# DLCQ bound states of $\mathcal{N}=(2,2)$ super-Yang-Mills theory at finite and large $N$

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(Received 26 August 1998; published 12 November 1998)

We consider the $(1+1)$-dimensional $\mathcal{N}=(2,2)$ supersymmetric matrix model which is obtained by dimensionally reducing $\mathcal{N}=1$ super Yang-Mills theory from four to two dimensions. The gauge groups we consider are $\mathrm{U}(N)$ and $\mathrm{SU}(N)$, where $N$ is finite but arbitrary. We adopt light-cone coordinates, and choose to work in the light-cone gauge. Quantizing this theory via discretized light-cone quantization (DLCQ) introduces an integer $K$ which restricts the light-cone momentum-fraction of constituent quanta to be integer multiples of $1/K$. Solutions to the DLCQ bound state equations are obtained for $2\leqslant K\leqslant 6$ by discretizing the light-cone supercharges, which results in a supersymmetric spectrum. Our numerical results imply the existence of normalizable massless states in the continuum limit $K\to\infty$, and therefore the absence of a mass gap. The low energy spectrum is dominated by string-like (or many parton) states. Our results are consistent with the claim that the theory is in a screening phase. [S0556-2821(98)04624-4]

PACS number(s): 11.10.Kk, 11.15.Tk

## I. INTRODUCTION

Supersymmetric gauge theories in low dimensions have been shown to be related to non-perturbative objects in M or string theory [1], and are therefore of particular interest nowadays. More dramatically, there is a growing body of evidence suggesting that gauged matrix models in $0+1$ and $1+1$ dimensions may offer a non-perturbative formulation of string theory [2,3]. There is also a suggestion that large $N$ gauge theories in various dimensions may be related to theories with gravity [4].

It is therefore interesting to study directly the non-perturbative properties of a class of supersymmetric matrix models at finite and large $N$, where $N$ is the number of gauge colors. In previous work [5,6], we focused on two dimensional matrix models, since the numerical technique of discrete light-cone quantization (DLCQ [7]) may be implemented to determine bound state wave functions and masses.

In this work, we extend these investigations by solving the DLCQ bound state equations for a two dimensional supersymmetric matrix model with $\mathcal{N}=(2,2)$ supersymmetry. Such a theory may be obtained by dimensionally reducing $\mathcal{N}=1$ super Yang-Mills theory from four to two space-time dimensions [8]. Various studies related to this model can be found in the literature [9,10], and it has recently been shown that this theory exhibits a screening phase [11].

The contents of this paper are as follows. In Sec. II, we formulate the $\mathcal{N}=(2,2)$ supersymmetric matrix model in the light-cone frame, and quantize the theory by imposing canonical (anti)commutation relations for fermions and bosons respectively. In Sec. III, we briefly describe the DLCQ numerical procedure, and present our numerical results for the bound state spectrum. The structure of bound state wave functions is also discussed. A summary of our work appears in the discussion in Sec. IV. Details of the underlying four dimensional $\mathcal{N}=1$ super Yang-Mills theory (i.e. before dimensional reduction) can be found in the Appendix.

## II. LIGHT-CONE QUANTIZATION AND DLCQ AT FINITE $N$

The two dimensional $\mathcal{N}=(2,2)$ supersymmetric gauge theory we are interested in may be formally obtained by dimensionally reducing $\mathcal{N}=1$ super Yang-Mills theory from four to two dimensions. For the sake of completeness, we review the underlying four dimensional Yang-Mills theory in the Appendix.

Dimensional reduction of the four dimensional Yang-Mills action (A14) given in the Appendix is carried out by stipulating that all fields be independent of the *two* transverse coordinates[1] $x^I=-x_I, I=1,2$. We may therefore assume that the fields depend only on the light-cone variables $\sigma^{\pm}=(1/\sqrt{2})(x^0\pm x^3)$. The resulting two dimensional theory may be described by the action

$$S^{LC}_{1+1}=\int d\sigma^+ d\sigma^- \ \mathrm{tr}\Bigg(\frac{1}{2}D_\alpha X_I D^\alpha X_I+\frac{g^2}{4}[X_I,X_J]^2 -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}+\mathrm{i}\theta_R^T D_+\theta_R+\mathrm{i}\theta_L^T D_-\theta_L +\sqrt{2}g\,\theta_L^T\epsilon_2\beta_I[X_I,\theta_R]\Bigg), \tag{1}$$

where the repeated indices $\alpha,\beta$ are summed over light-cone indices $\pm$, and $I,J$ are summed over transverse indices 1,2. The two scalar fields $X_I(\sigma^+,\sigma^-)$ represent $N\times N$ Hermitian matrix-valued fields, and are remnants of the transverse components of the four dimensional gauge field $A_\mu$, while $A_\pm(\sigma^+,\sigma^-)$ are the light-cone gauge field components of the residual two dimensional $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ gauge symmetry. The two component spinors $\theta_R$ and $\theta_L$ are remnants of the right-moving and left-moving projections of a four com-

[1]The space-time points in four dimensional Minkowski space are parametrized, as usual, by coordinates $(x^0,x^1,x^2,x^3)$.

 

ponent real spinor in the four dimensional theory. The components of $\theta_R$ and $\theta_L$ transform in the adjoint representation of the gauge group. $F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha+\mathrm{i}g[A_\alpha,A_\beta]$ is the two dimensional gauge field curvature tensor, while $D_\alpha=\partial_\alpha+\mathrm{i}g[A_\alpha,\cdot]$ is the covariant derivative for the (adjoint) spinor fields. The two $2\times 2$ real symmetric matrices $\beta_I$, and anti-symmetric matrix $\epsilon_2$, are defined in the Appendix.

Since we are working in the light-cone frame, it is natural to adopt the light-cone gauge $A_-=0$. With this gauge choice, the action (1) becomes

$$\tilde{S}^{LC}_{1+1}=\int d\sigma^+ d\sigma^-\ \mathrm{tr}\Big(\partial_+ X_I\partial_- X_I+\mathrm{i}\theta_R^T\partial_+\theta_R+\mathrm{i}\theta_L^T\partial_-\theta_L+\frac{1}{2}(\partial_- A_+)^2+gA_+J^++\sqrt{2}g\,\theta_L^T\epsilon_2\beta_I[X_I,\theta_R]+\frac{g^2}{4}[X_I,X_J]^2\Big), \tag{2}$$

where $J^+=\mathrm{i}[X_I,\partial_- X_I]+2\theta_R^T\theta_R$ is the longitudinal momentum current. The (Euler-Lagrange) equations of motion for the $A_+$ and $\theta_L$ fields are now

$$\partial_-^2 A_+=gJ^+, \tag{3}$$

$$\sqrt{2}\mathrm{i}\partial_-\theta_L=-g\epsilon_2\beta_I[X_I,\theta_R]. \tag{4}$$

These are evidently constraint equations, since they are independent of the light-cone time $\sigma^+$. The ''zero mode'' of the constraints above provides us with the conditions

$$\int d\sigma^- J^+=0 \quad\text{and}\quad \int d\sigma^-\epsilon_2\beta_I[X_I,\theta_R]=0, \tag{5}$$

which will be imposed on the Fock space to select the physical states in the quantum theory. The first constraint above is well known in the literature, and projects out the colorless states in the quantized theory [12]. The second (fermionic) constraint is perhaps lesser well known, but certainly provides non-trivial relations governing the small-$x$ behavior of light-cone wave functions[2] [13].

At any rate, Eqs. (3),(4) permit one to eliminate the non-dynamical fields $A_+$ and $\theta_L$ in the expression for the light-cone Hamiltonian $P^-$, which is a particular feature of light-cone gauge theories. There are no ghosts. We may therefore write down explicit expressions for the light-cone momentum $P^+$ and Hamiltonian $P^-$ exclusively in terms of the physical degrees of freedom represented by the two scalar fields $X_I$ and two-component spinor $\theta_R$:

$$P^+=\int d\sigma^-\ \mathrm{tr}(\partial_- X_I\partial_- X_I+\mathrm{i}\theta_R^T\partial_-\theta_R), \tag{6}$$

$$P^-=g^2\int d\sigma^-\ \mathrm{tr}\Big(-\frac{1}{2}J^+\frac{1}{\partial_-^2}J^+-\frac{1}{4}[X_I,X_J]^2+\frac{\mathrm{i}}{2}(\epsilon_2\beta_I[X_I,\theta_R])^T\frac{1}{\partial_-}\epsilon_2\beta_J[X_J,\theta_R]\Big). \tag{7}$$

The light-cone Hamiltonian propagates a given field configuration in light-cone time $\sigma^+$, and contains all the non-trivial dynamics of the interacting field theory.

Let us denote the two components of the spinor $\theta_R$ by the fermion fields $u^\alpha$, $\alpha=1,2$. Then, in terms of their Fourier modes, the physical fields may be expanded at light-cone time $\sigma^+=0$ to give[3]

$$X^I_{pq}(\sigma^-)=\frac{1}{\sqrt{2\pi}}\int_0^\infty\frac{dk^+}{\sqrt{2k^+}}\times[a^I_{pq}(k^+)e^{-\mathrm{i}k^+\sigma^-}+a^{I\dagger}_{qp}(k^+)e^{\mathrm{i}k^+\sigma^-}],\quad I=1,2, \tag{8}$$

$$u^\alpha_{pq}(\sigma^-)=\frac{1}{\sqrt{2\pi}}\int_0^\infty\frac{dk^+}{\sqrt{2}}\times[b^\alpha_{pq}(k^+)e^{-\mathrm{i}k^+\sigma^-}+b^{\alpha\dagger}_{qp}(k^+)e^{\mathrm{i}k^+\sigma^-}],\quad \alpha=1,2. \tag{9}$$

For the gauge group U($N$), the (anti)commutation relations take the form

$$[a^I_{pq}(k^+),a^{J\dagger}_{rs}(k'^+)]=\delta^{IJ}\delta_{pr}\delta_{qs}\delta(k^+-k'^+), \tag{10}$$

$$\{b^\alpha_{pq}(k^+),b^{\beta\dagger}_{rs}(k'^+)\}=\delta^{\alpha\beta}\delta_{pr}\delta_{qs}\delta(k^+-k'^+), \tag{11}$$

while for SU($N$), we have the corresponding relations

$$[a^I_{pq}(k^+),a^{J\dagger}_{rs}(k'^+)]=\delta^{IJ}\Big(\delta_{pr}\delta_{qs}-\frac{1}{N}\delta_{pq}\delta_{rs}\Big)\delta(k^+-k'^+), \tag{12}$$

$$\{b^\alpha_{pq}(k^+),b^{\beta\dagger}_{rs}(k'^+)\}=\delta^{\alpha\beta}\Big(\delta_{pr}\delta_{qs}-\frac{1}{N}\delta_{pq}\delta_{rs}\Big)\delta(k^+-k'^+). \tag{13}$$

An important simplification of the light-cone quantization is that the light-cone vacuum is the Fock vacuum $|0\rangle$, defined by

$$a^I_{pq}(k^+)|0\rangle=b^\alpha_{pq}(k^+)|0\rangle=0, \tag{14}$$

for all positive longitudinal momenta $k^+>0$. We therefore have $P^+|0\rangle=P^-|0\rangle=0$.

[2]If we introduce a mass term, such relations become crucial in establishing finiteness conditions. See [13], for example.

[3]The symbol † denotes quantum conjugation, and does not transpose matrix indices.

The ''charge-neutrality'' condition [first integral constraint from Eq. (5)] requires that all the color indices must be contracted for physical states. Thus physical states are formed by color traces of the boson and fermion creation operators $a^{I\dagger}, b^{\alpha\dagger}$ acting on the light-cone vacuum. A single trace of these creation operators may be identified as a single closed string, where each creation operator (or ''parton''), carrying some longitudinal momentum $k^+$, represents a ''bit'' of the string. A product of traced operators is then a multiple string state, and the quantity $1/N$ is analogous to a string coupling constant [14].

At this point, we may determine explicit expressions for the quantized light-cone operators $P^\pm$ by substituting the mode expansions (8),(9) into Eqs. (6),(7). The mass operator $M^2 \equiv 2P^+P^-$ may then be diagonalized to solve for the bound state mass spectrum. However, as was pointed out in [15], it is more convenient to determine the quantized expressions for the supercharges, since this leads to a regularization prescription for $P^-$ that preserves supersymmetry even in the discretized theory.

In order to elaborate upon this last remark, first note that the continuum theory possesses four supercharges, which may be derived from the dimensionally reduced form of the four dimensional $\mathcal{N}=1$ supercurrent [Eq. (A6) in the Appendix):

$$Q_\alpha^+ = 2^{5/4}\int_{-\infty}^{\infty} d\sigma^- \ \mathrm{tr}(\partial_- X_I \beta_{I\alpha\eta} u_\eta) \tag{15}$$

$$Q_\alpha^- = g\int_{-\infty}^{\infty} d\sigma^- \ \mathrm{tr}\bigg( -2^{3/4} J^+ \frac{1}{\partial_-}\epsilon_{2\alpha\eta} u_\eta + 2^{-1/4}\mathrm{i}[X_I, X_J](\beta_I\beta_J\epsilon_2)_{\alpha\eta} u_\eta \bigg), \tag{16}$$

where $\alpha=1,2$, and repeated indices are summed. The two $2\times 2$ real symmetric matrices $\beta_I$ are discussed in the Appendix. By explicit calculation or otherwise, these charges satisfy the following relations:[4]

$$\{Q_\alpha^+, Q_\beta^+\} = \delta_{\alpha\beta} 2\sqrt{2} P^+ \tag{17}$$

$$\{Q_\alpha^-, Q_\beta^-\} = \delta_{\alpha\beta} 2\sqrt{2} P^- \tag{18}$$

$$\{Q_\alpha^-, Q_\beta^+\} = 0, \quad \alpha,\beta = 1,2. \tag{19}$$

If we substitute the mode expansions (8),(9) into Eqs. (15),(16) for the light-cone supercharges $Q_\alpha^\pm$, we obtain the following ''momentum representations'' for these charges:

$$Q_\alpha^+ = 2^{1/4}\mathrm{i}\int_0^\infty dk\,\sqrt{k}\,\beta_{I\alpha\eta}[a_{Iij}^\dagger(k) b_{\eta ij}(k) - b_{\eta ij}^\dagger(k) a_{Iij}(k)] \tag{20}$$

and

$$\begin{aligned} Q_\alpha^- = \frac{\mathrm{i}2^{-1/4} g}{\sqrt{\pi}} &\int_0^\infty dk_1 dk_2 dk_3\, \delta(k_1+k_2-k_3)(\epsilon_2)_{\alpha\eta} \\ &\times \Bigg\{ \frac{1}{2\sqrt{k_1 k_2}}\left(\frac{k_2-k_1}{k_3}\right)[b_{\eta ij}^\dagger(k_3) a_{Iim}(k_1) a_{Imj}(k_2) - a_{Iim}^\dagger(k_1) a_{Imj}^\dagger(k_2) b_{\eta ij}(k_3)] \\ &+ \frac{1}{2\sqrt{k_1 k_3}}\left(\frac{k_1+k_3}{k_2}\right)[a_{Iim}^\dagger(k_1) b_{\eta mj}^\dagger(k_2) a_{Iij}(k_3) - a_{Iij}^\dagger(k_3) a_{Iim}(k_1) b_{\eta mj}(k_2)] \\ &+ \frac{1}{2\sqrt{k_2 k_3}}\left(\frac{k_2+k_3}{k_1}\right)[a_{Iij}^\dagger(k_3) b_{\eta im}(k_1) a_{Imj}(k_2) - b_{\eta im}^\dagger(k_1) a_{Imj}^\dagger(k_2) a_{Iij}(k_3)] \\ &- \frac{1}{k_1}[b_{\beta ij}^\dagger(k_3) b_{\eta im}(k_1) b_{\beta mj}(k_2) + b_{\eta im}^\dagger(k_1) b_{\beta mj}^\dagger(k_2) b_{\beta ij}(k_3)] \\ &- \frac{1}{k_2}[b_{\beta ij}^\dagger(k_3) b_{\beta im}(k_1) b_{\eta mj}(k_2) + b_{\beta im}^\dagger(k_1) b_{\eta mj}^\dagger(k_2) b_{\beta ij}(k_3)] \\ &+ \frac{1}{k_3}[b_{\eta ij}^\dagger(k_3) b_{\beta im}(k_1) b_{\beta mj}(k_2) + b_{\beta im}^\dagger(k_1) b_{\beta mj}^\dagger(k_2) b_{\eta ij}(k_3)] + [(\beta_I\beta_J - \beta_J\beta_I)\epsilon_2]_{\alpha\beta} \end{aligned}$$

[4] Surface terms contributing to the central charge are assumed to be vanishing.

$$\times\left(\frac{1}{4\sqrt{k_1k_2}}[b^\dagger_{\beta ij}(k_3)a_{Iim}(k_1)a_{Jmj}(k_2)+a^\dagger_{Jim}(k_1)a^\dagger_{Imj}(k_2)b_{\beta ij}(k_3)]\right.$$

$$+\frac{1}{4\sqrt{k_2k_3}}[a^\dagger_{Jij}(k_3)b_{\beta im}(k_1)a_{Imj}(k_2)+b^\dagger_{\beta im}(k_1)a^\dagger_{Jmj}(k_2)a_{Iij}(k_3)]$$

$$\left.\left.+\frac{1}{4\sqrt{k_3k_1}}[a^\dagger_{Iij}(k_3)a_{Jim}(k_1)b_{\beta mj}(k_2)+a^\dagger_{Iim}(k_1)b^\dagger_{\beta mj}(k_2)a_{Jij}(k_3)]\right)\right\},\tag{21}$$

where repeated indices are always summed: $\alpha,\beta,\eta=1,2$ (spinor indices), $I,J=1,2$ [SO(2) vector indices], and $i,j,m=1,\ldots,N$ (matrix indices).

In order to implement the DLCQ formulation[5] of the bound state problem—which is tantamount to imposing periodic boundary conditions $\sigma^-\sim\sigma^-+2\pi R$—we simply restrict the momentum variable(s) appearing in the expressions for $Q^\pm_\alpha$ [Eqs. (20),(21)] to the following discretized set of momenta: $\{(1/K)P^+,(2/K)P^+,(3/K)P^+,\ldots\}$. Here, $P^+$ denotes the total light-cone momentum of a state, and may be thought of as a fixed constant, since it is easy to form a Fock basis that is already diagonal with respect to the quantum operator $P^+$ [7]. The integer $K$ is called the ''harmonic resolution,'' and $1/K$ measures the coarseness of our discretization—we recover the continuum by taking the limit $K\to\infty$. Physically, $1/K$ represents the smallest positive[6] unit of longitudinal momentum-fraction allowed for each parton in a Fock state.

Of course, as soon as we implement the DLCQ procedure, which is specified unambiguously by the harmonic resolution $K$, the integrals appearing in the definitions for $Q^\pm_\alpha$ are replaced by finite sums, and the eigenequation $2P^+P^-|\Psi\rangle=M^2|\Psi\rangle$ is reduced to a finite matrix diagonalization problem. In this last step, we use the fact that $P^-$ is proportional to the square of any one of the two supercharges $Q^-_\alpha$, $\alpha=1,2$ [Eq. (18)], and so the problem of diagonalizing $P^-$ is equivalent to diagonalizing any one of the two supercharges $Q^-_\alpha$. As was pointed out in [15], this procedure yields a supersymmetric spectrum for any resolution $K$. In the present work, we are able to perform numerical diagonalizations for $2\leqslant K\leqslant 6$ with the help of MATHEMATICA and a desktop PC.

In the next section, we discuss the details of our numerical results.

[5]It might be useful to consult [12,15,16,17] for an elaboration of DLCQ in models with adjoint fermions. An extensive list of references on DLCQ and light-cone field theories appears in the review in [18].

[6]We exclude the zero mode $k^+=0$ in our analysis; the massive spectrum is not expected to be affected by this omission [19], but there are issues concerning the light-cone vacuum that involve $k^+=0$ modes [20,21].

## III. DLCQ BOUND STATE SOLUTIONS

We consider discretizing the light-cone supercharge $Q^-_\alpha$ for a particular $\alpha\in\{1,2\}$, and for the values $2\leqslant K\leqslant 6$. For a given resolution $K$, the light-cone momenta of partons in a given Fock state must be some positive integer multiple of $P^+/K$, where $P^+$ is the total light-cone momentum of the state.

Of course, the fact that we may choose any one of the two supercharges to calculate the spectrum provides a strong test for the correctness of our computer program and consistency of the DLCQ formulation. It turns out, however, that there are a few surprises in store. First of all, the supersymmetry algebra [Eqs. (17)–(19)] is certainly true in a continuum space-time, but there is no obvious reason to expect that these relations should also hold exactly in a discretized version of the theory. From our numerical studies, however, we find that relations (17) and (19) are indeed exactly satisfied in the discretized theory.

A potential problem arises, however, in relation (18). First of all, one finds that $Q^-_1$ and $Q^-_2$ do not anti-commute: $\{Q^-_1,Q^-_2\}\neq 0$. However, this is not too surprising, since one can trace this anomaly to the truncation of momenta (i.e. there is a non-zero lower bound on $k^+$) following from the DLCQ procedure. In particular, as we increase the resolution, the non-zero matrix entries in $\{Q^-_1,Q^-_2\}$ become more and more sparsely distributed, and we expect them to occupy a subset of measure zero in the continuum limit $K\to\infty$. This is substantiated by direct inspection of the matrix $\{Q^-_1,Q^-_2\}$ for different values of $K$.

We also encounter a further anomaly when computing the difference $(Q^-_1)^2-(Q^-_2)^2$. According to relation (18), this difference is precisely zero in the continuum, but in the discretized theory, it is non-vanishing. As we discussed above, this can be understood as an artifact of the truncated momenta in the DLCQ formulation, and disappears in the continuum limit $K\to\infty$.

Nevertheless, we should worry at this stage about the definition of the light-cone Hamiltonian. Relation (18) suggests that we may define the DLCQ light-cone Hamiltonian as the square of any one of the supercharges. Because the difference $(Q^-_1)^2-(Q^-_2)^2$ is non-vanishing in the discretized theory, we have two possible choices for defining the light-cone Hamiltonian: $P^-_1=(1/\sqrt{2})(Q^-_1)^2$ or $P^-_2=(1/\sqrt{2})(Q^-_2)^2$. Surprisingly, after diagonalizing each of these operators for different $K$, the spectrum of eigenvalues

TABLE I. SU($N$) bound state masses $M^2$ in units $g^2N/\pi$ for resolution $K=3$ (six significant figures). When expressed in these units, the masses are independent of $N$ (i.e. there are no $1/N$ corrections at this resolution), and so these results are applicable for any $N>1$. The notation ''4+4'' above implies an exact 8-fold mass degeneracy in the DLCQ spectrum with 4 bosons and 4 fermions. In total, there are 20 bosons and 20 fermions.

| Bound state masses $M^2$ for $K=3$ | |
|---|---|
| $M^2$ | Mass degeneracy |
| 0 | 8+8 |
| 1.30826 | 4+4 |
| 12.6273 | 4+4 |
| 22.0645 | 4+4 |

turns out to be identical. This is certainly another attractive feature of DLCQ that deserves further study.[7]

Of course, this implies that the spectrum of the theory for finite $K$ is independent of the choice of supercharge, and therefore well defined. It would be interesting to investigate whether other physical observables are independent of the observed anomaly in the supersymmetry algebra.

Let us begin with a discussion of massless states. First, for gauge symmetry U($N$), the U(1) degrees of freedom explicitly decouple, and this provides trivial examples of massless states, which can be seen in the DLCQ analysis. Consequently, all the non-trivial dynamics is contained in the SU($N$) gauge theory. For $K=2$, the SU($N$) Fock space consists of two parton states only. Moreover, since $Q^-_\alpha$ increases or decreases the number of partons by 1, it necessarily annihilates all Fock states, and so all states are massless. However, for $K\geqslant 3$, determining the existence (or not) of massless states in the SU($N$) theory turns out to be a highly non-trivial problem involving the full dynamics of the Hamiltonian. We will therefore restrict our attention to the bound state spectrum of the SU($N$) gauge theory.

The results of our DLCQ numerical diagonalization of $(Q^-_\alpha)^2$ at resolution $K=3$ are summarized in Table I. To test our numerical algorithm, we diagonalize (the square of) each supercharge, and find the same spectrum—which is consistent with supersymmetry. Let us now consider resolution $K=4$. The results of our numerical diagonalizations are presented in Table II. If we express the masses in units $g^2N/\pi$, then there are no $1/N$ corrections at resolutions $K=3$ and $K=4$. However, for $K\geqslant 5$, one sees $1/N$ effects in the spectrum. In Table III, we list bound state masses for $N=3$, 10, 100 and 1000 at resolution $K=5$. At this resolution, there are 472 bosons and 472 fermions. Table III illustrates mass splittings that occur in the spectrum as a result of $1/N$ interactions, which become increasingly important as we decrease $N$. For example, at $N=1000$, there is an apparent degeneracy in the numerical spectrum at $M^2=2.18043$ which is visibly broken when $N=10$.

It turns out that these states are easily identified as weakly bound multi-particles at large (but finite) $N$. To show this, note that bound states at $K=2$ are necessarily massless—$M^2(K=2)=0$—while for $K=3$, the lightest non-zero mass state satisfies $M^2(K=3)=1.30826$. The mass squared $M^2(K=5)$ of a freely interacting *multi-particle* composed of

TABLE II. SU($N$) bound state masses $M^2$ in units $g^2N/\pi$ for resolution $K=4$ (six significant figures). When expressed in these units, the masses are independent of $N$ (i.e. there are no $1/N$ corrections at this resolution), and so these results are applicable for any $N>1$. In total, there are 92 bosons and 92 fermions at this resolution.

| Bound state masses $M^2$ for $K=4$ | |
|---|---|
| $M^2$ | Mass degeneracy |
| 0 | 32+32 |
| 1.20095 | 8+8 |
| 4.00943 | 4+4 |
| 12.2424 | 4+4 |
| 12.2962 | 8+8 |
| 15.0490 | 4+4 |
| 15.2822 | 4+4 |
| 19.5028 | 8+8 |
| 20 | 4+4 |
| 22.5321 | 4+4 |
| 23.1272 | 4+4 |
| 28.6177 | 4+4 |
| 28.6955 | 4+4 |

TABLE III. SU($N$) bound state masses $M^2$ in units $g^2N/\pi$ for resolution $K=5$ (six significant figures), and for $N=3$, 10, 100 and 1000. We have selected the lightest 10 states in each case, with mass degeneracy $4+4$ for non-zero masses. Massless states have degeneracy $92+92$. Note that if a state has degeneracy $8+8$, then we include it twice (e.g. for $N=3$, there is a bound state with mass squared $M^2=1.13442$ with degeneracy $8+8$). Convergence in the large $N$ limit is evident.

| Bound state masses $M^2$ for $K=5$, and $N=3,10,100,1000$ | | | |
|---|---|---|---|
| $M^2$ | | | |
| $N=3$ | $N=10$ | $N=100$ | $N=1000$ |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0442062 | 0.0112824 | 0.00679546 | 0.00674981 |
| 0.658859 | 0.634820 | 0.630485 | 0.630441 |
| 1.13442 | 1.08578 | 1.08135 | 1.08131 |
| 1.13442 | 1.11224 | 1.10995 | 1.10993 |
| 1.23157 | 1.56551 | 1.57314 | 1.57321 |
| 1.29964 | 2.09691 | 2.17960 | 2.18043 |
| 1.55373 | 2.10814 | 2.17971 | 2.18043 |
| 1.76132 | 2.14535 | 2.18009 | 2.18043 |
| 1.77999 | 2.14571 | 2.18009 | 2.18043 |

[7]One suggestion is that these ''discrepancies'' in the operator algebra are related to large gauge transformations arising from the light-like compactification in DLCQ, and are therefore expected to vanish in the continuum limit $K\rightarrow\infty$. For finite $K$, the operator ''anomalies'' we observed may be gauge equivalent to zero.

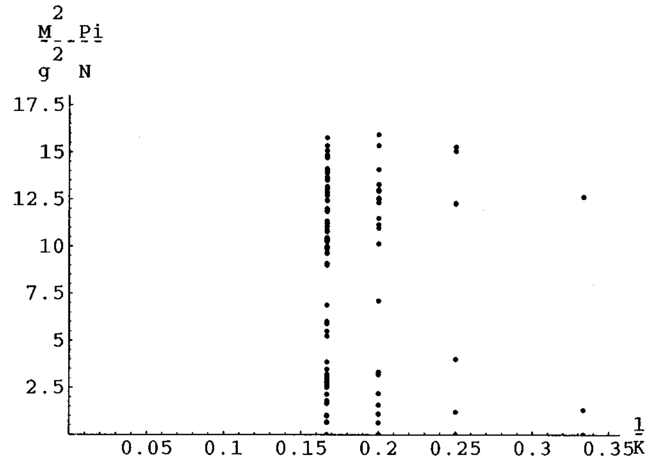


FIG. 1. Bound state masses $M^2$ (in units $g^2N/\pi$) versus $1/K$ for $N=1000$. We only plot masses satisfying $M^2<16$, but there are many bound states above this for $K\geqslant 5$.

one $K=2$ particle and one $K=3$ particle now follows from simple kinematics [22]:

$$\frac{M^2(K=5)}{5}=\frac{M^2(K=3)}{3}+\frac{M^2(K=2)}{2}. \tag{22}$$

The result is $M^2(K=5)=2.18043$, after inserting the observed values for $M^2(K=3)$ and $M^2(K=2)$. Note that this value for $M^2(K=5)$ is a *prediction* for the mass of two freely interacting particles at resolution $K=5$ (or, equivalently, carrying $K=5$ units of light-cone momentum). Thus, for large enough $N$ (or for sufficiently small coupling $1/N$), we expect to see bound states approaching this two-free-body mass. Table III confirms this prediction. Such predictions of multi-particle masses were also carried out for the $\mathcal{N}=(1,1)$ model [5], and are a strong consistency test of the (typically complex) numerical algorithms adopted in this work.

Note, in general, that the $1/N$ interactions increase the masses of light particles, and decrease the mass of heavy particles (see Table III).

For $K=6$, there are over 4500 states in the Fock basis. The resulting DLCQ spectrum for $N=1000$ appears in Fig. 1, together with bound state masses obtained at the lower resolutions $K=3$, 4 and 5. It is apparent from this graph that as we increase $K$ (i.e. as we move from right to left), the DLCQ spectrum seems to approach some dense subset of the positive real (vertical) axis. One may infer that in the continuum limit $K\rightarrow\infty$, the spectrum does indeed ''fill up'' the vertical axis, which is certainly compatible with a recent study that suggested this theory should be in a screening phase [11].

As we pointed out earlier, decreasing $N$ introduces noticeable splittings in the spectrum[8] which has the effect of smearing out the points in Fig. 1. The qualitative features of the spectrum expected from a screening theory are therefore also present for smaller values of $N$.

So far, we have only discussed properties of the DLCQ spectrum—such as bound state masses and their corresponding degeneracies—but solving the DLCQ bound state equations also involves deriving explicit numerical expressions for bound state *wave functions*. This is of particular interest to us here since we would like to know whether such a theory exhibits a mass gap or not. In the context of our DLCQ analyses, this involves establishing, in addition to the trivial light-cone vacuum, the existence—or not—of a *normalizable* massless state in the continuum theory $K\rightarrow\infty$.

According to the literature, the $\mathcal{N}=(2,2)$ model is believed *not* to have a mass gap [1]. We now outline how our numerical results support such a claim.[9]

First, at resolution $K=3$ and $N=1000$, one identifies a massless boson (and its superpartners) that has the form[10]

$$\mathrm{tr}\left[a_1^\dagger\left(\frac{P^+}{3}\right)a_2^\dagger\left(\frac{2P^+}{3}\right)\right]|0\rangle+\mathrm{tr}\left[a_2^\dagger\left(\frac{P^+}{3}\right)a_1^\dagger\left(\frac{2P^+}{3}\right)\right]|0\rangle, \tag{23}$$

where $P^+$ is the total (fixed) momentum. At resolution $K=4$, one identifies a massless boson of the form

$$\begin{aligned}&0.497134\ \mathrm{tr}\left[a_1^\dagger\left(\frac{2P^+}{4}\right)a_2^\dagger\left(\frac{2P^+}{4}\right)\right]|0\rangle\\&+0.580827\ \mathrm{tr}\left[a_1^\dagger\left(\frac{P^+}{4}\right)a_2^\dagger\left(\frac{3P^+}{4}\right)\right]|0\rangle\\&+0.495501\ \mathrm{tr}\left[a_2^\dagger\left(\frac{P^+}{4}\right)a_1^\dagger\left(\frac{3P^+}{4}\right)\right]|0\rangle\\&+\text{additional Fock states,}\end{aligned} \tag{24}$$

where ''additional Fock states'' represents a superposition of two and four parton Fock states with amplitudes less than 0.25 (typically, very small). It is therefore natural to identify the bound state solution above for $K=4$ with the $K=3$ solution [Eq. (23)]. At $K=5$, something seems to go wrong; there are no massless states that may be characterized as a superposition of predominantly two-parton states, as in Eqs.

[8]Some splittings are in fact present for $N=1000$, but are not seen in the numerical spectrum since we quote masses to only six significant figures.

[9]The $\mathcal{N}=(8,8)$ model, in contrast, was shown to have a mass gap [6].

[10]We choose not to normalize states to unity, since it is very time consuming computationally when working at finite $N$, and not necessary when solving for spectra. A simpler procedure is just to renormalize each Fock state by $1/N^{(q/2)}$, where $q$ is the number of partons in the Fock state (implicitly implied in this work). Then the *relative* size of each Fock state wave function indicates the relative importance of the Fock state to the overall bound state.

(23) and (24). However, there is a state with mass squared[11] $M^2=0.0067$ ($N=1000$), which has the explicit form

$$\begin{aligned}
&0.52344\ \mathrm{tr}\left[a_1^\dagger\left(\frac{4P^+}{5}\right)a_2^\dagger\left(\frac{P^+}{5}\right)\right]|0\rangle\\
&+0.468159\ \mathrm{tr}\left[a_1^\dagger\left(\frac{2P^+}{5}\right)a_2^\dagger\left(\frac{3P^+}{5}\right)\right]|0\rangle\\
&+0.468159\ \mathrm{tr}\left[a_1^\dagger\left(\frac{3P^+}{5}\right)a_2^\dagger\left(\frac{2P^+}{5}\right)\right]|0\rangle\\
&+0.52344\ \mathrm{tr}\left[a_2^\dagger\left(\frac{4P^+}{5}\right)a_1^\dagger\left(\frac{P^+}{5}\right)\right]|0\rangle\\
&+\text{additional Fock states,}
\end{aligned}\tag{25}$$

where ''additional Fock states'' above represents a superposition of four parton Fock states with relatively small amplitudes. Evidently, it is natural to associate this bound state with the massless bound states (23) and (24) observed at lower resolutions.

At this point, we would like to know whether this non-zero mass will persist in the continuum limit $K\to\infty$, or whether it is an artifact of the Fock state truncation enforced by the DLCQ procedure.

As it turns out, solving the DLCQ bound state equations at resolution $K=6$ reveals an exactly massless (bosonic) solution that is essentially a superposition of two-parton Fock states [as in Eqs. (23), (24) and (25)], with wave functions that have approximately the same shape, relative magnitude and sign as the wave functions appearing at lower resolutions. The ''glitch'' in the spectrum observed at $K=5$, therefore, appears to be an artifact of the numerical truncation, although it would be desirable to probe larger values of $K$ (e.g. $K\geqslant 7$) to confirm this viewpoint.

At any rate, we have identified a series of DLCQ solutions that is expected to converge in the limit $K\to\infty$ to a massless bound state. This continuum solution would rule out the possibility of a mass gap, in agreement with [1].

We should remark at this point that the Fock state representation of the lowest energy states in this model appears to be significantly more complicated than solutions found in other field theories with massless particles—such as the 't Hooft pion or Schwinger particle. A theory with complex adjoint fermions studied relatively recently [16,17] revealed many massless states with constant wave function solutions. In contrast, it turns out that any normalizable state in the (continuum) supersymmetric model studied here must be a superposition of an infinite number of Fock states [23]. An analogous situation occurred in the model with $\mathcal{N}=(1,1)$ supersymmetry [5].

Finally, we comment on the ''string-like'' nature of bound states that dominate the low energy spectrum. Although we focused on a massless state composed of mainly two-parton Fock states, one can always find a massless bound state dominated by Fock states with an arbitrarily large number of partons for sufficiently large $K$. The structure of the low energy spectrum is similar, consisting of bound states of all lengths permitted by the truncation parameter $K$. Such qualitative features of the spectrum were exhibited also in the $\mathcal{N}=(1,1)$ supersymmetric model [5,15].

## IV. DISCUSSION

To summarize, we have performed a detailed analysis of the DLCQ bound state spectrum of an $\mathcal{N}=(2,2)$ supersymmetric matrix model, which may be heuristically derived by dimensionally reducing $\mathcal{N}=1$ super-Yang-Mills theory from four to two space-time dimensions. The gauge group is SU($N$), and we allow $N$ to be finite, but arbitrarily large.

We discretize the light-cone supercharges via DLCQ, and find that certain supersymmetry relations are exactly satisfied even in the discretized formulation. In particular, relations (17) and (19) hold exactly in DLCQ. We find, however, that relation (18) holds only approximately, although the discrepancy diminishes as the resolution $K$ is increased. As a consequence, the difference between the discrete light-cone supercharges $(Q_1^-)^2-(Q_2^-)^2$ is non-zero. Surprisingly, however, the eigenvalues of $(Q_1^-)^2$ and $(Q_2^-)^2$ turn out to be *identical* at any resolution, and so the DLCQ spectrum of the theory has an unambiguous definition as the eigenvalues of either supercharge squared. With this definition, the DLCQ spectrum turns out to be *exactly supersymmetric at any resolution* (see Tables I and II, for example). It would be desirable to understand why—in the DLCQ formulation—the supersymmetry operator algebra is only approximately satisfied, while the spectrum itself appears to reflect unbroken supersymmetry. Perhaps the observed anomaly in the algebra cancels for physical observables,[12] which would be consistent with the idea that certain quantities are ''gauge equivalent'' to zero.

In Table III, we illustrate the dependence of bound state masses on the number of gauge colors, $N$. Convergence is evident at large $N$ if we keep $g^2N$ constant. We also resolve mass splittings in the spectrum as a result of $1/N$ interactions (see Table III). It appears that decreasing $N$ (i.e. increasing the strength of the $1/N$ interactions) has the generic effect of decreasing particle masses, except for very light particle states. Note that there is no $N$ dependence of the spectrum at resolutions $K=3$ and 4—one needs to consider $K\geqslant 5$ to observe any variation with $N$.

In Fig. 1, we plotted the DLCQ spectrum for resolutions $K=3$, 4, 5 and 6, and observed that the supersymmetric spectrum approaches a dense subset of the positive real axis.

[11]This is not a $1/N$ effect. If we let $N\to\infty$, the mass squared $M^2$ does *not* converge to zero.

[12]We might be able to redefine the supercharges $Q_1^-\to RQ_1^-R^{-1}$, $Q_2^-\to SQ_2^-S^{-1}$, for appropriate matrices $R,S$, so that relation (18) holds exactly even in the discretized theory. However, this would imply a non-vanishing result for relation (19). Nevertheless, it would be tempting to argue that this non-vanishing contribution (or ''central charge'') reflects the topology induced by compactification of the light-like circle in DLCQ [24].

This is compatible with the recent claim that the theory is in a screening phase [11].

By carefully studying the Fock state content of certain bound states at different resolutions, we argued for the existence of a normalizable massless state in the continuum limit $K\to\infty$. A mass gap is therefore expected to be absent in this theory.

We also observed that the low energy spectrum is dominated by states with arbitrarily many partons—a constituent picture involving few-parton Fock states is obviously an inadequate representation for capturing the full low energy dynamics of this model.

In light of this highly complex bound state structure, it is tempting to suggest that we are probing a dynamical system that might be more adequately (and simply) described by an effective field theory in higher dimensions.[13] Following the remarkable proposals of matrix theory and the anti–de Sitter and conformal field theory correspondence, it would be interesting to pursue this speculation further.


### ACKNOWLEDGMENTS

F.A. is grateful for the hospitality bestowed by the Max-Planck-Institute (Heidelberg) where this work was begun.


### APPENDIX: SUPER-YANG-MILLS THEORY IN FOUR DIMENSIONS

Let us start with $\mathcal{N}=1$ super Yang-Mills theory in $3+1$ dimensions with gauge group U($N$) or SU($N$):

$$S_{3+1}=\int d^4x\ \mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{\mathrm{i}}{2}\bar{\Psi}\Gamma^{\mu}D_{\mu}\Psi\right), \quad \text{(A1)}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+\mathrm{i}g[A_\mu,A_\nu], \quad \text{(A2)}$$

$$D_\mu\Psi=\partial_\mu\Psi+\mathrm{i}g[A_\mu,\Psi], \quad \text{(A3)}$$

and $\mu,\nu=0,\ldots,3$. The Majorana spinor $\Psi$ transforms in the adjoint representation of the gauge group. The (flat) space-time metric $g_{\mu\nu}$ has signature $(+,-,-,-)$, and we adopt the normalization $\mathrm{tr}(T^aT^b)=\delta^{ab}$ for the generators of the gauge group.

The supersymmetry transformations

$$\delta A_\mu=\mathrm{i}\bar{\epsilon}\Gamma_\mu\Psi \quad \text{(A4)}$$

$$\delta\Psi=\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon \quad \text{(A5)}$$

where $\Gamma^{\mu\nu}=\frac{1}{2}[\Gamma^\mu,\Gamma^\nu]$ give rise to the following supercurrent:

$$J^\mu=\frac{\mathrm{i}}{2}\ \mathrm{tr}(\Gamma^{\rho\sigma}\Gamma^\mu F_{\rho\sigma}\Psi). \quad \text{(A6)}$$

In order to realize the four dimensional Dirac algebra $\{\Gamma_\mu,\Gamma_\nu\}=2g_{\mu\nu}$ in terms of Majorana matrices, we use as building blocks the following three $2\times2$ real anti-commuting matrices:

$$\epsilon_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \epsilon_2=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \epsilon_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \quad \text{(A7)}$$

We may now define four $4\times4$ pure-imaginary matrices as tensor products of the above matrices,

$$\Gamma^0=\mathrm{i}\epsilon_2\otimes\epsilon_1,\quad \Gamma^1=\mathrm{i}\epsilon_1\otimes\mathbf{1},\quad \Gamma^2=\mathrm{i}\epsilon_3\otimes\mathbf{1},\quad \Gamma^3=\mathrm{i}\epsilon_2\otimes\epsilon_2, \quad \text{(A8)}$$

and it follows that these matrices satisfy $\{\Gamma_\mu,\Gamma_\nu\}=2g_{\mu\nu}$, as required. In our numerical work, we use this particular representation.

To formulate the theory in light-cone coordinates, it is convenient to introduce matrices

$$\Gamma^\pm=\frac{1}{\sqrt{2}}(\Gamma^0\pm\Gamma^3),\quad \Gamma^I=\begin{pmatrix}\mathrm{i}\beta_I&0\\0&\mathrm{i}\beta_I\end{pmatrix}, \quad \text{(A9)}$$

where the two $2\times2$ real-symmetric matrices $\beta_I$, $I=1,2$, are defined by writing $\beta_1=\epsilon_1$ and $\beta_2=\epsilon_3$.

It is now straightforward to verify that $P_L\equiv\frac{1}{2}\Gamma^+\Gamma^-$ and $P_R\equiv\frac{1}{2}\Gamma^-\Gamma^+$ project out the left- and right-moving components of the four-component spinor $\Psi$. Defining $\psi$ by a rescaling, $\Psi=2^{1/4}\psi$, we introduce left- and right-moving spinors $\psi_{L,R}$ as follows:

$$\psi=\psi_R+\psi_L,\quad \psi_R=P_R\psi,\quad \psi_L=P_L\psi. \quad \text{(A10)}$$

This decomposition is particularly useful when working with light-cone coordinates, since in the light-cone gauge one can express the left-moving component $\psi_L$ in terms of the right-moving component $\psi_R$ by virtue of the fermion constraint equation. We will derive this result shortly. In terms of the usual four dimensional Minkowski space-time coordinates, the light-cone coordinates are given by

$$x^+=\frac{1}{\sqrt{2}}(x^0+x^3),\quad \text{``time coordinate,''} \quad \text{(A11)}$$

$$x^-=\frac{1}{\sqrt{2}}(x^0-x^3),\quad \text{``longitudinal space coordinate,''} \quad \text{(A12)}$$

$$\mathbf{x}^\perp=(x^1,x^2),\quad \text{``transverse coordinates.''} \quad \text{(A13)}$$

Note that the ''raising'' and ''lowering'' of the $\pm$ indices is given by the rule $x^\pm=x_\mp$, while $x^I=-x_I$ for $I=1,2$, as usual. It is now a routine task to demonstrate that the Yang-Mills action (A1) is equivalent to

[13] The presence of two transverse scalar fields suggests a non-critical string theory in four dimensions.

$$S^{LC}_{3+1}=\int dx^+dx^-d\mathbf{x}^\perp \mathrm{tr}\left(\frac{1}{2}F^2_{+-}+F_{+I}F_{-I}-\frac{1}{4}F^2_{IJ}\right.$$
$$\left.+\mathrm{i}\psi^T_R D_+\psi_R+\mathrm{i}\psi^T_L D_-\psi_L-\mathrm{i}\sqrt{2}\psi^T_L\epsilon_2\beta_I D_I\psi_R\right), \tag{A14}$$

where the repeated indices $I,J$ are summed over $\{1,2\}$. Some surprising simplifications follow if we now choose to work in the *light-cone gauge* $A^+=A_-=0$. In this gauge $D_-\equiv\partial_-$, and so the (Euler-Lagrange) equation of motion for the left-moving field $\psi_L$ is simply

$$\partial_-\psi_L=\frac{1}{\sqrt{2}}\epsilon_2\beta_I D_I\psi_R, \tag{A15}$$

which is evidently a non-dynamical constraint equation, since it is independent of the light-cone time. We may therefore eliminate any dependence on $\psi_L$ (representing unphysical degrees of freedom) in favor of $\psi_R$, which carries the eight physical fermionic degrees of freedom in the theory. In addition, the equation of motion for the $A_+$ field yields Gauss' law

$$\partial^2_- A_+=\partial_-\partial_I A_I+gJ^+ \tag{A16}$$

where $J^+=\mathrm{i}[A_I,\partial_- A_I]+2\psi^T_R\psi_R$, and so the $A_+$ field may also be eliminated to leave the two bosonic degrees of freedom $A_I$, $I=1,2$. Note that the two fermionic degrees of freedom exactly match the bosonic degrees of freedom associated with the transverse polarization of a four dimensional gauge field, which is of course consistent with the supersymmetry. We should emphasize that unlike the usual covariant formulation of Yang-Mills theory, the light-cone formulation here permits one to remove *explicitly* any unphysical degrees of freedom in the Lagrangian (or Hamiltonian); there are no ghosts, and supersymmetry is manifest.

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