) \to +\infty$, $\tilde{\mathcal{J}}_{Ti}(\vec{k},s)$ must have singularities somewhere in the left half-plane. Otherwise, it will be an entire function which is zero at infinity and therefore identically zero.] The singularities of $\tilde{\mathcal{J}}_{Ti}(\vec{k},s)$ in the left-half s plane yield contributions to $\mathcal{A}_{Ti}(\vec{k},t)$ through Eq. (4.5) which decrease exponentially in time as $e^{-\alpha t}$. They are therefore subdominant compared with the Landau cut contributions (4.18).

The source term contribution from the small $s = i\omega$ region can be understood simply as follows. Within the above hypothesis, the source term $\tilde{\mathcal{J}}_{Ti}(\vec{k},i\omega)$ can be expanded in a Taylor series for small ω

$$\tilde{\mathcal{J}}_{Ti}(\vec{k},i\omega) = J_0(\vec{k}) + J_1(\vec{k})\omega + \mathcal{O}(\omega^2).$$

The even term, $J_0(\vec{k})$ can be absorbed into a shift of $\dot{\mathcal{A}}_{Ti}(\vec{k},0)$, while the odd term $J_1(\vec{k})$ can be absorbed into a shift of $\mathcal{A}_{Ti}(\vec{k},0)$, Eq. (4.5). Thus as before the asymptotic large time behavior is completely dominated by the branch points at $\omega = \pm k$. Under the above assumptions on the Laplace transform of the external current, we find the general time dependence to be given by

$$\mathcal{A}_{Ti}^{cut}(\vec{k},t) \overset{t\to\infty}{=} -\frac{12}{e^2T^2kt^2}\{[k\mathcal{A}_{Ti}(\vec{k},0) - S_i(k)]\cos(kt) + [\dot{\mathcal{A}}_{Ti}(\vec{k},0) + C_i(k)]\sin(kt)\}\left[1 + \mathcal{O}\left(\frac{1}{t}\right)\right]. \tag{4.20}$$

Here $C_i(\omega)$ and $S_i(\omega)$ stand for the Fourier cosine and sine transforms of the source $\mathcal{J}_{Ti}(\vec{k},t)$, respectively:

$$C_i(\omega)=\int_0^\infty dt\cos(\omega t)\mathcal{J}_{Ti}(\vec{k},t),$$

$$S_i(\omega)=\int_0^\infty dt\sin(\omega t)\mathcal{J}_{Ti}(\vec{k},t).$$

We emphasize that to this order in the HTL resummation, the region near $\omega\approx 0$ of the spectral density is regular (no branch singularities) and therefore *does not* contribute to the long time dynamics, which is completely determined by the end point (branch point) at $\omega=k$.

B. Longitudinal part

An analysis very similar to the one outlined above yields the following expressions for the longitudinal component of the photon self-energy in the case of an impulsive source $\mathcal{J}_L(\vec{x},t)=\delta^3(\vec{x})\delta(t)$ with spatial Fourier and Laplace transform given by $\tilde{\mathcal{J}}_L(\vec{k},s)=1$. The case of a more complicated source can be obtained by convolution. In this case we obtain

$$\mathcal{A}_0(\vec{k},s)=\frac{1}{k^2+\Sigma^l(s)} \tag{4.21}$$

where

$$\Sigma^l(s)=\frac{e^2T^2}{3}-\frac{e^2T^2}{6}\frac{s}{ik}BP\ln\left(\frac{is-k}{is+k}\right). \tag{4.22}$$

Thus the longitudinal self-energy along the imaginary axis in the s-plane, when approaching from the right is obtained as before to be

$$\Sigma^l(i\omega+0^+)=\frac{e^2T^2}{3}\left[1-\frac{\omega}{2k}\ln\left|\frac{k+\omega}{k-\omega}\right|\right]-i\frac{e^2T^2\pi}{6}\frac{\omega}{k}\Theta(k^2-\omega^2). \tag{4.23}$$

The location of the longitudinal quasiparticle or plasmon poles is given by the following dispersion relation,

$$k^2+\frac{e^2T^2}{3}\left[1-\frac{\omega}{2k}\ln\left|\frac{k+\omega}{k-\omega}\right|\right]-i\frac{e^2T^2\pi}{6}\frac{\omega}{k}\Theta(k^2-\omega^2)=0. \tag{4.24}$$

For soft external momenta ($k\ll eT$) the plasmon poles can be seen to be at $s=\pm i\omega_0\simeq\pm ieT/3$. The real time dependence of the longitudinal condensate is then found by inverting the transform using

$$\mathcal{A}_0(\vec{k},t)=\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{st}\tilde{\mathcal{A}}_0(\vec{k},s) \tag{4.25}$$

where the contour is to the right of all the singularities as in Eq. (4.5). As in the transverse case, at high temperatures the singularity structure is dominated by the discontinuity across the cut that runs from $-ik$ to $+ik$ corresponding to Landau damping, and the two plasmon poles at $\pm i\omega_0$. The Bromwich contour is then deformed to pick up the cut and pole contributions so that

$$\mathcal{A}_0(\vec{k},t)=\mathcal{A}_0^{pole}(\vec{k},t)+\mathcal{A}_0^{cut}(\vec{k},t), \tag{4.26}$$

where

$$\mathcal{A}_0^{pole}(\vec{k},t)=-Z^l[T]\frac{\sin(\omega_0t)}{\omega_0}, \tag{4.27}$$

$$Z^l[T]=\left[\frac{\partial\Sigma^l(i\omega)}{\partial\omega^2}\right]^{-1}_{\omega=\omega_o\simeq eT/3} \tag{4.28}$$

and

$$\mathcal{A}_0^{cut}(\vec{k},t)=-\frac{2}{\pi}\int_0^k d\omega\frac{\Sigma^l_I(i\omega+0^+)\sin(\omega t)}{[k^2+\Sigma^l_R(i\omega)]^2+[\Sigma^l_I(i\omega+0^+)]^2}. \tag{4.29}$$

Unlike the transverse components for which the sum rule is a consequence of the canonical commutation relations, for the longitudinal component there is no equivalent sum rule because the field $A_0(x)$ is a non-propagating Lagrange multiplier.

The long time, asymptotic behavior of the longitudinal condensate is dominated by the end-points of the integral (4.29). The end-point $\omega=0$ yields contributions that vanish for long t faster than any negative power of t, as it was the case for the transverse part (4.16). We find that the long time asymptotics for $t\gg 1/k$ is dominated by the end-point contribution at $\omega=k$,

$$\mathcal{A}_0^{cut}(\vec{k},t)=a^{cut}_{asymp}(\vec{k},t)\left[1+\mathcal{O}\left(\frac{1}{t}\right)\right],$$

$$a^{cut}_{asymp}(\vec{k},t)\equiv-\frac{12}{e^2T^2}\int_0^\infty dx\times e^{-x}\frac{\cos[kt+\alpha(x,t)]}{\sqrt{\log^4\frac{cx}{kt}+\frac{5\pi^2}{2}\log^2\frac{cx}{kt}+\frac{9\pi^4}{16}}} \tag{4.30}$$

$$\alpha(x,t)\equiv\arctan\frac{\pi\log\frac{cx}{kt}}{\frac{3\pi^2}{4}+\log^2\frac{cx}{kt}} \tag{4.31}$$

$$c\equiv\frac{1}{2}\exp\left[2+\frac{6k^2}{e^2T^2}\right]. \tag{4.32}$$

The integral in Eq. (4.30) cannot be expressed in terms of elementary functions and it is related to the $\nu(x)$ function

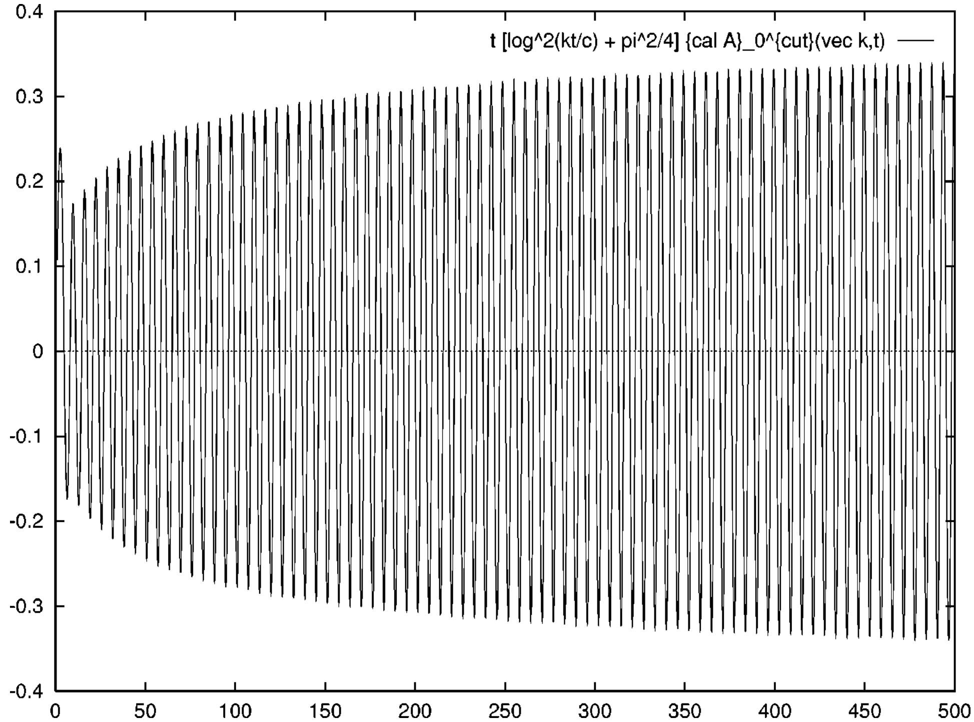

FIG. 3. $t[\log^2(kt/c)+\pi^2/4]\mathcal{A}_0^{cut}(\vec{k},t)$ as a function of t for $e^2T^2=6$ and $k=1$ [see Eqs. (4.30)–(4.33)].

[47]. For large t, one can derive an asymptotic expansion in inverse powers of $\log[kt/c]$ by integrating Eq. (4.30) by parts:

$$a_{asymp}^{cut}(\vec{k},t) \overset{t\to\infty}{=} -\frac{12}{e^2T^2t}\frac{\cos(kt+\beta(kt))}{\log^2\frac{kt}{c}+\frac{\pi^2}{4}}\left[1+\mathcal{O}\left(\frac{1}{\log t}\right)\right] \tag{4.33}$$

$$\beta(kt)\equiv -\arctan\frac{\pi\log kt}{\log^2 kt-\frac{\pi^2}{4}}. \tag{4.34}$$

This expansion is not very good quantitatively unless t is very large. For example, for $kt=500$, $a_{asymp}^{cut}(\vec{k},t)$ given by Eq. (4.30) approximates $\mathcal{A}_0^{cut}(\vec{k},t)$ up to 0.1%, whereas, the dominant term in Eq. (4.33) is about 30% smaller than $a_{asymp}^{cut}(\vec{k},t)$. Figure 3 shows $t[\log^2(kt/c)+\pi^2/4]\mathcal{A}_0^{cut}(\vec{k},t)$ and Fig. 4 shows $(12/e^2T^2)\cos[kt+\beta(kt)]$ [see Eqs. (4.30)–(4.33)], both figures should coincide for large enough t but we see numerically a sizable discrepancy even for $t=500$ due to the slow convergence of the asymptotic expansion (4.33).

A saddle point analysis yields an intermediate asymptotics of the form

$$a_{asymp}^{cut}(\vec{k},t)\approx\frac{3\sqrt{\pi}\cos(kt)}{e^2T^2t\ln^{2.5}[kt]} \tag{4.35}$$

which although is a relatively good estimate within a time window (see Fig. 5) is seen numerically to have a discrepancy of the same order as the dominant term above in a wider range of time. Nevertheless, the exact expression (4.30) is easily obtained numerically for arbitrary range of parameters.

The case of a general source requires a convolution. The cut contribution takes now the form

$$\mathcal{A}_0^{cut}(\vec{k},t)=-\frac{2}{\pi}\int_0^k d\omega\frac{\Sigma_I^l(i\omega+0^+)[C_L(\omega)\sin(\omega t)-S_L(\omega)\cos(\omega t)]}{[k^2+\Sigma_R^l(i\omega)]^2+[\Sigma_I^l(i\omega+0^+)]^2},$$

where $C_L(\omega)$ and $S_L(\omega)$ stand for the cosine and sine Fourier coefficients of the source $\mathcal{J}_L(\vec{k},t)$, respectively.

For a longitudinal source fullfilling the same general analyticity assumptions as the transverse source we can expand in series for small ω as

$$\mathcal{J}_L(\vec{k},i\omega)=J_{0L}+J_{1L}\omega+\mathcal{O}(\omega^2).$$

As for Eq. (4.29) the end-point $\omega=0$ of the integral (4.29) yields contributions that vanish for long t faster than any

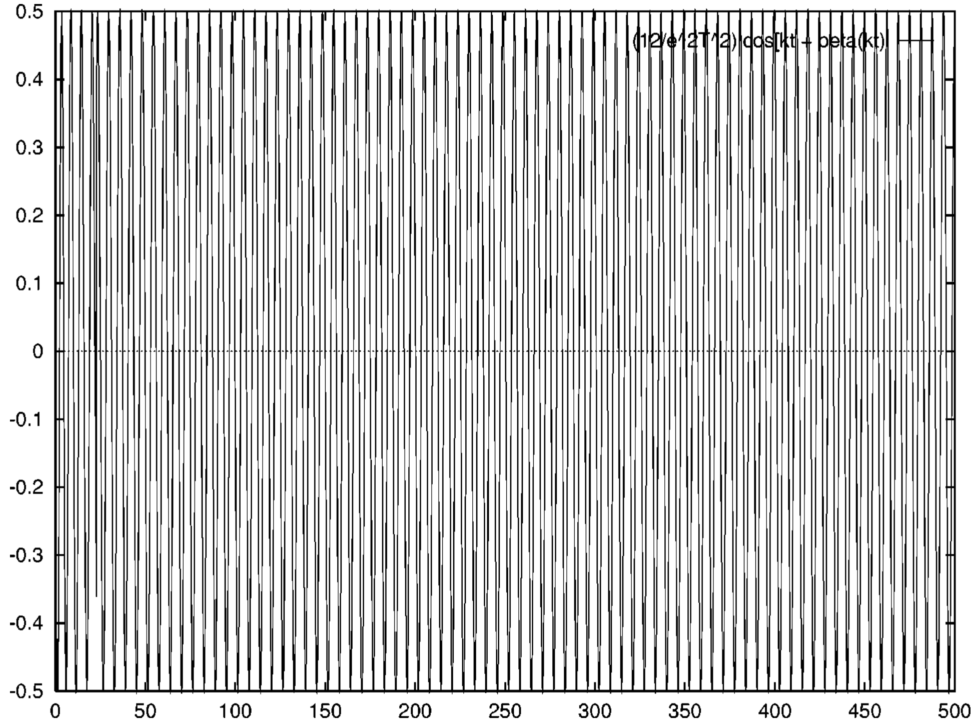

FIG. 4. $(12/e^2T^2)\cos[kt+\beta(kt)]$ a function of t for $e^2T^2=6$ and $k=1$ [see Eqs. (4.30)–(4.33)].

negative power of t. Just as in the transverse case, this result is a consequence of the fact that to this order in the HTL resummation, the spectral density is regular (no branch points) near $\omega=0$.

To summarize, we gather the final results for the asymptotic real-time evolution of the transverse and longitudinal non-equilibrium expectation value in the linear approximation

Transverse (no external source):

$$\mathcal{A}_{Ti}(\vec{k},t)=\mathcal{A}_{Ti}(\vec{k},0)\left[Z^t-\frac{12}{e^2T^2}\,\frac{\cos(kt)}{e^2T^2}\,\frac{\cos(kt)}{t^2}+\mathcal{O}\left(\frac{1}{t^3}\right)\right]$$
$$+\dot{\mathcal{A}}_{Ti}(\vec{k},0)\left[\frac{Z^t}{\omega_p}-\frac{12}{e^2T^2}\,\frac{\sin(kt)}{e^2T^2}\,\frac{\sin(kt)}{kt^2}+\mathcal{O}\left(\frac{1}{t^3}\right)\right]. \quad (4.36)$$

The sum rule (4.17) implies that the coherent field configu-

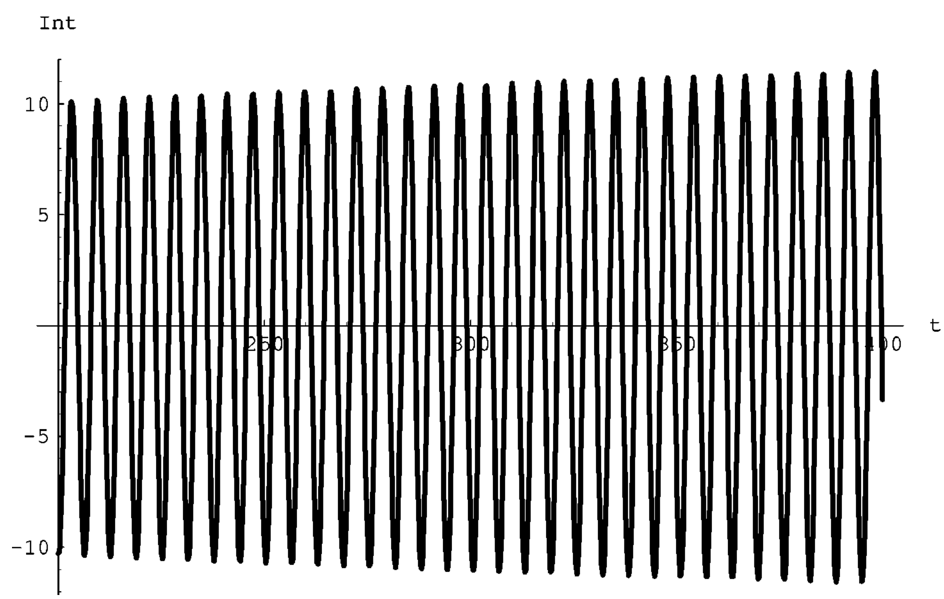

FIG. 5. $t\times(\ln[t])^{2.5}\times\mathcal{A}_0^{cut}(k,t)$ vs t (in units of 1/k) for $m_D^2/k^2=2$.

ration relaxes to an asymptotic amplitude which is *smaller* than the initial, and the ratio of the final to the initial amplitude is completely determined by the thermal wave function renormalization.

Longitudinal (impulsive external source):

$$\mathcal{A}_0(\vec{k},t) = -Z^l[T]\frac{\sin(\omega_0 t)}{\omega_0} + a^{cut}_{asymp}(\vec{k},t)\left[1+\mathcal{O}\left(\frac{1}{t}\right)\right]. \tag{4.37}$$

The transverse and longitudinal wave function renormalizations $Z^t[T], Z^l[T]$ are given by Eqs. (4.15) and (4.28) respectively and $a^{cut}_{asymp}(\vec{k},t)$ is given by Eqs. (4.30), (4.33) while the positions of the poles ω_p, ω_0 can be obtained by solving Eqs. (4.11) and (4.24) respectively.

In summary: we find that the long time dynamics is dominated by the Landau damping thresholds at $\omega = \pm k$, not by the $\omega \approx 0$ region of the spectral density. The early time dynamics is determined by moments of the total spectral density. Long time power law tails had recently found in the current correlators in Ref. [28].

V. LANGEVIN DESCRIPTION FOR THE SOFT MODES AND FLUCTUATION-DISSIPATION RELATION

A. Langevin description

So far we have studied the damping of soft modes in the plasma by the hard particles from the microscopic point of view. In this section we provide a stochastic description of the relaxation of gauge fields via a semiclassical Langevin equation with a Markovian damping kernel and a Gaussian white noise.

A semiclassical description treats the hard modes as a ''bath'' and the soft modes as the ''system.'' The bath degrees of freedom are integrated out, their main effect being encoded in a dissipative kernel and a stochastic noise inhomogeneity in the resulting Langevin equation. The dissipative kernel is related to the stochastic correlation function of the noise via a generalized fluctuation-dissipation relation. Physically the stochasticity arises because the hard scales which are integrated out in the HTL scheme and are responsible for Landau damping will also provide random kicks to the soft degrees of freedom.

This section is devoted to a *microscopic derivation* of the Langevin equation for the inhomogeneous gauge field configuration, to leading order in the hard thermal loop program. This is achieved explicitly by integrating out the hard modes which provide a natural realization of the ''bath'' variables while the soft modes are to be treated as the ''system'' variables. The methodology of this approach is based on the Feynman-Vernon influence functional [48] which has already been used to describe dissipation and decoherence in quantum systems from a microscopic theory [44,49,50].

The non-equilibrium partition function for the full field theory is

$$\mathcal{Z} = \int \mathcal{D}A^{(\pm)}_{Ti}\mathcal{D}\Phi^{(\pm)}\mathcal{D}\Phi^{(\pm)\dagger} \times \exp\Bigg[i\int d^4x\Bigg\{ \partial_\mu\Phi^{(+)\dagger}\partial^\mu\Phi^{(+)} + \frac{1}{2}\partial_\mu\vec{A}^{(+)}_T\cdot\partial^\mu\vec{A}^{(+)}_T - e\vec{A}^{(+)}_T\cdot\vec{j}^{(+)}_T - e^2\vec{A}^{(+)}_T\cdot\vec{A}^{(+)}_T\Phi^{(+)\dagger}\Phi^{(+)} - [+\to-]\Bigg\}\Bigg] \tag{5.1}$$

and the effective action for the ''system'' (the soft photons) follows by performing the path integral over the ''bath'' (the hard scalars) treating the ''system'' degrees of freedom as background fields. This means that all subsequent expectation values will be evaluated in the reduced density matrix which defines the effective field theory for the system degrees of freedom. It is then convenient to introduce the ''center of mass'' and ''relative'' coordinates

$$A^{(\pm)}_{Ti} = \mathcal{A}_{Ti} \pm \frac{R_{Ti}}{2}. \tag{5.2}$$

In terms of these redefined fields the effective action can be obtained to one-loop order via a systematic loop-expansion of the reduced partition function:

$$\begin{aligned} S_{eff}[\mathcal{A}_{Ti},R_{Ti}] = \int d^4x\Bigg\{ & -R_{Ti}\partial^2\mathcal{A}_{Ti} \\ & -2e^2R_{Ti}\mathcal{A}_{Ti}\langle\Phi^\dagger\Phi\rangle \\ & +\frac{ie^2}{4}\int d^4x' R_{Ti}(x)R_{Tj}[\langle j^{(+)}_{Ti}(x)j^{(-)}_{Tj}(x')\rangle \\ & +\langle j^{(-)}_{Ti}(x)j^{(-)}_{Tj}(x')\rangle]R_{Tj}(x') \\ & +ie^2\int d^4x' R_{Ti}(x)[\langle j^{(-)}_{Ti}(x)j^{(+)}_{Tj}(x')\rangle \\ & -\langle j^{(+)}_{Ti}(x)j^{(-)}_{Tj}(x')\rangle]\Theta(x_0-x_0')\mathcal{A}_{Tj}(x')\Bigg\}. \end{aligned} \tag{5.3}$$

The transverse current $j_{Ti}(x)$ was introduced in Eq. (2.2) and the curent-current correlators can be calculated easily using the defining formulas for the free scalar propagators in Eqs. (2.8)–(2.11). In order to make explicit the soft momentum scales of interest we perform a spatial Fourier transform, in terms of which the reduced effective action, including the influence functional of the hard modes is given by

$$S_{eff}[\mathcal{A}_{Ti},R_{Ti}] = \int \frac{d^3k}{(2\pi)^3}\int dt\Bigg\{ -R_{Ti}(\vec{k},t)\left(\frac{d^2}{dt^2}+k^2+2e^2\langle\Phi^\dagger\Phi\rangle\right) \times \mathcal{A}_{Ti}(-\vec{k},t) - \int^t dt' R_{Ti}(\vec{k},t)\mathcal{D}_{ij}(k;t,t')\mathcal{A}_{Tj}(-\vec{k},t') + i\int dt' R_{Ti}(\vec{k},t)\mathcal{N}_{ij}(k;t,t')R_{Tj}(-\vec{k},t')\Bigg\} \tag{5.4}$$

where $\mathcal{D}_{ij}$ and $\mathcal{N}_{ij}$ will be shown to be the dissipation and noise kernels respectively. The dissipation kernel which is given by

$$\mathcal{D}_{ij}(\vec{k};t,t') = 4ie^2\int\frac{d^3p}{(2\pi)^3}p_{Ti}p_{Tj}[G^<_p(t,t')G^<_{k+p}(t,t') - G^>_p(t,t')G^>_{k+p}(t,t')]\Theta(t-t') = 2e^2\int\frac{d^3p}{(2\pi)^3\omega_p\omega_{k+p}}p_{Ti}p_{Tj}\{(1+n_p + n_{p+k})\sin[(\omega_p+\omega_{k+p})(t-t')] + (n_p - n_{p+k})\sin[(\omega_{k+p}-\omega_p)(t-t')]\} \times\Theta(t-t') \tag{5.5}$$

clearly gives a causal contribution to the effective action as seen in Eq. (5.4). The fact that it is real follows from the properties of the non-equilibrium Green's functions [Eqs. (2.10) and (2.11)] which imply that

$$G^<_p(t,t')G^<_{k+p}(t,t') - G^>_p(t,t')G^>_{k+p}(t,t') \sim 2i\,\mathrm{Im}[G^<_p(t,t')G^<_{k+p}(t,t')].$$

Furthermore the dissipation kernel is in fact precisely the one-loop self-energy which appears in the effective action equation of motion Eq. (3.6). The noise kernel on the other hand is *acausal* and gives an *imaginary* contribution to the effective action

$$\mathcal{N}_{ij}(\vec{k};t,t') = -e^2\int\frac{d^3p}{(2\pi)^3}p_{Ti}p_{Tj}[G^<_p(t,t')G^<_{k+p}(t,t') + G^>_p(t,t')G^>_{k+p}(t,t')] = \frac{e^2}{2}\int\frac{d^3p}{(2\pi)^3\omega_p\omega_{k+p}}p_{Ti}p_{Tj}\{(1+n_p+n_{p+k}+2n_pn_{p+k})\cos[(\omega_p+\omega_{k+p})(t-t')] + (n_p+n_{p+k}+2n_pn_{p+k})\cos[(\omega_{k+p}-\omega_p)(t-t')]\}. \tag{5.6}$$

Therefore,

$$\mathcal{N}_{ij} \propto \mathrm{Re}\left[\int d^3p\,p_{Ti}p_{Tj}G^<_p(t,t')G^<_{k+p}(t,t')\right] \tag{5.7}$$

which means that the noise-noise correlator and the dissipation kernel are the real and imaginary parts respectively of the same analytic function of $(t-t')$. Thus they are automatically related by dispersion relations which reveal the fluctuation-dissipation theorem within this context.

The imaginary, non-local, acausal part of the effective action gives a contribution to the path-integral that may be rewritten in terms of a stochastic field as [50]

$$\mathcal{Z} = \int \mathcal{D}R_{Ti}(\vec{k})\mathcal{D}R^*_{Ti}(\vec{k})\exp\left[-\int dt dt'\frac{d^3k}{(2\pi)^3}R^*_{Ti}(\vec{k},t)\mathcal{N}_{ij}(k;t,t')R_{Tj}(\vec{k},t')\right] \propto \int \mathcal{D}R_{Ti}(\vec{k})\mathcal{D}R^*_{Ti}(\vec{k})\mathcal{D}\xi_i(\vec{k})\mathcal{D}\xi^*_i(\vec{k})P[\xi]\exp\left[i\int dt\frac{d^3k}{(2\pi)^3}\xi_i(\vec{k},t)R_{Ti}(-\vec{k},t) + \mathrm{c.c.}\right] \tag{5.8}$$

where the noise has a Gaussian probability distribution

$$P[\xi] = \exp\left[-\int dt dt'\,\xi_i(\vec{k},t)\mathcal{N}^{-1}_{ij}(k;t,t')\xi_j(-\vec{k},t')\right] \tag{5.9}$$

with zero mean and non-Markovian correlations:

$$\langle\langle \xi_i(\vec{k},t)\rangle\rangle = 0;$$

$$\langle\langle \xi_i(\vec{k},t)\xi_j(\vec{k}',t')\rangle\rangle=(2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')\mathcal{N}_{ij}(\vec{k};t,t'), \tag{5.10}$$

where $\langle\langle\cdots\rangle\rangle$ is the *stochastic* average with the probability distribution $P[\xi]$.

In the hard thermal loop approximation the leading terms in the noise kernel can be calculated explicitly, giving

$$\mathcal{N}_{ij}(\vec{k};t,t')\simeq\frac{e^2T^3}{6}\mathcal{P}_{ij}(\vec{k})\left[\frac{\sin[k(t-t')]}{k^3(t-t')^3}-\frac{\cos[k(t-t')]}{k^2(t-t')^2}\right]. \tag{5.11}$$

Clearly these are *long range* correlations which cannot be replaced by local delta function in any regime of approximation. In fact the Fourier transform of the kernel yields

$$\tilde{\mathcal{N}}_{ij}(\vec{k},w)=\int_{-\infty}^{+\infty}d(t-t')\mathcal{N}_{ij}(\vec{k};t-t')e^{-i\omega(t-t')} \tag{5.12}$$

$$=\mathcal{P}_{ij}(\vec{k})\frac{e^2T^3}{24k}\left(1-\frac{\omega^2}{k^2}\right)\Theta(k^2-\omega^2)$$

$$=\mathcal{P}_{ij}(\vec{k})\frac{T}{2\omega}\Sigma_I(i\omega+0^+), \tag{5.13}$$

where $\Sigma_I(i\omega+0^+)$ is the imaginary part of the self-energy given by Eq. (4.10) responsible of the Landau damping processes.

This clearly shows Landau damping to be the origin of the noise correlation and also provides an explicit realization of the fluctuation-dissipation theorem. We draw attention to the factor of T/ω which arises from the high T limit of the Bose factor $(1+2n_\omega)$ (see below) and leads to a noise-noise correlation $\propto e^2T^3$. This result is in accord with the usual fluctuation-dissipation relation, in which the noise-noise correlation function has one more power of T than the dissipative contribution to the Langevin equation.

The Langevin equation itself is obtained via the saddle point condition [50]

$$\left.\frac{\delta S_{eff}}{\delta R_i(\vec{k},t)}\right|_{R=0}=\xi_i(\vec{k},t) \tag{5.14}$$

leading to

$$\frac{d^2}{dt^2}\mathcal{A}_{Ti}(\vec{k},t)+k^2\mathcal{A}_{Ti}(\vec{k},t)+2e^2\langle\Phi^\dagger\Phi\rangle\mathcal{A}_{Ti}(\vec{k},t)-\int_{-\infty}^{t}dt'\mathcal{D}_{ij}(k;t,t')\mathcal{A}_{Tj}(\vec{k},t')=\xi_i(\vec{k},t'). \tag{5.15}$$

B. Fluctuation-dissipation relation

The general form of the fluctuation-dissipation relation can be established at this level by retaining the complete expressions for the dissipative and noise kernels without performing the HTL approximation. A standard analysis of the spectral representation [39,51] of the equilibrium correlators leads to the following result for the Fourier transform of the noise-noise correlation function

$$\tilde{\mathcal{N}}_{ij}(\vec{k},\omega)=\frac{1}{4}\mathrm{Im}[\Sigma^{ret}_{ij}(\vec{k};\omega)]\coth\left[\frac{\beta\omega}{2}\right]. \tag{5.16}$$

Finally taking temporal Fourier transforms of both sides of the Langevin equation (5.15) and averaging over the noise with the Gaussian distribution function we find

$$\langle\langle\tilde{\mathcal{A}}_{Ti}(\vec{k},\omega)\tilde{\mathcal{A}}_{Ti}(-\vec{k},-\omega)\rangle\rangle=|\tilde{\mathcal{A}}^H_{Ti}(\vec{k},\omega)|^2+\rho(\vec{k},\omega)\coth\left[\frac{\beta\omega}{2}\right]$$

$$\rho(\vec{k},\omega)=\frac{1}{4}\,\frac{\mathrm{Im}[\Sigma^{ret}_{ij}(\vec{k},\omega)]}{(\omega^2-\omega_k^2-\mathrm{Re}[\Sigma^{ret}_{ij}(\vec{k},\omega)])^2+(\mathrm{Im}[\Sigma^{ret}_{ij}(\vec{k},\omega)])^2} \tag{5.17}$$

where the tildes stand for the Fourier transform and $\tilde{\mathcal{A}}^H_{Ti}(\vec{k},t)$ is a solution of the homogeneous equation, which is precisely given by Eq. (4.36) [assuming $\dot{\mathcal{A}}^H_{Ti}(\vec{k},t=0)=0$ which can be relaxed with the proper generalization]. The double brackets stand for averages over the noise with the probability distribution (5.9), (5.10). This is the form of the usual fluctuation-dissipation relation, which we obtained consistently by integrating out the hard modes and deriving the influence functional [48,49] for the transverse components of the gauge invariant fields. The semiclassical Langevin equation is useful in order to obtain semiclassical correlation functions by averaging the solution of a partial differential equation over a stochastic Gaussian noise. The question arises: to which correlation function of the microscopic theory are these stochastic averages related? The answer to this question is found by writing a spectral representation of the equilibrium correlator $\langle\mathcal{A}_{i,\vec{k}}(t)\mathcal{A}_{i,-\vec{k}}(t')\rangle$ in which the brackets stand for averages in the equilibrium density matrix. A straightforward but tedious exercise following the steps described in [39,51] reveals that this relation is given by

$$\langle\langle\tilde{\mathcal{A}}_{Ti}(\vec{k},\omega)\tilde{\mathcal{A}}_{Ti}(-\vec{k},-\omega)\rangle\rangle=\frac{1}{2}\{\langle A_i(\vec{k},\omega)A_i(-\vec{k},-\omega)\rangle+\langle A_i(-\vec{k},-\omega)A_i(\vec{k},\omega)\rangle\} \tag{5.18}$$

which again is a result known to be a consequence of the fluctuation-dissipation relation in simple systems. The correlation function (5.18) has a finite, non-trivial classical limit and agrees with the one proposed to be studied within the context of classical field theory in [23].

The real-time analysis presented here agrees with the general picture discussed in Refs. [27–29]. Moreover, our analysis reveals the precise range of the kernels.

Clearly the situation will be more complicated in QCD where a separation between hard and soft degrees of freedom must be implemented in order to obtain the influence functional for the soft degrees of freedom. However, the procedure detailed in this section can be carried out consistently once this separation is introduced.

Of course the main rationale for obtaining a Langevin equation is to provide a semiclassical scheme to implement the calculation of correlation functions from the solutions of stochastic differential equations. However, we note that unless a successful scheme to deal with the non-Markovian kernels is implemented the advantages of a Langevin description are at best formal. Ignoring the non-localities of the dissipative and noise kernels will clearly miss the important physics associated with Landau damping. A naive Markovian approximation is not only uncontrolled and unwarranted but clearly very untrustworthy in view of the fact that the relevant kernels are all long-ranged and it is precisely this long-ranged nature of these kernels which is responsible for the important dissipative effects of Landau damping. The importance and difficulties of keeping these non-localities in a classical lattice description has been recognized in [33–35].

VI. KINETICS OF LANDAU DAMPING

The real-time formulation of non-equilibrium quantum field theory allows us to obtain the corresponding kinetic equations for the relaxation of the occupation number or population of quanta. In particular our goal is to obtain the kinetic equation for the relaxation of the expectation value of the number of soft quanta. In keeping with the focus of this article we will only consider the population relaxation to lowest order in the HTL approximation and concentrate mainly on the understanding of relaxation via Landau damping.

Kinetic approaches towards describing transport phenomena and relaxational dynamics typically require a wide separation between microscopic time and length scales, namely the thermal (or Compton) wavelength (mean separation of particles) and the relaxation scales (mean free path and relaxation time). This approach which ultimately leads to the Boltzmann transport equations involves the identification of slow and fast variables which justifies a gradient expansion. This is a coarse graining procedure that averages over microscopic time scales and leads to irreversible time evolution.

In the collisional approach to Boltzmann kinetics only the distribution-changing processes that conserve energy and momentum are considered, and these are weighted by the corresponding Bose/Fermi statistical factors. Off-shell processes that occur on time scales shorter than the relaxation scale are not included; therefore only processes with asymptotically on-shell final states are accounted for in this description. Landau damping processes which contribute via thermal loops to forward scattering are not included in the typical Boltzmann equation. Thus we anticipate that the naive Boltzmann approach will yield no population relaxation via Landau damping. However this is conceptually inconsistent because we have learned in the previous sections that an initial coherent configuration will relax to an amplitude smaller than the initial by these processes and we would expect such a relaxation to contribute to a depletion of the number of quanta of the initial state. The resolution of this inconsistency requires one to go *beyond* a simple Boltzmann approach and to include off-shell processes in the kinetic description. This is the focus of this section.

The contribution of *off-shell* processes to the *non-equilibrium* evolution of particle distributions is ignored in most approaches towards kinetics. Recently an approach to kinetics that incorporates off-shell effects has been proposed to describe processes in which relaxation competes with other fast scales [40,41], in particular near phase transitions [40]. The importance of off-shell contributions to the evolution of particle distributions has also been recognized within the context of fast kinetics in semiconductors [42].

This section is devoted to a study of kinetics as an *initial value problem* [40–42] in order to reveal the role played by off-shell processes in transport phenomena and relaxation in the medium.

We *derive* a kinetic equation that takes into account microscopic time scales in the theory from first principles allowing us to analyze clearly the effect of *off-shell* non-collisional Landau damping processes on the evolution of the photon distribution. This framework will allow us to make contact with the relaxation of a coherent initial configuration studied in the previous sections. We want to study both the relaxation of an initial distribution of asymptotic photons with free dispersion relation $\Omega(k)=k$, as well as for the quasiparticles with dispersion relation $\Omega(k)=\omega_p(k)$ with $\omega_p(k)$ being the solution of the dispersion equation (4.11) i.e. the ''true,'' in-medium pole. We will distinguish between these two physically different cases and address them separately.

This approach begins by defining a suitable number operator [40]. In the case of asymptotic photons this is the usual number operator in terms of the canonical field and momenta which is given by the energy per momentum k divided by the frequency. In the case of quasiparticles the energy stored in the plasma has two components: the free field part plus the response from the medium. From classical electromagnetism of polarizable media [52] in linear response, the two contributions lead to an energy density in the medium that is quadratic in terms of the electric and magnetic fields, each term however, multiplied by a coefficient that involves derivatives of the dielectric and permeability tensors with respect to frequency [52]. The contribution from the plasma collective modes is obtained by evaluating these coefficients at the plasma frequency. Using Kramers Kronig dispersion relations [52], these coefficients are related to the residue of the dielectric constant at the plasma poles. This relation has been formalized at the field theoretical level by Migdal in his pioneering work on collective modes in medium where he developed the quantization procedure in medium in terms of quasiparticle operators [53]. This field theoretical treatment automatically leads to the identification of these coefficients with the residues of the retarded propagators at the plasmon poles, i.e. the wave function renormalization. Migdal obtains in this manner the energy density corresponding to on-shell collective modes in terms of the quasiparticle operators [53].

More recently the energy density of the plasma including the polarization effects has been obtained in terms of operators that create and destroy collective modes in the plasma by Blaizot and Iancu [54]. The results of these authors is consistent with the collective mode quantization and the energy density obtained by Migdal [53] and with the results of classical polarizable media [52]. Blaizot and Iancu [54] use the collective mode decomposition of the transverse gauge field

$$\vec{A}^t_{\vec{k}}(t)=\sqrt{\frac{Z^t_k}{2\omega_p(k)}}\sum_{\lambda=1,2}[\vec{\epsilon}_\lambda(\vec{k})a_\lambda(\vec{k})e^{-i\omega_p(k)t}+\vec{\epsilon}_\lambda(-\vec{k})a^\dagger_\lambda(-\vec{k})e^{i\omega_p(k)t}] \tag{6.1}$$

where the operators $a^\dagger_\lambda(\vec{k});a_\lambda(\vec{k})$ have a retarded propagator with unit residue at the plasmon pole and Z^t_k is the wave function renormalization.

Migdal [53] and Blaizot and Iancu [54] prove that the energy density associated with the collective modes in the medium can be written as

$$\mathcal{E}(k)=\omega_p(k)\sum_{\lambda=1,2}a^\dagger_\lambda(\vec{k})a_\lambda(\vec{k}). \tag{6.2}$$

This result is the same as that obtained in the classical theory of polarizable media when the electric and magnetic fields (transverse) are written in terms of collective modes [52].

Thus following Migdal [53] and Blaizot and Iancu [54] we introduce the Heisenberg number operator of on-shell collective modes

$$\hat{N}_{\vec{k}}=\frac{1}{4\Omega_k\mathcal{Z}^t_k}[\dot{\vec{A}}_T(\vec{k})\cdot\dot{\vec{A}}_T(-\vec{k})+\Omega^2_k\vec{A}_T(\vec{k})\cdot\vec{A}_T(-\vec{k})]-\frac{1}{2\mathcal{Z}^t_k} \tag{6.3}$$

where for asymptotic photons $\Omega_k=k;\mathcal{Z}^t_k=1$ (we neglect here the zero temperature contribution) and for collective modes $\Omega_k=\omega_p(k);\mathcal{Z}^t_k=Z^t[T]$ with $\omega_p(k)$ being the plasmon pole and $Z^t[T]$ the wave function renormalization in the HTL limit. This formulation now permits to treat the collective modes much in the same manner as the usual renormalized in and out fields in S-matrix theory, i.e. by rewriting the action in terms of the renormalized fields and introducing counterterms that reflect the true position of the pole and residue.

It turns out to be easier to work with the Heisenberg operator $\dot{\hat{N}}_k$ rather than the number operator and is obtained using the Heisenberg operator equations which are easily seen to be

$$\dot{\vec{A}}_T(\vec{k},t)=\vec{\Pi}_T(-\vec{k},t)$$

$$\dot{\vec{\Pi}}_T(-\vec{k},t)=-(k^2+2e^2\langle\Phi^\dagger\Phi\rangle+\delta\Omega^2_k)\vec{A}_T(\vec{k},t)-\vec{j}_T(\vec{k},t)$$

where the counterterm accounts for the definition of the number of quasiparticles and $\vec{\Pi}_T(\vec{k})$ represents the canonical momentum conjugate to the transverse electromagnetic field $\vec{A}_T(\vec{k})$.

We now consider an initial state described by a density matrix for which the *expectation value* of the above number operator for on-shell collective modes is non-vanishing.

Using these equations the expectation value of the Heisenberg rate operator in the initial density matrix is obtained in the following form which is rather convenient for subsequent calculations:

$$\langle\dot{\hat{N}}_k\rangle(t)=\dot{N}_k(t)=-\frac{1}{2\Omega_k\mathcal{Z}^t_k}\frac{\partial}{\partial t''}[\langle\vec{j}^+_T(\vec{k},t)\cdot\vec{A}^-_T(-\vec{k},t'')\rangle]_{t=t''}-\frac{2e^2\langle\Phi^\dagger\Phi\rangle+\delta\Omega^2_k}{2\Omega_k\mathcal{Z}^t_k}\frac{\partial}{\partial t''}[\langle\vec{A}^+_T(\vec{k},t)\cdot\vec{A}^-_T(\vec{k},t'')\rangle+\langle\vec{A}^+_T(\vec{k},t'')\cdot\vec{A}^-_T(-\vec{k},t)\rangle]_{t=t''}. \tag{6.4}$$

This formulation has been previously applied to the study of photon production in a strongly out of equilibrium phase transition [40].

The expectation values are calculated by inserting the operators into the closed time path integral and expanding in powers of α. Since $\delta\Omega^2_k$ is of order α the second term in Eq. (6.4) is calculated as a tadpole in free field theory and it vanishes identically. Let us consider the case in which the initial density matrix at time t_0 is diagonal in the basis of eigenstates of the number operator, and evolves subsequently with the interaction Hamiltonian. To lowest order in α we find the expectation value of the rate to be [40]

$$\dot{N}_k(t)=\frac{e^2}{4\pi^3\Omega_k\mathcal{Z}^t_k}\int^t dt'\int dp[p^2-(\vec{p}\cdot\hat{k})^2][G^>_p(t,t')G^>_{k+p}(t,t')\dot{\mathcal{G}}^<_k(t',t)-G^<_p(t,t')G^<_{k+p}(t,t')\dot{\mathcal{G}}^>_k(t',t)]\Theta(t-t'). \tag{6.5}$$

The theta function ensures that this expression is causal. The Green's functions for the scalars and the photons can be read off from Eqs. (2.10), (2.11), (2.12) and (2.13) but with the frequency Ω_k replacing the bare frequency. Finally the rate can be written as [40]

$$\dot{N}_{\vec{k}}(t)=\frac{e^2}{16\pi^3\Omega_k\mathcal{Z}_k^t}\int\frac{d^3p}{\omega_p\omega_{k+p}}p^2\sin^2\theta\int_{t_0}^{t}d\tau\{\cos[(\omega_p+\omega_{k+p}+\Omega_k)(t-\tau)][(1+N_k(t_0))(1+n_p)(1+n_{k+p})-N_k(t_0)n_pn_{k+p}]$$
$$+\cos[(\omega_p+\omega_{k+p}-\Omega_k)(t-\tau)][(1+N_k(t_0))n_pn_{k+p}-N_k(t_0)(1+n_p)(1+n_{k+p})]+\cos[(\omega_p-\omega_{k+p}+\Omega_k)(t-\tau)]$$
$$\times[(1+N_k(t_0))(1+n_p)n_{k+p}-N_k(t_0)n_p(1+n_{k+p})]+\cos[(\omega_p-\omega_{k+p}-\Omega_k)(t-\tau)][(1+N_k(t_0))(1+n_{k+p})n_p$$
$$-N_k(t_0)n_{k+p}(1+n_p)]\}. \tag{6.6}$$

We note that the expression above depends on the occupation number of the gauge field at the initial time t_0 only—this is obviously a consequence of the fact that perturbation theory at lowest order, neglects the change in occupation number. Recently we have proposed [40] a Dyson-like resummation of the perturbative expansion that includes off-shell effects in the relaxation of the population. This resummation scheme is obtained by the replacement $N_k(t_0)\rightarrow N_k(\tau)$ in Eq. (6.6) resulting in a non-Markovian description. The resulting kinetic equation with memory is akin to that obtained via the generalized Kadanoff-Baym approximation [42] in non-relativistic many body systems. This approximation has been recently shown to coincide with the exact result in the weak coupling limit in a solvable model of relaxation [56,57] and will be shown below to imply a Dyson-resummation of the perturbative series. The kinetic Eq. (6.6) has an obvious interpretation in terms of gain minus loss processes [40], but the retarded time integrals and the cosine functions replace the more familiar energy conserving delta functions. Taking the occupation number outside the integral and integrating to large times, thereby replacing the cosines by delta functions as in a Boltzmann description would lead to a vanishing right hand side since none of the resulting energy conserving delta functions can be satisfied. However, the non-Markovian kinetic equation (6.6) will lead to non-trivial relaxational dynamics that will be studied in detail below.

Let us consider the situation in which the initial state has been prepared far in the past, i.e. $t_0\rightarrow-\infty$.

An equilibrium solution is simply

$$N_k^{eq}=\text{const} \tag{6.7}$$

where the constant is arbitrary because none of the resulting energy-conserving delta functions can be satisfied for the values of Ω_k either corresponding to the bare frequencies or the quasiparticle poles. This is a consequence of the off-shell processes, a detailed understanding of this feature will be provided elsewhere [56,57].

Let us now consider departures from this equilibrium solution and study the relaxation of a disturbance in the distribution function introduced in the system at $t=0$ so that

$$N_k(t=0)=N_k^{eq}+\delta N_k(0). \tag{6.8}$$

Denoting the particle distribution for $t>0$ by

$$N_k(t>0)=N_k^{eq}+\delta N_k(t) \tag{6.9}$$

we obtain a rate equation for $\delta N_k(t)$ which is now the same as in the previous step except that the time integrals stretch from 0 to t instead of $t_0\rightarrow-\infty$ to t

$$\frac{d}{dt}\delta N_{\vec{k}}(t)=\frac{e^2}{16\pi^3\Omega_k\mathcal{Z}_k^t}\int\frac{d^3p}{\omega_p\omega_{k+p}}p^2\sin^2\theta\int_0^t d\tau\{(1+n_p+n_{p+k})[\cos[(\omega_p+\omega_{k+p}+\Omega_k)(t-\tau)]-\cos[(\omega_p+\omega_{k+p}-\Omega_k)$$
$$\times(t-\tau)]](n_{p+k}-n_p)[\cos[(\omega_p-\omega_{k+p}+\Omega_k)(t-\tau)]-\cos[(\omega_p-\omega_{k+p}-\Omega_k)(t-\tau)]]\}\delta N_k(\tau). \tag{6.10}$$

Terms independent of N_k vanish identically since the time integrals for those terms yield delta functions which are never satisfied.

Although this non-Markovian but linear equation will be solved exactly by Laplace transform below, it is illuminating to compare the different approximations that are obtained under the assumption that the relaxation time scale for δN_k is much longer than the time scale of the non-local kernels. Under this assumption, which will be analyzed below, $\delta N_k(\tau)$ can be replaced by $\delta N_k(t)$ and taken outside of the integral leading to a Markovian description. A further approximation, taking the upper limit of the remaining integral to $t\rightarrow\infty$ leads to the familiar Boltzmann equation, thus the two approximations to be compared with the ''exact'' solution are the following.

Markovian:

$$\frac{d}{dt}\delta N_k^M(t) = -\Gamma(t)\,\delta N_k^M(t) \tag{6.11}$$

$$\Gamma(t) = -\frac{e^2}{16\pi^3\Omega_k \mathcal{Z}_k^t}\int \frac{d^3p}{\omega_p\omega_{k+p}}p^2\sin^2\theta\int_0^t d\tau\{(1+n_p+n_{p+k})[\cos[(\omega_p+\omega_{k+p}+\Omega_k)(t-\tau)]-\cos[(\omega_p+\omega_{k+p}-\Omega_k)(t-\tau)]]$$
$$\times(n_{p+k}-n_p)[\cos[(\omega_p-\omega_{k+p}+\Omega_k)(t-\tau)]-\cos[(\omega_p-\omega_{k+p}-\Omega_k)(t-\tau)]]\}, \tag{6.12}$$

with solution

$$\delta N_k^M(t) = e^{-\int_0^t\Gamma(t')dt'}\,\delta N_k^M(0). \tag{6.13}$$

Boltzmann:

$$\frac{d}{dt}\delta N_k^B(t) = -\Gamma(\infty)\,\delta N_k^B(t) \tag{6.14}$$

$$\delta N_k^B(t) = e^{-\Gamma(\infty)t}\delta N_k^B(0). \tag{6.15}$$

Taking the limit $t\to\infty$ in $\Gamma(t)$, the cosines become energy conserving delta functions and comparing with the expression for the transverse self-energy given by Eq. (4.7), it is straightforward to see that in the Boltzmann approximation we obtain

$$\frac{d}{dt}\delta N_k^B(t) \overset{t\to\infty}{=} -\frac{\Sigma_I^t(\Omega_k)}{\Omega_k}\delta N_k^B(t)\equiv 0, \tag{6.16}$$

with $\Sigma_I^t(\Omega_k)$ being the imaginary part of the transverse self-energy evaluated at Ω_k. This result is the familiar relationship between the relaxation rate of the particle distribution $\Gamma(\infty)$ and the damping rate, which is determined by the imaginary part of the self-energy *on-shell* and vanishes in the present case because the damping processes are *off-shell*.

The solution of the non-Markovian equation (6.10) is obtained by Laplace transform and given in general by

$$\delta N_k(t) = \delta N_k(0)\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\,\frac{e^{st}}{s-\mathcal{S}_k(s)}, \tag{6.17}$$

where c is a real constant chosen so that the contour is to the right of the singularities of the integrand and $\mathcal{S}_k(s)$ is the Laplace transform of the non-local kernel given by

$$\mathcal{S}_k(s) = \frac{e^2}{16\pi^3\Omega_k}\int\frac{d^3p}{\omega_p\omega_{k+p}}p^2 \times\sin^2\theta\left\{\left[\frac{s}{s^2+(\omega_p+\omega_{k+p}+\Omega_k)^2} - \frac{s}{s^2+(\omega_p+\omega_{k+p}-\Omega_k)^2}\right]\right.$$
$$\left.\times(1+n_p+n_{k+p}) + \left[\frac{s}{s^2+(\omega_p-\omega_{k+p}+\Omega_k)^2} - \frac{s}{s^2+(\omega_p-\omega_{k+p}-\Omega_k)^2}\right](n_{p+k}-n_p)\right\}. \tag{6.18}$$

The expression (6.17) clearly shows that the non-Markovian rate equation (6.10) implies a Dyson-like resummation of the perturbative series as anticipated before.

Comparing the non-Markovian rate equation (6.10) to the Markovian approximation (6.11) one can clearly see that the Markovian approximation averages over the time scales of the kernel, whereas the non-Markovian equation includes the contribution of coherent processes throughout the history of the kernel. If the range of the kernel was indeed shorter than the relaxation time of the population, the real-time solutions of both equations will differ by terms of the order of the ratio of the time scale of the kernel to the relaxation time scale of the distribution. However, in situations in which the kernel is long-ranged as is the case under consideration, the non-Markovian expression allows the inclusion of coherent effects in the relaxation.

The dominant contribution to $\mathcal{S}(s)$ in the HTL limit arises from the Landau damping term leading to the simplified expression

$$\mathcal{S}_k(s) = -\frac{e^2T^2ks}{12\Omega_k}\int_{-1}^{1}\frac{x(1-x^2)}{s^2+(kx-\Omega_k)^2}dx. \tag{6.19}$$

More explicitly,

$$\mathcal{S}_k(s) = -\frac{e^2T^2}{12k}\left\{4s+\frac{s(k^2+s^2-3\Omega_k^2)}{2k^2\Omega_k}\times\log\frac{s^2+(k-\Omega_k)^2}{s^2+(k+\Omega_k)^2}+\frac{k^2-\Omega_k^2+3s^2}{2ik^2}\times\log\frac{(s+ik)^2+\Omega_k^2}{(s-ik)^2+\Omega_k^2}\right\}. \tag{6.20}$$

It is easy to show that this form of $\mathcal{S}_k(s)$ is in fact related to the self-energy in the HTL limit by the following illuminating equation:

$$\mathcal{S}_k(s) = \frac{\Sigma^t(s-i\Omega_k)-\Sigma^t(s+i\Omega_k)}{2i\Omega_k}. \tag{6.21}$$

The real-time solutions to Eqs. (6.10), (6.17) and (6.18) in the HTL limit with Eq. (6.19) will now be given for the cases of quasiparticle and bare particle respectively. This analysis will reveal the range of validity of the assumptions leading to the Markovian approximation.

A. Quasiparticle: $\Omega_k = \omega_p(k)$

When Ω_k is chosen to be the quasi-particle pole, $\omega_p(k) > k$ we note that the integrand of Eq. (6.17) has a single isolated pole at $s=0$. Indeed, the limit of $s \to 0$ in $\mathcal{S}_k(s)$ would lead to delta functions in Eq. (6.19) which, however, cannot be satisfied for $\Omega_k = \omega_p(k)$. Therefore, $\mathcal{S}_k(0)=0$ and $s=0$ is an isolated single pole and completely determines the asymptotic limit of the real-time solution. This can also be seen by looking at Eq. (6.21), from which the analytic structure is explicit. Clearly, $\mathcal{S}_k(s)$ vanishes at $s=0$ because $\Sigma^t(-i\omega_p(k)) = \Sigma^t(i\omega_p(k))$. Furthermore from the known singularity structure of $\Sigma^t(s)$ one concludes that $\mathcal{S}_k(s)$ must have branch cuts for $i(\omega_p(k)-k) < s < i(\omega_p(k)+k)$ and $-i(\omega_p(k)-k) < s < -i(\omega_p(k)+k)$. Using Eq. (6.21), it is now a straightforward exercise to see that the residue at this pole at $s=0$ is given by $(1-2\partial\Sigma^t(i\Omega)/\partial\Omega^2)^{-1}$ which to this order is $\approx Z^t[T]^2$ where $Z^t[T]$ the (transverse) wave function renormalization given in Eq. (4.15).

Again the long time behavior is completely dominated by the end points of the cut, leading to the asymptotic result

$$\delta N_k(t) \overset{t\to\infty}{=} \delta N_k(0) \Bigg\{ Z^t[T] + \frac{e^2T^2\pi^2}{12 Z^t[T] k \omega_p(k) t^2} \times \Bigg[\frac{\cos(\omega_p(k)+k)t}{(\omega_p(k)+k)^2 \left(1+\frac{e^2T^2D_+}{3k^2}\right)} - \frac{\cos(\omega_p(k)-k)t}{(\omega_p(k)-k)^2\left(1+\frac{e^2T^2D_-}{3k^2}\right)} \Bigg] \Bigg\} \tag{6.22}$$

where

$$D_\pm \equiv 1 + \left(\frac{1}{2} \pm \frac{\omega_p(k)}{k}\right) \log\left(1 \pm \frac{k}{\omega_p(k)}\right).$$

We clearly see that asymptotically the population has relaxed to a smaller value and the ratio of the asymptotic to the initial population is determined by the square of the thermal wave-function renormalization. This is in agreement with the analysis of the relaxation of the expectation value of the field—since $N_k \propto A_T^2$ it is expected that the ratio of the asymptotic value of the quasi-particle population to the initial value be proportional to the square of the same relation for the expectation value of the field.

Therefore the relaxation of the quasiparticle number has its origin in Landau damping, this is consistent with the results of Blaizot and Iancu [54] who proved that the time derivative of the total energy is related to Landau damping. Since the quasiparticle number is related to the energy of the collective modes the relaxation of the quasiparticle number is directly related to Landau damping.

This intuition is also borne out in the usual Boltzmann approach wherein the relaxation rate of the distribution function (in the relaxation time approximation) is twice the damping rate for the quasiparticle.

This relaxation has a simple interpretation. Consider the case of a physical electron with interpolating operators defined to create single electrons with the physical mass and unit amplitude out of the in or out vacuum states. The asymptotic correlation function in real time of these interpolating operators has the oscillatory parts corresponding to the physical pole (with unit residue), but there are power law corrections arising from the overlap of the states created by these operators with the multiparticle continuum [55]. Whereas at zero temperature the multiparticle continuum is beyond the two particle threshold, at finite temperature and in the case under consideration, the leading contribution is obtained from Landau damping corresponding to intermediate states with space-like momenta.

For the Markovian approximation (6.13) we find, for $t \to \infty$,

$$-\int_0^t \Gamma(t')dt' \approx 2 \left.\frac{\partial \Sigma_t(\omega)}{\partial \omega^2}\right|_{\omega_p(k)} + \mathcal{O}\left[\left(\frac{e^2T^2}{k^2}\right)\left(\frac{\cos[(\omega_k-k)t]}{t^2(\omega_k-k)^2} + \frac{\cos[(\omega_k+k)t]}{t^2(\omega_k+k)^2}\right)\right]. \tag{6.23}$$

For $e^2T^2/k^2 \ll 1$ we see that to lowest order in e^2T^2/k^2 the perturbative expansion of the Markovian solution coincides with the solution of the non-Markovian equation. However for soft momenta such an expansion is not valid and the validity of the Markovian approximation must be questioned.

B. Failure of the Markovian and Boltzmann approximation

To assess whether the Markovian and Boltzmann approximations will be reliable we must understand the different time scales, in particular the range of the kernel.

In the hard thermal approximation we find that the non-Markovian kinetic equation (6.10) reduces to

$$\frac{d}{dt}\delta N_k(t) \simeq -\frac{e^2T^2k}{12\Omega_k}\int_{-1}^1 dx(1-x^2)x\int_0^t d\tau \sin[kx(t-\tau)] \times \sin[\Omega_k(t-\tau)]\delta N_k(\tau). \tag{6.24}$$

The integral over the variable x inside the kernel can be performed and we find that the kernel falls off as $1/(t-\tau)^2 + \cdots$. Then if $eT/k \ll 1$ the relaxation time scale of the population is longer than the range of the kernel and the Markovian approximation is warranted. In this case the discrepancies between the non-Markovian and Markovian results are perturbatively small. On the other hand, for soft scales the relaxation time scales become comparable to the time scale of the kernel and a Markovian approximation is certainly unjustified. The non-Markovian equation for relaxation includes the coherent effects on similar time scales that are averaged out (coarse-grained) in the Markovian approximation.

C. Bare particle, or hard quasiparticle

For the case of the bare particle, the dispersion relation is simply $\Omega_k = k$ – this is also the case for the large k limit of the quasiparticle dispersion relation [31,32,37]. In this case, the quantity $\mathcal{S}_k(s)/s$ in Eq. (6.18) has a logarithmic singularity as $s \to 0$, because the position of the putative pole Ω_k has moved to the tip of the cut and there is a pinching singularity. There is no longer a pole at $s=0$ in the Laplace transform $[s-\mathcal{S}_k(s)]^{-1}$, rather it diverges as $(s \ln s)^{-1}$ as $s \to 0$. This logarithmic divergence arising from the pinching singularity is very similar to that recently studied within the context of hard fermions [58].

In the Markovian approximation (6.11) and in the hard thermal loop limit, the rate equation becomes

$$\frac{d}{dt}\delta N_{\vec{k}}(t) = \frac{e^2}{4\pi^2}\int dp\,\frac{dn_p}{dp}\int_{-1}^{1} dx(1-x^2)xp^2 \times \int_0^t d\tau\{\cos[k(1-x)(t-\tau)]\}\delta N_k(t)$$
$$\simeq -\frac{e^2T^2}{12k}\left[\frac{\sin(2kt)}{k^2t^2} + \frac{2}{kt} - \frac{2}{k^3t^3} + \frac{2\cos(2kt)}{k^3t^3}\right]\delta N_k(t) \qquad (6.25)$$

which for long times yields a power law with an anomalous exponent:

$$\delta N_k(t) \sim \delta N_k(0)(kt)^{-e^2T^2/6k^2}. \qquad (6.26)$$

An anomalous exponent somewhat similar to this one has been found in [58] in the case of a hard fermion and it has the same origin, i.e., a pinching infrared singularity. The expression (6.25) reveals that the kernel is long ranged, falling off with an inverse power of time in this case. Therefore a Markovian approximation will be justified when $e^2T^2/k^2 \ll 1$ because only in this weak coupling limit is the population relaxation *slower* than the fall off of the kernel.

The solution of the non-Markovian equations (6.10), (6.18) in the HTL limit (only the Landau damping contribution is considered) is obtained again by inversion of the Laplace transform. We find that the long time behavior is given by the end-points. In this case the $\omega=0$ end-point dominates yielding

$$\delta N_k(t) = \delta N_k(0)\,\frac{e^2T^2}{6\,k^2}\mathrm{Re}\int_0^\infty \frac{dy}{y} \times \frac{e^{-y}}{\left[1+\frac{e^2T^2}{6\,k^2}\log\frac{2kt}{iy\bar{e}}\right]^2 + \left(\frac{\pi e^2T^2}{12\,k^2}\right)^2}\left[1+\mathcal{O}\left(\frac{1}{t}\right)\right] \qquad (6.27)$$

where $\bar{e} = 2.718\ldots$ is the base of the natural logarithms.

It is illuminating to point out that as $\Omega_k \to k$ the logarithmic singularities in the real part of the transverse self-energy imply that (for fixed finite k)

$$\left.\frac{\partial\Sigma^t(\omega)}{\partial\omega^2}\right|_{\omega\to k} \to \infty \Rightarrow Z^t[T] \to 0. \qquad (6.28)$$

As the pole approaches the tip of the cut (for finite k) the residue becomes smaller until it vanishes exactly when the pole merges with the continuum. It is remarkable that in this limit distortions of the distribution function relax completely and vanish asymptotically. Although for the case of a hard quasiparticle with $k \gg eT$ the dispersion relation approaches that of a free particle, $Z^t[T] \to 1$ and physics is perturbative, there is, however, slow relaxation. We note that none of these effects can be captured by a simple Boltzmann approach since all of these phenomena are associated with off-shell effects.

In the case of the distribution function for bare particles, we can interpret this anomalous relaxation as the dressing effect from the medium, i.e. at long times the bare particles are completely dressed by the medium and disappear from the spectrum.

The interpretation is different for the case of a hard quasiparticle, in which case the free dispersion relation is obtained in the limit $k \gg eT$. In this limit the effective coupling $eT/k \ll 1$ and the relaxation is slow. This result could be important in understanding the relaxation of a distribution of photons produced in bremmstrahlung processes at high energy in the quark gluon plasma. A full study of the relaxation of hard quasiparticles is beyond the realm of this article and will be studied in detail elsewhere [57].

VII. DISCUSSION AND CONCLUSIONS

Our goal in this article was to provide a detailed analysis of the real-time relaxation of soft gauge invariant non-equilibrium expectation value through the off-shell process of Landau damping. These determine the leading contributions to the thermal propagators in the hard thermal loop limit and are the dominant contributions to the long-time asymptotics. The off-shell nature of these processes determine the non-Markovian nature of the relaxation phenomena associated with them. We focussed our study on the leading HTL contributions to the relaxation of gauge invariant transverse and longitudinal non-equilibrium expectation value in scalar electrodynamics. These results will also apply to fermion electrodynamics and non-Abelian theories since the structure of the retarded Green's function is the same up to this order in the HTL expansion. After providing an elegant re-derivation of the well known HTL effective action using the tools of non-equilibrium quantum field theory in the linear amplitude approximation [36], we moved on to the main goals of this article:

(i) to study in detail the relaxation in real time, in the linear approximation (small amplitude) of the transverse and longitudinal gauge invariant non-equilibrium expectation value. The off-shell process of Landau damping results in power law relaxation of the transverse and logarithmic relaxation of the longitudinal (plasmon or charge-density) excitations. Both types of non-equilibrium expectation value relax to an asymptotic amplitude that depends on the thermal wave-function renormalization, which is completely determined by Landau damping in the HTL limit. One of the main conclusions of this detailed analysis is that the relaxational dynamics asymptotically at long times and to leading order in HTL resummation is completely determined by the behavior of the spectral density near *the Landau damping thresholds* at $\omega = \pm k$ (a branch cut singularity), the contribution from the region $\omega \approx 0$ are regular (no branch point singularities) and therefore lead to subleading corrections to dynamics at long times and high temperature. This result is for both longitudinal and transverse non-equilibrium expectation value and is confirmed by an exhaustive analytic and numerical study. The short time evolution of the non-equilibrium expectation value is determined by moments of the total spectral density. Therefore a complete understanding of the global analytic structure of the retarded propagator, in particular the complete cut contribution from Landau damping processes is required.

This is special to the HTL resummation at one-loop: at higher orders, a branch point could develop at $\omega = 0$. Such a branch point would produce pure power like tails, with no oscillations, and so dominate at large times.

We restricted ourselves in this paper to small amplitude non-equilibrium expectation value so that we were confined to the linear regime. New phenomena beyond the HTL scheme are expected in the non-linear amplitude regime. Such regimes can be studied within self-consistent Hartree-type approximations in the out of equilibrium framework [44].

(ii) We have obtained the influence functional (non-equilibrium effective action) for the soft gauge invariant degrees of freedom by integrating out the hard degrees of freedom to leading order in the HTL approximation. This allowed us to obtain the Langevin equation for the soft degrees of freedom to leading order in HTL and to provide a microscopic *ab initio* calculation of the dissipative and noise kernels in the HTL limit. Both kernels display the non-localities associated with Landau damping and we find that there is no region of time scales in which a Markovian approximation describes the dynamics correctly. As a byproduct we obtained the fluctuation-dissipation relation and recognized the correlation function that emerges in the classical limit. We established in detail that a Markovian description of relaxation of transverse or longitudinal non-equilibrium expectation value is unwarranted to this order in the HTL resummation.

(iii) Having understood the relaxation of coherent non-equilibrium expectation value through off-shell effects of Landau damping we asked how these processes can be incorporated in the relaxation of the distribution function for the transverse fields. A Boltzmann approach would yield no relaxation to lowest order in the HTL because there is no imaginary part on-shell for the transverse or longitudinal quasiparticles. Therefore a kinetic description of the relaxation of the distribution function must necessarily go beyond a Boltzmann collision approximation.

We provided a novel description of the kinetics of relaxation of the distribution function for transverse degrees of freedom that includes off-shell effects and goes far beyond the usual Boltzmann approach. We have compared the relaxation obtained from this non-Markovian kinetic equation to that described by the usual Boltzmann (which yields a trivial result) and a Markovian version that includes coarse-grained details of the off-shell processes. We have found that the distribution function for soft quasiparticles relaxes as a power $(1/t^2)$, and found an unusual dressing dynamics for bare particles. This kinetic approach also reveals unusual logarithmic real time relaxation for hard quasiparticles resulting from infrared pinching singularities similar to those found in the case of hard fermions [58].

The body of these results reveals new and unusual features of relaxation of soft degrees of freedom in gauge theories. These will obviously have to be taken seriously into account in a full description of relaxational processes in the QGP and should also be important to clarify better the role played by damping in the sphaleron rate in the symmetric phase.

In higher order corrections there will arise contributions from collisional processes that provide an imaginary part on-shell. At next order for example both Compton scattering and pair-annihilation will contribute to collisional relaxation and will provide a collisional width both to the transverse and longitudinal (plasmon) degrees of freedom. The imaginary part of the self-energy on-shell is typically associated with a damping rate and associated with an exponential decay of the amplitude. Excepting intermediate time scales, this exponential relaxation, however is not a proper description either at early or long times where power laws dominate the dynamics [36]. The relaxation associated with Landau damping at lowest order, will have to be balanced with the next order corrections which yield an approximate exponential relaxation and the resulting dynamics will depend on the details of the competition between these different processes, both Landau damping and collisional. Which process dominates will depend on the particular time scale of interest and the time scales for competition between the two different type of phenomena will depend on the details of the perturbative contributions. The kinetic approach introduced here could also prove useful to study the energy loss of quarks and leptons via off-shell processes in the QGP.

We plan to address this competition between Landau damping and collisional phenomena, along with an extension of the treatment presented in this article to leptons and a more detailed study of non-Markovian kinetics in future work.

ACKNOWLEDGMENTS

D.B. thanks the N.S.F. for partial support through grant PHY-9605186; he also thanks J. Smit, G. Aarts, P. Arnold, D. Son, W. Buchmuller, A. Jakovac and A. Patkos for conversations. R.H. and S.P.K. were supported by DOE grant DE-FG02-91-ER40682. S.P.K. would like to thank BNL for hospitality during the progress of this work. The authors acknowledge support from NATO.

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