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Path integral for the Hilbert-Palatini and Ashtekar gravity

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To write down a path integral for the Ashtekar gravity one must solve three fundamental problems. First, one must understand the rules of complex contour functional integration with holomorphic action. Second, one should find which gauges are compatible with reality conditions. Third, one should evaluate the Faddeev-Popov determinant produced by these conditions. In the present paper we derive the BRST path integral for Hilbert-Palatini gravity. We show that for a certain class of gauge conditions this path integral can be rewritten in terms of the Ashtekar variables. Reality conditions define contours of integration. For our class of gauges all ghost terms coincide with what one could write naively, just ignoring any Jacobian factors arising from the reality conditions. [S0556-2821(98)06824-6]

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I. INTRODUCTION

The invention of complex canonical variables [1] opened a new avenue for the nonperturbative treatment of quantum general relativity. In these new variables all constraints were made polynomial at the expense of introducing reality conditions. Afterwards, many gravitational theories were reformulated in a similar way, including even 11-dimensional supergavity [2]. Quite spectacular success was achieved in loop quantum gravity [3]. In view of the recent progress in nonperturbative methods it seems especially important to develop a path integral formulation of Ashtekar gravity which could serve as a bridge between perturbative and nonperturbative results.

The constraint structure of Ashtekar gravity has been studied in some detail (for reviews, see [4] and [5]). The Becchi-Rowet-Stora-Tyutin (BRST) charge was constructed [6]. An earlier attempt to study the path integral in Ashtekar variables was given by Torre [7]. However, in this paper subtleties coming from integration over complex variables were ignored. It is known that any restriction imposed on integration variables may lead to Faddeev-Popov ghosts [8]. It is unclear what kind of ghost action is induced by the reality conditions.

It is obvious that the path integral for Ashtekar gravity will have a somewhat unusual form. In the case of complex scalar fields the action is real and one integrates over the whole complex plane. In the case of Ashtekar gravity the action is holomorphic. Thus one may expect some sort of contour integration. The position of the contour must be defined by using the reality conditions. However, it is not known yet which gauges are compatible with these conditions.

Our strategy is rather simple. We derive the path integral for Hilbert-Palatini gravity and then rewrite it in terms of the Ashtekar variables. By itself, the first part of our work does not have great novelty. The Hamiltonian structure of Hilbert-Palatini gravity has been analyzed in a number of papers [9–11,4,5]. Given this analysis construction of the path integral is quite straightforward. However, the transition to the Ashtekar variables requires a complex canonical transformation which is not well defined in the path integral. We would also like to avoid any gauge fixing at intermediate steps before the path integral is written down. Thus we are forced to choose a basis in the Hilbert-Palatini action different from the ones used earlier and redo calculations of the constraint algebra, BRST charge, etc. A price to pay for the relatively easy transition to the Ashtekar variables in the path integral is an ugly form of the Hamiltonian constraint of the Hilbert-Palatini action. It leads to lengthy calculations at intermediate steps, which are reported here in some detail to make the paper self-contained.

As our main result, we transformed the Hilbert-Palatini path integral to the Ashtekar variables. This can be done successfully for a restricted class of gauges only. One is not allowed to impose gauge conditions on the connection variables. Therefore, path integral quantization of Ashtekar gravity in an arbitrary gauge remains an open problem.

The paper is organized as follows. In the next section some preliminary information on the self-dual Hilbert-Palatini action is collected. We introduce variables which will be convenient for the construction of the path integral, rederive the Ashtekar action, and give some useful equations. In Sec. III we reconsider the constraint structure of Hilbert-Palatini gravity in terms of our variables. Section IV is devoted to the BRST quantization of Hilbert-Palatini gravity. In Sec. V we establish a relation between first and second class constraints of the Hilbert-Palatini action and the reality conditions and vanishing of the imaginary part of the Ashtekar action. In Sec. VI we rewrite the path integral in terms of the Ashtekar variables. This represents our main result. Readers who do not want to go into the technicalities of the BRST quantization will find a simple derivation of the Faddeev path integral for Ashtekar gravity in Sec. VII. In the last

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section some perspectives are briefly discussed. Technical details are collected in the Appendixes.

II. SELF-DUAL HILBERT-PALATINI ACTION

Let $\Omega^{\gamma\delta}=d\omega^{\gamma\delta}+\omega_\gamma{}^\alpha\wedge\omega^{\alpha\delta}$; ω and e are connection and tetrad one-forms, respectively. The signature of the metric is $(-,+,+,+)$. The Levi-Civita tensor is defined by the equation $\epsilon_{0123}=1$. Define the star operator as $\star\omega^{\alpha\beta}=1/2\epsilon^{\alpha\beta}{}_{\gamma\delta}\omega^{\gamma\delta}$. Define

$$A^{\alpha\beta}=\frac{1}{2}\,(\omega^{\alpha\beta}-i\star\omega^{\alpha\beta}),$$

$$\mathcal{F}^{\alpha\beta}=dA^{\alpha\beta}+A^{\alpha}{}_{\gamma}\wedge A^{\gamma\delta}=\frac{1}{2}\,(\Omega^{\alpha\beta}-i\star\Omega^{\alpha\beta}). \tag{1}$$

These fields satisfy $\star A=iA$, $\star\mathcal{F}=i\mathcal{F}$. Let us start with the self-dual Hilbert-Palatini action expressed in terms of self-dual connection only [11–14]:

$$S_{SD}=\int\epsilon_{\alpha\beta\gamma\delta}e^{\alpha}\wedge e^{\beta}\wedge\mathcal{F}^{\gamma\delta}. \tag{2}$$

Let us split coordinates x^μ into ''time'' t and ''space'' x^i and introduce the notation

$$e^0=Ndt+\chi_a E_i^a dx^i,\quad e^a=E_i^a dx^i+E_i^a N^i dt,$$

$$A_i^a=\epsilon^{abc}A_{bci},\quad A_0^a=\epsilon^{abc}A_{bc0},$$

$$F_{ij}^a=\epsilon^{abc}\mathcal{F}_{ij,bc}, \tag{3}$$

where $a,b,c=1,2,3$ are flat SO(3) indices. E_a^i will denote the inverse of E_i^a. We also need weighted fields:

$$\tilde{E}_a^i=\sqrt{h}E_a^i,\quad \underset{\sim}{N}=(\sqrt{h})^{-1}N, \tag{4}$$

$\sqrt{h}=\det E_i^a$. After long but elementary calculations we can represent Eq. (2) in the following form:

$$S_{SD}=2\int dt\, d^3x(P_a^i\partial_t A_i^a+A_0^a\mathcal{G}_a+N^i\mathcal{H}_i+\underset{\sim}{N}\mathcal{H}),$$

$$P_a^i=i(\tilde{E}_a^i-i\epsilon_a{}^{bc}\tilde{E}_b^i\chi_c),$$

$$\mathcal{G}_a=\nabla_i P_a^i=\partial_i P_a^i-\epsilon_{abc}A_i^b P^{ci},$$

$$\mathcal{H}_i=-2i\tilde{E}_a^k F_{ik}^a-\epsilon_{ijk}\tilde{E}_a^j\tilde{E}_b^k\epsilon^{lmn}\underset{\sim}{E}{}_l^d\chi_d F_{mn}^{ab},$$

$$\mathcal{H}=2\tilde{E}_a^i\tilde{E}_b^k F_{ik}^{ab}, \tag{5}$$

$\underset{\sim}{E}{}_i^a=h^{-1/2}E_i^a$. By a suitable redefinition of Lagrange multipliers χ^a can be removed from the action:

$$\mathcal{N}_D^i=N^i+\frac{\tilde{E}_a^i\chi^a(N^j\underset{\sim}{E}{}_j^b\chi_b-\underset{\sim}{N})}{1-\chi^2},\quad \underset{\sim}{\mathcal{N}}=\frac{\underset{\sim}{N}-N^i\underset{\sim}{E}{}_i^a\chi_a}{1-\chi^2}. \tag{6}$$

The action (5) now reads

$$S_{SD}=S_A=2\int dt\, d^3x(P_a^i\partial_t A_i^a+A_0^a\mathcal{G}_a+\mathcal{N}_D{}^i H_i+\underset{\sim}{\mathcal{N}}H),$$

$$H_i=-2P_a^k F_{ik}^a,$$

$$H=-2P_a^i P_b^k F_{ik}^{ab}. \tag{7}$$

All χ dependence is hidden in the canonical variables. We arrived at the Ashtekar action (7) (later denoted as S_A). The absence of χ in S_A leads to a first class primary constraint $p_\chi=0$, where p_χ is the canonical momentum for χ. This constraint generates shifts of χ by an arbitrary function and originates from the Lorentz boosts.

One must bear in mind that not all the components of Re P_a^i are independent. To restore the correct form of P_a^i one needs a condition Im $P_a^{(i}$ Re $P_a^{j)}=0$ or, equivalently,

$$\mathrm{Im}(P_a^i P_a^j)=0. \tag{8}$$

Equation (8) is known as the first metric reality condition. Being supplemented by the second metric reality condition

$$\partial_t\mathrm{Im}(P_a^i P_a^j)=0 \tag{9}$$

on an initial hypersurface, it ensures the real evolution of the metric [15–17]. As usual, the triad field $\tilde{E}$ should be nondegenerate.

Define the smeared constraints

$$\mathcal{G}(n)=\int d^3x\, n^a\mathcal{G}_a,\quad H^A(\underset{\sim}{N})=\int d^3x\,\underset{\sim}{N}H,$$

$$\mathcal{D}(\vec{N})=\int d^3x\, N^i(H_i+2A_i^a\mathcal{G}_a). \tag{10}$$

They obey the following algebra:

$$\{\mathcal{G}(n),\mathcal{G}(m)\}_C=-\mathcal{G}(n\times m),$$

$$\{\mathcal{D}(\vec{N}),\mathcal{D}(\vec{M})\}_C=-2\mathcal{D}([\vec{N},\vec{M}]),$$

$$\{\mathcal{D}(\vec{N}),\mathcal{G}(n)\}_C=-2\mathcal{G}(N^i\partial_i n),$$

$$\{H^A(\underset{\sim}{N}),\mathcal{G}(n)\}_C=0,$$

$$\{\mathcal{D}(\vec{N}),H^A(\underset{\sim}{N})\}_C=-2H^A(\mathcal{L}_{\vec{N}}\underset{\sim}{N}),$$

$$\{H^A(\underset{\sim}{N}),H^A(\underset{\sim}{M})\}_C=2\mathcal{D}(\vec{K})-2\mathcal{G}(2K^jA_j), \tag{11}$$

where

$$(n\times m)^a=\epsilon^{abc}n^b m^c,\quad \mathcal{L}_{\vec{N}}\underset{\sim}{N}=N^i\partial_i\underset{\sim}{N}-\underset{\sim}{N}\partial_i N^i,$$

$$[\vec{N},\vec{M}]^i=N^k\partial_k M^i-M^k\partial_k N_i\,, \tag{12}$$

$$K^j=(\underset{\sim}{N}\partial_i\underset{\sim}{M}-\underset{\sim}{M}\partial_i\underset{\sim}{N})P^i_a P^j_a\,. \tag{13}$$

We introduced the subscript C to distinguish the Poisson brackets $\{\cdot,\cdot\}_C$ of the complex Ashtekar theory from those of the real Hilbert-Palatini action.

III. HAMILTONIAN FORM OF THE HILBERT-PALATINI ACTION

Let us start with the Hilbert-Palatini action

$$S=\frac{1}{2}\int\epsilon_{\alpha\beta\gamma\delta}e^\alpha\wedge e^\beta\wedge\Omega^{\gamma\delta}. \tag{14}$$

Recall that the Ashtekar action is obtained from the Hilbert-Palatini one by adding a pure imaginary term $-i\frac{1}{2}\int\epsilon_{\alpha\beta\gamma\delta}e^\alpha\wedge e^\beta\wedge\star\Omega^{\gamma\delta}$. Therefore,

$$S=\text{Re}\,S_A=2\int dt d^3x(\tilde{E}^i_a\partial_t\omega^{0a}_i+Z^i_a\partial_t\xi^a_i+n^a_G\text{Re}\,\mathcal{G}_a$$
$$+n^a_L\text{Im}\,\mathcal{G}_a+\mathcal{N}_D{}^i\text{Re}\,H_i+\underset{\sim}{\mathcal{N}}\,\text{Re}\,H), \tag{15}$$

where

$$n^a_G=\text{Re}\,A^a_0\,,\quad n^a_L=-\text{Im}\,A^a_0\,, \tag{16}$$

$$Z^i_a=\epsilon_a{}^{bc}\tilde{E}^i_b\chi_c\,, \tag{17}$$

$$\xi^a_i=\tfrac{1}{2}\epsilon^a{}_{bc}\omega^{bc}_i\,. \tag{18}$$

In order to simplify the constraint algebra we replace Re H_i by the modified vector constraint. To this end we shift the Lagrange multipliers:

$$n^a_G=\mathcal{N}^a_G+2\mathcal{N}^i_D\xi^a_i\,,\quad n^a_L=\mathcal{N}^a_L+2\mathcal{N}^i_D\omega^{0a}_i\,. \tag{19}$$

We see that $\tilde{E}^i_a$ plays the role of the momentum for ξ^a_i whereas Z^i_a is the momentum conjugate to ω^{0a}_i. Here Z^i_a has three independent components only. To have time derivatives of true dynamical variables we replace

$$\omega^{0a}_i=\eta^a_i+\epsilon^{abc}\xi^b_i\chi_c\,. \tag{20}$$

Then the kinetic term reads $\tilde{E}^i_a\partial_t\eta^a_i-(\epsilon^{abc}\xi^b_i\tilde{E}^i_c)\partial_t\chi_a$. By a suitable change of variables we can bring this term to the standard form $p\partial_t q$. Let us introduce a basis in the space of 3×3 matrices:

$$(r_A)^a_i=\underset{\sim}{E}{}^b_i(\beta_A)^a_b\,,\quad(\gamma_a)^b_i=\tfrac{1}{2}\epsilon_{abc}\underset{\sim}{E}{}^c_i\,, \tag{21}$$

where β_A are six symmetric 3×3 matrices. Define

$$\xi^a_i=r^a_i+(\gamma_b)^a_i\omega^b,\quad r^a_i=(r_A)^a_i\lambda^A. \tag{22}$$

ω and λ will be treated as new canonical variables.

We arrive at the following expression for the Hilbert-Palatini action:

$$\frac{1}{2}S=\int dt\,d^3x(\tilde{E}^i_a\partial_t\eta^a_i+\chi_a\partial_t\omega^a+\mathcal{N}^a_G\Phi^G_a+\mathcal{N}^a_L\Phi^L_a+\mathcal{N}^i_D\Phi^D_i+\underset{\sim}{N}\Phi^H),$$

$$\Phi^G_a=\partial_i(\epsilon_a{}^{bc}\tilde{E}^i_b\chi_c)-\epsilon_{ab}{}^c\eta^b_i\tilde{E}^i_c-\epsilon_{ab}{}^c\omega^{zb}\chi_c\,,$$

$$\Phi^L_a=\partial_i\tilde{E}^i_a+\epsilon_{abc}\eta^b_i\epsilon^{cgf}\tilde{E}^i_g\chi_f-(\delta_{ab}-\chi_a\chi_b)\omega^b,$$

$$\Phi^D_i=-2[\tilde{E}^j_a\partial_i\eta^a_j-\partial_j(\tilde{E}^j_a\eta^a_i)-\omega^a\partial_i\chi_a],$$

$$\Phi^H=\epsilon^{abc}\tilde{E}^i_b\tilde{E}^j_c(\delta_{ad}-\chi_a\chi_d)\epsilon^d{}_{gf}\eta^g_i\eta^f_j+2\tilde{E}^i_a\tilde{E}^j_b\chi^b(\partial_i\eta^a_j-\partial_j\eta^a_i)-(1-\chi^2)[2\partial_i(\tilde{E}^i_a\omega^a)-h^{-1}\omega^a\partial_i(h\tilde{E}^i_a)]$$
$$+\omega^a\chi^b(\tilde{E}^i_a\partial_i\chi^b+\tilde{E}^i_b\partial_i\chi_a)+\tilde{E}^j_a\omega^b(\chi_a\eta^b_j-\chi_b\eta^a_j)-\omega^a\chi_a(\tilde{E}^j_b\eta^c_j\chi^b\chi_c-\chi^2\tilde{E}^j_b\eta^b_j)-\frac{1}{2}(1-\chi^2)\omega^a\omega^b(\delta_{ab}-\chi_a\chi_b)$$
$$+2\epsilon^{abc}\tilde{E}^i_b\tilde{E}^j_c[(1-\chi^2)\partial_i r^a_j+r^d_j\chi_d\partial_i\chi_a-(1-\chi^2)\chi_a r^d_j\eta^d_i+(\delta_{ag}-\chi_a\chi_g)\eta^g_i r^d_j\chi_d]$$
$$-(1-\chi^2)\epsilon^{abc}\tilde{E}^i_b\tilde{E}^j_c(\delta_{ad}-\chi_a\chi_d)\epsilon^d{}_{gf}r^g_i r^f_j\,. \tag{23}$$

We see that λ_A has no conjugate momentum, and thus is nondynamical. We observe also that λ_A is contained in Φ^H only.

Let us analyze constraints of the theory along the lines of the usual Dirac procedure [18]. Since all steps are completely standard, we omit irrelevant technical details (cf. [11,4]). First we note that $\tilde{E}_a^i$ and χ_a are conjugate momenta to η_i^a and ω^a, respectively. By analyzing the consistency conditions we get the following set of constraints:

$$p_\alpha^{(n)}=0, \quad p_A^{(\lambda)}=0, \quad \Phi_\alpha=0, \quad \underset{\sim}{\mathcal{N}}\frac{\partial\Phi^H}{\partial\lambda_A}=0, \tag{24}$$

where $p^{(q)}$ denotes the momentum conjugate to the variable q, (n) are all Lagrange multipliers, and $\Phi_\alpha=(\Phi_a^G,\Phi_a^L,\Phi_i^D,\Phi^H)$. Introduce

$$\Phi'_\alpha=\Phi_\alpha-\frac{1}{2}\,p_A^\lambda\mathcal{A}_{AB}^{-1}\left\{\Phi_\alpha,\frac{\partial\Phi^H}{\partial\lambda_B}\right\}, \tag{25}$$

where $\mathcal{A}_{AB}=-\frac{1}{2}(\partial^2\Phi^H/\partial\lambda_A\partial\lambda_B)$. Then Φ'_α and $p_\alpha^{(n)}$ are first class constraints.

The remaining constraints $p_A^{(\lambda)}$ and $\underset{\sim}{\mathcal{N}}(\partial\Phi^H/\partial\lambda_A)$ are second class constraints with a nontrivial matrix of commutators. This matrix is nondegenerate and can be used to construct Dirac's brackets. To avoid using such an object one should solve the second class constraints explicitly.

The constraints $p_A^{(\lambda)}=0$ are solved trivially giving us back Φ_α as first class constraints. Since Φ^H is quadratic in λ, it can be represented as

$$\Phi^H=\Phi_0^H+2\mathcal{B}_A\lambda_A-\lambda_A\mathcal{A}_{AB}\lambda_B. \tag{26}$$

The remaining second class constraints give the equations

$$0=\frac{\delta\Phi^H}{\delta\lambda^A}=2(-\mathcal{A}_{AB}\lambda_B+\mathcal{B}_A), \tag{27}$$

which can be solved for λ, resulting in expressions for nondynamical components r_i^a in terms of other canonical variables. Here we give final results only; some intermediate steps are reported in Appendix A:

$$\begin{aligned}
r_i^a=\frac{1}{2(1-\chi^2)}\,&(-X_{ad}\epsilon^{dbc}\tilde{E}_b^k\tilde{E}_c^jX_{gf}\underset{\sim}{E}_i^g\partial_k\underset{\sim}{E}_j^f+X_{ag}\underset{\sim}{E}_i^g\epsilon^{dbc}\tilde{E}_b^k\tilde{E}_c^jX_{df}\partial_k\underset{\sim}{E}_j^f-\epsilon^{dbc}\tilde{E}_b^k\tilde{E}_c^jX_{dg}\underset{\sim}{E}_i^gX_{af}\partial_k\underset{\sim}{E}_j^f\\
&-\chi_a\epsilon_{dbc}\tilde{E}_b^jX_{cg}\underset{\sim}{E}_i^g\partial_i\chi_d+\epsilon^{dbc}\chi_b\partial_i\chi_c-\epsilon^{abc}\chi_b\tilde{E}_c^j\underset{\sim}{E}_i^d\partial_j\chi_d+\epsilon^{abc}\chi_b\eta_i^c+\epsilon^{dbc}\tilde{E}_a^j\underset{\sim}{E}_i^d\chi_b\eta_j^c),
\end{aligned} \tag{28}$$

where $X_{ab}=(\delta_{ab}-\chi_a\chi_b)$. The Hamiltonian constraint reads

$$\begin{aligned}
\Phi^H=&\,\Phi_0^H+\mathcal{B}_A\mathcal{A}_{AB}^{-1}\mathcal{B}_B\\
=&-\frac{1}{2}\,(1-\chi^2)\omega^a\omega^bX_{ab}-(1-\chi^2)[2\partial_i(\tilde{E}_a^i\omega^a)-h^{-1}\omega^a\partial_i(h\tilde{E}_a^i)]+\omega^a\chi_b(\tilde{E}_a^i\partial_i\chi_b+\tilde{E}_b^i\partial_i\chi_a)\\
&+[\tilde{E}_a^i\omega^b(\chi_a\eta_i^b-\chi_b\eta_i^a)-\omega^a\chi_a(\tilde{E}_b^i\chi^b\eta_i^c\chi_c-\chi^2\tilde{E}_b^i\eta_i^b)]\\
&+\frac{1}{2}\,\{-\epsilon^{abc}\tilde{E}_b^i\tilde{E}_c^jX_{ad}\epsilon^{dpq}\tilde{E}_p^k\tilde{E}_q^lX_{gf}\partial_i\underset{\sim}{E}_j^g\partial_k\underset{\sim}{E}_l^f+\epsilon^{abc}\tilde{E}_b^i\tilde{E}_c^jX_{ag}\partial_i\underset{\sim}{E}_j^g\epsilon^{dpq}\tilde{E}_p^k\tilde{E}_q^lX_{df}\partial_k\underset{\sim}{E}_l^f\\
&-\epsilon^{abc}\tilde{E}_b^i\tilde{E}_c^jX_{ag}\partial_k\underset{\sim}{E}_l^g\epsilon^{dpq}\tilde{E}_p^k\tilde{E}_q^lX_{df}\partial_i\underset{\sim}{E}_j^f\}\\
&+\{-\epsilon^{abc}\tilde{E}_b^i\tilde{E}_c^j\chi_a\epsilon^{dpq}\tilde{E}_p^k\partial_k\chi_dX_{qg}\partial_i\underset{\sim}{E}_j^g+\epsilon^{abc}\tilde{E}_b^i\tilde{E}_c^j\partial_i\underset{\sim}{E}_j^a\epsilon^{dpq}\tilde{E}_p^k\partial_k\underset{\sim}{E}_l^f\chi_q-\epsilon^{abc}\tilde{E}_b^i\tilde{E}_c^j\epsilon^{adp}\tilde{E}_p^k\partial_k\chi_d\partial_i\underset{\sim}{E}_j^g\chi_g\}\\
&-\frac{\chi^2}{2(1-\chi^2)}\,\epsilon^{abc}\tilde{E}_b^i\partial_i\chi_aX_{cq}\epsilon^{dpq}\tilde{E}_p^j\partial_j\chi_d+\epsilon^{abc}\tilde{E}_a^k\tilde{E}_b^i\tilde{E}_c^j\epsilon^{dpq}\chi_d\eta_k^p\partial_i\underset{\sim}{E}_j^q-\epsilon^{abc}\tilde{E}_b^i\partial_i\tilde{E}_c^j\epsilon^{apq}\chi_p\eta_j^q\\
&+\frac{1}{1-\chi^2}\,\epsilon^{abc}\tilde{E}_b^i\partial_i\chi_a\epsilon^{cpq}\chi_p\eta_j^q\tilde{E}_d^j\chi_d\\
&+\Bigg\{2\tilde{E}_a^i\tilde{E}_b^j\chi_b(\partial_i\eta_j^a-\partial_j\eta_i^a)+\epsilon^{abc}\tilde{E}_b^i\tilde{E}_c^j\epsilon_{apq}\eta_i^p\eta_j^q+\frac{1}{2}\,\epsilon^{abc}\chi_a\tilde{E}_b^i\eta_j^c\epsilon^{dpq}\chi_d\tilde{E}_p^j\eta_i^q-\frac{1}{2}\,\epsilon^{abc}\chi_a\eta_i^b\epsilon^{cpq}\chi_p\eta_j^q\tilde{E}_g^i\tilde{E}_g^j\\
&-\frac{1}{2(1-\chi^2)}\,\epsilon^{abc}\chi_a\eta_i^b\epsilon^{cpq}\chi_p\eta_j^q\tilde{E}_g^i\chi_g\tilde{E}_f^j\chi_f\Bigg\}.
\end{aligned} \tag{29}$$

We finish up this section with some useful commutators. Introduce the smeared first class constraints

$$G(n)=\int d^3x\, n^a\Phi^G_a, \quad L(m)=\int d^3x\, m^b\Phi^L_b,$$

$$D(\vec{N})=\int d^3x\, N^i\Phi^D_i, \quad H(\underset{\sim}{N})=\int d^3x\, \underset{\sim}{N}\Phi^H. \tag{30}$$

Here all the constraints are taken from Eqs. (23), except for the Hamiltonian constraint Φ^H which is now given by Eq. (29). Here ξ^a_i is expressed in terms of canonical variables by means of Eqs. (22) and (28).

The transformations of the connection fields are

$$\{G(n),\xi^d_j\}=\epsilon^{dab}n^a\xi^b_j+\partial_j n^d,$$

$$\{G(n),\eta^d_j+\epsilon^{dpq}\xi^p_j\chi_q\}=\epsilon^{dab}n^a(\eta^b_j+\epsilon^{bpq}\xi^p_j\chi_q),$$

$$\{L(m),\xi^d_j\}=-\epsilon^{dab}m^a(\eta^b_j+\epsilon^{bpq}\xi^p_j\chi_q),$$

$$\{L(m),\eta^d_j+\epsilon^{dpq}\xi^p_j\chi_q\}=\epsilon^{dab}m^a\xi^b_j+\partial_j m^d,$$

$$\{D(\vec{N}),\xi^d_j\}=2(N^i\partial_i\xi^d_j+\xi^d_i\partial_j N^i),$$

$$\{D(\vec{N}),\eta^d_j+\epsilon^{dpq}\xi^p_j\chi_q\}=2[N^i\partial_i(\eta^d_j+\epsilon^{dpq}\xi^p_j\chi_q)+(\eta^d_i+\epsilon^{dpq}\xi^p_i\chi_q)\partial_j N^i]. \tag{31}$$

The Poisson brackets between the constraints are straightforward to evaluate. One obtains

$$\{G(n),G(m)\}=-G(n\times m),$$

$$\{L(n),L(m)\}=G(n\times m),$$

$$\{G(n),L(m)\}=-L(n\times m),$$

$$\{D(\vec{N}),D(\vec{M})\}=-2D([\vec{N},\vec{M}]),$$

$$\{D(\vec{N}),G(n)\}=-2G(N^i\partial_i n),$$

$$\{D(\vec{N}),L(m)\}=-2L(N^i\partial_i m),$$

$$\{H(\underset{\sim}{N}),G(n)\}=0,$$

$$\{H(\underset{\sim}{N}),L(m)\}=0,$$

$$\{D(\vec{N}),H(\underset{\sim}{N})\}=-2H(\mathcal{L}_{\vec{N}}\underset{\sim}{N}),$$

$$\{H(\underset{\sim}{N}),H(\underset{\sim}{M})\}=2D(\vec{K})-2G(2K^j\xi_j)-2L[2K^j(\eta_j+\xi_j\times\chi)], \tag{32}$$

where

$$K^j[\underset{\sim}{N},\underset{\sim}{M}]=(\underset{\sim}{N}\partial_i\underset{\sim}{M}-\underset{\sim}{M}\partial_i\underset{\sim}{N})K^{ij},$$

$$K^{ij}=-(\tilde{E}^i_a\tilde{E}^j_a(1-\chi^2)+\tilde{E}^i_a\chi_a\tilde{E}^j_b\chi_b). \tag{33}$$

Other notation is taken from Eq. (12). Here K^i is in fact the same as in Eq. (13) but written in different variables.

Φ^H will be called the Hamiltonian constraint. Φ^D generates diffeomorphisms of the three-surface and will be called the diffeomorphism constraint. Φ^G and Φ^L generate the SO(3,R) rotations and the Lorentz boosts, respectively. They will be called the Gauss law constraint and the Lorentz constraint, respectively.

There is a set of remarkable relations between the Poisson brackets of Hilbert-Palatini gravity and that of Ashtekar gravity:

$$\{\mathcal{G}(n),P^j_a\}_C=\{G(n),P^j_a\}=\{iL(n),P^j_a\},$$

$$\{\mathcal{G}(n),A^a_j\}_C=\{G(n),A^a_j\}=\{iL(n),A^a_j\},$$

$$\{\mathcal{D}(\vec{N}),P^j_a\}_C=\{D(\vec{N}),P^j_a\}, \quad \{\mathcal{D}(\vec{N}),A^a_j\}_C=\{D(\vec{N}),A^a_j\},$$

$$\{H^A(N),P^j_a\}_C=\{H(N),P^j_a\}. \tag{34}$$

Note that last relation holds for P^j_a only.

In a different context the relation between Hilbert-Palatini and Ashtekar brackets was considered recently by Khatsymovsky [19].

IV. BRST QUANTIZATION OF HILBERT-PALATINI GRAVITY

In this section we construct the BRST path integral [20] for Hilbert-Palatini gravity. Here we follow the review in [21]. Consider a dynamical system with phase space variables (q^s,p_s), Hamiltonian H_0, and constraints Φ_α. Let n^α be the Lagrange multipliers associated with the constraints Φ_α, and π_α be the canonically conjugate momenta. The extended phase space is defined by introducing extra ghost and antighost fields $(b^\alpha,\overline{c}_\alpha,c^\alpha,\overline{b}_\alpha)$, obeying the following nonvanishing antibrackets:

$$\{b^\alpha,\overline{c}_\beta\}_+=-\delta^\alpha_\beta, \quad \{c^\alpha,\overline{b}_\beta\}_+=-\delta^\alpha_\beta.$$

$c^\alpha,\overline{c}_\alpha$ are real, whereas $b^\alpha,\overline{b}_\alpha$ are imaginary.

It is convenient to define an additional structure on the extended phase space, that of ''ghost number.'' This is done by attributing the following ghost number to the canonical variables: c^α,b^α have ghost number 1, $\overline{c}_\alpha,\overline{b}_\alpha$ have ghost number -1. All other variables have ghost number 0.

On this space one can construct a BRST generator Ω and a BRST-invariant Hamiltonian H. They are determined by the following conditions.

(a) Ω is real and odd; (b) Ω has ghost number 1; (c) $\Omega=-ib^\alpha\pi_\alpha+c^\alpha\Phi_\alpha+$ ''higher ghost terms;'' (d) $\{\Omega,\Omega\}_+=0$.

(a) H is real and even; (b) H has ghost number 0; (c) H coincides with H_0 up to higher ghost terms; (d) $\{H,\Omega\}=0$.

If H_0 weakly vanishes (as in our case), one can take $H=0$ since the formalism supports an arbitrariness in the definition of observables: $H_0\sim H_0+k^\alpha\Phi_\alpha$.

The BRST generator is fully defined by structure functions of the constraint algebra:

$$\Omega=-ib^{\alpha}\pi_{\alpha}+\sum_{n\geqslant 0}c^{\alpha_{n+1}}\cdots c^{\alpha_1}U^{(n)\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_{n+1}}\overline{b}_{\beta_n}\cdots\overline{b}_{\beta_1}.$$

The structure functions for Hilbert-Palatini gravity are constructed in Appendix B. As a result, we obtain

$$\Omega=-ib^{\alpha}\pi_{\alpha}+c^{\alpha}\Phi_{\alpha}+\tfrac{1}{2}\,c^{\alpha}c^{\beta}C^{\gamma}_{\alpha\beta}\overline{b}_{\gamma}+c^{\alpha}c^{\beta}c^{\gamma}U^{(2)\delta\lambda}_{\alpha\beta\gamma}\overline{b}_{\delta}\overline{b}_{\lambda}, \tag{35}$$

where $U^{(2)}$ is taken from Eq. (B8). Note that for Yang-Mills theory the term with $U^{(2)}$ is absent in the BRST charge. This is also the case of Ashtekar gravity [6].

The quantization is based on the generating functional for the Green functions which is represented in the form

$$Z[j,J,\lambda]=\int\mathcal{D}\mu\,e^{i\int dt(L_{eff}+j_sq^s+J^sp_s+\lambda_{\alpha}n^{\alpha})}, \tag{36}$$

where

$$L_{eff}=\dot{q}^sp_s+\dot{n}^{\alpha}\pi_{\alpha}+\dot{c}^{\alpha}\overline{b}_{\alpha}+\dot{b}^{\alpha}\overline{c}_{\alpha}-H_{eff},$$
$$H_{eff}=H-\{\psi,\Omega\}_{+}. \tag{37}$$

Here ψ is an odd and imaginary function which has ghost number -1 and plays the role of a gauge-fixing function, whereas $\mathcal{D}_{\mu}$ is the usual measure (product over time of the Liouville measure of the extended phase space).

Let us choose

$$\psi=-\overline{b}_{\alpha}n^{\alpha}+i\overline{c}_{\alpha}\left(\frac{1}{\gamma}f^{\alpha}(q,p)+\frac{1}{\gamma}g^{\alpha}(n)\right). \tag{38}$$

By substituting Eqs. (35) and (38) into Eqs. (37) and putting $H=0$ one obtains

$$\begin{aligned}H_{eff}=&-n^{\alpha}\Phi_{\alpha}-i\overline{b}_{\alpha}b^{\alpha}+c^{\alpha}n^{\beta}C^{\gamma}_{\alpha\beta}\overline{b}_{\gamma}-3c^{\alpha}c^{\beta}n^{\gamma}U^{(2)\delta\lambda}_{\alpha\beta\gamma}\overline{b}_{\delta}\overline{b}_{\lambda}\\&+\frac{1}{\gamma}\Bigg\{(f^{\alpha}+g^{\alpha})\pi_{\alpha}-\overline{c}_{\alpha}\frac{\partial g^{\alpha}}{\partial n^{\beta}}b^{\beta}-i\overline{c}_{\alpha}\{f^{\alpha},\Phi_{\beta}\}c^{\beta}\\&-i\overline{c}_{\alpha}\{f^{\alpha},C^{\delta}_{\beta\gamma}\}c^{\beta}c^{\gamma}\overline{b}_{\delta}-i\overline{c}_{\alpha}\{f^{\alpha},U^{(2)\xi\eta}_{\beta\gamma\delta}\}c^{\beta}c^{\gamma}c^{\delta}\overline{b}_{\xi}\overline{b}_{\eta}\Bigg\}.\end{aligned} \tag{39}$$

Let us make the change of variables with unit Jacobian:

$$\pi_{\alpha}\to\gamma\pi_{\alpha},\quad \overline{c}_{\alpha}\to\gamma\overline{c}_{\alpha}.$$

Then let $\gamma\to 0$. In this limit integration over π_{α}, b^{α}, and $\overline{b}_{\alpha}$ is easily performed, giving

$$Z[j,J,\lambda]=\int\mathcal{D}q\,\mathcal{D}p\,\mathcal{D}n\,\mathcal{D}c\,\mathcal{D}\overline{c}\,\delta(f^{\alpha}+g^{\alpha})\times e^{i\int dt(L'_{eff}+j_sq^s+J^sp_s+\lambda_{\alpha}n^{\alpha})}, \tag{40}$$

where

$$\begin{aligned}L'_{eff}=&\dot{q}^sp_s+n^{\alpha}\Phi_{\alpha}\\&-i\overline{c}_{\beta}\left(\frac{\partial g^{\beta}}{\partial n^{\alpha}}\partial_t-\frac{\partial g^{\beta}}{\partial n^{\gamma}}C^{\gamma}_{\alpha\lambda}n^{\lambda}+\{\Phi_{\alpha},f^{\beta}\}\right)c^{\alpha}\\&-\overline{c}_{\xi}\overline{c}_{\eta}\left(\frac{\partial g^{\eta}}{\partial n^{\delta}}\{f^{\xi},C^{\delta}_{\alpha\beta}\}+3\frac{\partial g^{\xi}}{\partial n^{\delta}}\frac{\partial g^{\eta}}{\partial n^{\lambda}}U^{(2)\delta\lambda}_{\alpha\beta\gamma}n^{\gamma}\right)c^{\alpha}c^{\beta}\\&-i\overline{c}_{\alpha}\overline{c}_{\xi}\overline{c}_{\eta}\frac{\partial g^{\xi}}{\partial n^{\lambda}}\frac{\partial g^{\eta}}{\partial n^{\sigma}}\{f^{\alpha},U^{(2)\lambda\sigma}_{\beta\gamma\delta}\}c^{\beta}c^{\gamma}c^{\delta}\end{aligned} \tag{41}$$

and $q^s=(\eta^a_i,\omega^a)$, $p_s=(\tilde{E}^i_a,\chi_a)$.

This completes the construction of the path integral for Hilbert-Palatini gravity. One can see that the dependence of the structure constants on the canonical variables leads to the appearance of multighost interaction terms in Eq. (41). By an appropriate choice of gauge-fixing functions one can eliminate these terms. All nonvanishing components of $U^{(2)}$ have upper indices corresponding to the Gauss or Lorentz constraints. Therefore, if the functions g^{α} do not depend on the Lagrange multipliers $\mathcal{N}_G$ and $\mathcal{N}_L$, all terms with $U^{(2)}$ disappear. If, furthermore, the functions f^{α} do not depend on the canonical coordinates q^s, the Poisson brackets $\{f^{\xi},C^{\delta}_{\alpha\beta}\}$ vanish and the remaining higher ghost terms disappear also. In such a case, the general structure of the path integral is identical to that of rank-1 Yang-Mills theory. For short, these gauges will be called the Yang-Mills (YM) gauges. They play an important role in the path integral quantization of Ashtekar gravity.

V. CONSTRAINTS VERSUS REALITY CONDITIONS

In this section we establish the relation between solutions of the constraints in the real Hilbert-Palatini formulation and the reality conditions (8) and (9) of Ashtekar gravity. Let us recall expressions for the complex canonical variables P and A in terms of the real canonical variables:

$$P^i_a=i(\tilde{E}^i_a-i\epsilon_a{}^{bc}\tilde{E}^i_b\chi_c),$$

$$A^a_j=\xi^a_j-i(\eta^a_j+\epsilon^{abc}\xi^b_j\chi_c),$$

$$\xi^a_j=r^a_j-\tfrac{1}{2}\,\epsilon^{abc}\omega_b\underset{\sim}{E}{}^c_j, \tag{42}$$

r^a_j is given by Eq. (28).

Here it will be demonstrated that the reality conditions (8) and (9) are satisfied by Eqs. (42) provided the canonical variables of the real theory satisfy the Gauss law and the Lorentz constraint. Moreover, we shall prove that the Ashtekar action is real under the same conditions. The last statement is not completely trivial even though the real Hilbert-Palatini action is related to the complex Ashtekar action by a canonical transformation. The point is that this transformation is not canonical on the whole phase space [4]. Thus for our basis in the phase space reality of the Ashtekar action must be checked independently.

The first reality condition (8) is satisfied trivially. Let us rewrite Eq. (9) in a more explicit form. The time evolution $P^l_a P^j_a$ is given by the Poisson brackets of the total complex Hamiltonian (7) and $P^l_a P^j_a$:

$$\partial_t(P^l_a P^j_a)=\left\{\int dt\; d^3x(A^a_0\mathcal{G}_a+\mathcal{N}_D{}^i H_i+\underset{\sim}{N}H),P^l_a P^j_a\right\}_C$$
$$=-2[2P^l_a P^j_a\partial_i\mathcal{N}_D{}^i-P^k_a P^l_a\partial_k\mathcal{N}_D{}^j$$
$$-P^k_a P^j_a\partial_k\mathcal{N}_D{}^l+\mathcal{N}_D{}^i\partial_i(P^l_a P^j_a)]$$
$$+2(\nabla_k P^k_a)(\mathcal{N}_D{}^j P^l_a+\mathcal{N}_D{}^l P^j_a)$$
$$-2\underset{\sim}{N}\epsilon^{abc}P^i_a(P^j_c\nabla_i P^l_b+P^l_c\nabla_i P^j_b). \tag{43}$$

The first line of Eq. (43) is real for real $\mathcal{N}_D{}^i$ due to the first reality condition (8). The second line disappears due to the Gauss law constraint. Therefore, to ensure real metric evolution one must require

$$\mathrm{Im}[\epsilon^{abc}P^i_a(P^j_c\nabla_i P^l_b+P^l_c\nabla_i P^j_b)]=0. \tag{44}$$

The condition (44) can be presented as $\mathrm{Im}\{P^l_aP^j_a,H\}_C=0$. It is clear that this condition is invariant under *complex* SO(3) transformations. These transformations can be used to put $\chi=0$. One can easily demonstrate that for the fields (42) the condition (44) is satisfied.

Now let us prove that under the same conditions

$$\mathrm{Im}\ H_i=\mathrm{Im}(H_i+2A^a_i\mathcal{G}_a)=0. \tag{45}$$

From Eqs. (11) and (34) one can see that $\{\mathcal{G},\mathcal{G}\}_C\sim\mathcal{G}$ and $\{\Phi^D,\mathcal{G}\}\sim\mathcal{G}$. Hence the surface $\mathcal{G}=0$ is invariant under complex SO(3) transformations and real diffeomorphisms. Since $\{\mathcal{G},H_i+2A^a_i\mathcal{G}_a\}_C\sim\mathcal{G}$ and $\{\Phi^D,\mathrm{Im}(H_i+2A^a_i\mathcal{G}_a)\}\sim\mathrm{Im}(H_i+2A^a_i\mathcal{G}_a)$, these transformations map solutions of Eq. (45) to themselves inside the surface $\mathcal{G}=0$. One can use SO(3) transformations and diffeomorphisms to impose the condition $\chi=0$ everywhere, and $\partial_k\tilde{E}^j_a=0$ at a certain point. At this point one must only check the cancellation of the second derivatives of $\tilde{E}$. This is straightforward to do by using Eqs. (42), (28) and the explicit form (23) of the constraint $\mathcal{G}=\Phi^G+i\Phi^L$.

To prove that Im $H=0$ one can use the Lorentz boosts to put $\chi=0$. This makes the calculations quite elementary even without further gauge fixing.

By straightforward calculations one can demonstrate that the imaginary part of the kinetic term $P^j_a\partial_t A^a_j$ is a total derivative and thus can be discarded in quantization. This is done in Appendix C.

As was advertised at the beginning of this section, we demonstrated that the complex canonical variables satisfy the reality conditions on the surface of Eqs. (42), the second class constraint (27), and the two first class constraints Φ^G and Φ^L. Note that the reality conditions admit more solutions. For example, one can interchange the real and imaginary parts of P^j_a.

VI. PATH INTEGRAL QUANTIZATION OF ASHTEKAR GRAVITY

In this section we derive a path integral for Ashtekar gravity from the one for Hilbert-Palatini gravity.

Consider the functional (40) in a YM gauge:

$$Z[j,J]=\int \mathcal{D}\eta^a_i\,\mathcal{D}\tilde{E}^i_a\,\mathcal{D}\omega^a\,\mathcal{D}\chi_a\,\mathcal{D}\mathcal{N}_G\,\mathcal{D}\mathcal{N}_L\,\mathcal{D}\mathcal{N}^i_D\,\mathcal{D}\underset{\sim}{N}\,\mathcal{D}c^\alpha\,\mathcal{D}\bar{c}_\alpha\;\delta(f^\alpha+g^\alpha)\exp\left(i\int dt(L'_{eff}+j^i_a\eta^a_i+J^a_i\tilde{E}^i_a)\right). \tag{46}$$

We drop the sources for the Lagrange multipliers, χ and ω. A discussion of the source terms is postponed to the end of this section.

Since the gauge-fixing function g^α does not depend on the Gauss and Lorentz Lagrange multipliers, integration over these Lagrange multipliers gives δ functions of the corresponding constraints, $\delta(\Phi^G_a)\delta(\Phi^L_a)$. This means that in fact we are working on the surface of these constraints. In the previous section it is shown that on this surface the imaginary part of the Ashtekar action vanishes. Thus one can write

$$L'_{eff}=L_A(P,A)-i\bar{c}_\beta\left(\frac{\partial g^\beta}{\partial n^\alpha}\,\partial_t-\frac{\partial g^\beta}{\partial n^\gamma}\,C^\gamma_{\alpha\lambda}n^\lambda+\{\Phi_\alpha,f^\beta\}\right)c^\alpha. \tag{47}$$

We assume that complex canonical variables are expressed in terms of real canonical variables by means of Eqs. (42).

One can integrate over ω^a by using the delta function of the Lorentz constraint Φ^L. This is equivalent in effect to the substitution

$$\omega^a(\tilde{E},\eta,\chi):=\left(\delta_{ab}+\frac{\chi_a\chi_b}{1-\chi^2}\right)\partial_i\tilde{E}^i_b+\tilde{E}^i_a\eta^b_i\chi_b-\frac{\chi_a}{1-\chi^2}\,(\tilde{E}^i_b\eta^b_i-\tilde{E}^i_b\chi_b\eta^c_i\chi_c). \tag{48}$$

The path integral measure is multiplied by

$$\Delta_1=\det{}^{-1}(\delta_{ab}-\chi_a\chi_b)=\prod_{x,t}\frac{1}{1-\chi^2}. \tag{49}$$

Now we are ready to change the integration variables in Eq. (46):

$$\tilde{E}^i_a \to P^i_a = i\tilde{E}^i_a + \epsilon^{abc}\tilde{E}^i_b\chi_c , \qquad \eta^a_i \to A^a_i = \xi^a_i - i(\eta^a_i + \epsilon^{abc}\xi^b_i\chi_c). \tag{50}$$

This gives rise to a determinant

$$\Delta_2 = \det{}^{-1}(1 i\,\delta^i_j\delta^b_a + \delta^i_j\epsilon^{abc}\chi_c)\det{}^{-1}\Bigg[\frac{1}{2(1-\chi^2)}\,[-2\,\delta^j_i\epsilon^{abc}\chi_c + \epsilon^{apq}\tilde{E}^j_q(\delta_{pb}-\chi_p\chi_b)\underset{\sim}{E}{}^d_i\chi_d - \chi_a\epsilon^{dpq}\underset{\sim}{E}{}^d_i\tilde{E}^j_q(\delta_{pb}-\chi_p\chi_b)]$$
$$- i\left(\delta^j_i\delta^a_b + \frac{1}{2(1-\chi^2)}\,[2\,\delta^j_i(\delta^a_b\chi^2 - \chi_a\chi_b) + (1-\chi^2)\tilde{E}^j_a\chi_b\underset{\sim}{E}{}^c_i\chi_c - (\delta_{ab}-\chi_a\chi_b)\underset{\sim}{E}{}^c_i\chi_c\tilde{E}^j_d\chi_d]\right)\Bigg] = \prod_{x,t}\left(-\frac{1}{1-\chi^2}\right). \tag{51}$$

Note that if all the gauge-fixing functions f depend on the real fields χ and $\tilde{E}$ through P only, the ghost action becomes degenerate [see Eqs. (34)]. This is a manifestation of the fact that the Lorentz constraint is ''superfluous'' in complex Ashtekar gravity. Therefore, we must fix the corresponding gauge freedom by means of a condition on χ:

$$\chi^a = \chi^a_{(0)}(\tilde{E}), \tag{52}$$

where $\chi_{(0)}$ is a given function.

Before integrating over χ let us rewrite Eq. (52) in a different form. By inverting the first equation in Eqs. (42), one obtains

$$\tilde{E}^i_a = \left(\frac{\epsilon^{abc}\chi_c}{1-\chi^2} - i\,\frac{\delta_{ab}-\chi_a\chi_b}{1-\chi^2}\right)P^i_b = \pi^b_a(\chi)P^i_b . \tag{53}$$

As a result of Eq. (52), one can replace χ by $\chi_{(0)}(\tilde{E})$. The right hand side of Eq. (53) becomes $\tilde{E}$ dependent. This dependence, however, can be removed at least locally by means of a formal power series expansion. As a result, we obtain

$$\tilde{E}^i_a = \bar{\pi}^b_a(P)P^i_b , \tag{54}$$

where $\bar{\pi}$ is a function of P but not of P^*, which depends on the choice of gauge-fixing function $\chi^{(0)}$. For the present analysis the explicit form of $\bar{\pi}$ is of no importance. Note that the simple relation $\tilde{E} = \mathrm{Im}\,P$ would not work, because it depends both on P and its complex conjugate.

One can replace Eq. (52) by the condition

$$\chi = \chi_{(0)}(\bar{\pi}P) = \bar{\chi}(P). \tag{55}$$

The two conditions (52) and (55) are equivalent since they select the same surfaces in phase space. However, the ghost terms and Jacobian factors appearing due to the delta functions of the gauge conditions are different for Eqs. (52) and (55). In the final result these differences compensate each other, as one can easily show using a geometric interpretation of the Faddeev-Popov determinant.

Let us integrate over χ with the help of the delta function $\delta(\chi - \bar{\chi}(P))$. Since we already changed variables to P and A, no Jacobian factor appears.

Integration over P and A should be understood as a contour integration in complex space. One integrates along the lines defined by the reality conditions and Eqs. (52) and (48). As usual, there are real parameters which label points of the contours in the complex planes. These are $\tilde{E}$ and η. Since the fields ω and χ are already excluded, we do not integrate over the position of the contours.

Consider the ghost action. Integration over $\bar{c}$ and c gives the following functional determinant:

$$\det\left(\frac{\partial g^\beta}{\partial n^\alpha}\,\partial_t - \frac{\partial g^\beta}{\partial n^\gamma}\,C^\gamma_{\alpha\lambda}n^\lambda + \{\Phi_\alpha, f^\beta\}\right). \tag{56}$$

Let us separate indices corresponding to the Lorentz boosts: $\{\Phi_\alpha\} = \{\Phi^L_a;\Phi_\mu\}$, $\{f^\alpha\} = \{\chi^a - \bar{\chi}^a(P); f^\mu(\chi,P)\}$, $\{g^\alpha\} = \{0;g^\mu\}$. Greek indices from the middle of the alphabet correspond to the Gauss law, diffeomorphism, and Hamiltonian constraints. The matrix elements in Eq. (56) contain the following brackets:

$$\{\Phi_\mu, f^\nu(\chi,P)\} = \{\Phi_\mu, P\}\,\frac{\delta f^\nu}{\delta P} + \frac{\delta\Phi_\mu}{\delta\omega}\,\frac{\delta f^\nu}{\delta\chi},$$
$$\{\Phi_\mu, \chi - \bar{\chi}(P)\} = \frac{\delta\Phi_\mu}{\delta\omega} - \{\Phi_\mu, P\}\,\frac{\delta\bar{\chi}}{\delta P}, \tag{57}$$

where summation indices are suppressed. Let us multiply the lines corresponding to $\chi^a - \bar{\chi}^a$ by $-\delta f^\nu/\delta\chi^a$ and add them to the f^ν lines. This produces the matrix elements

$$\frac{\partial g^\nu}{\partial n^\mu}\,\partial_t - \frac{\partial g^\nu}{\partial n^\rho}\,C^\rho_{\mu\sigma}n^\sigma + \{\Phi_\mu, P\}\left(\frac{\delta f^\nu}{\delta P} + \frac{\delta f^\nu}{\delta\chi}\,\frac{\delta\bar{\chi}}{\delta P}\right)$$
$$= \frac{\partial g^\nu}{\partial n^\mu}\,\partial_t - \frac{\partial g^\nu}{\partial n^\rho}\,C^\rho_{\mu\sigma}n^\sigma + \{\Phi^{[C]}_\mu, f^\nu(\bar{\chi}(P),P)\}_C . \tag{58}$$

$\Phi^{[C]}_\mu$ is the Ashtekar constraint corresponding to Φ_μ, $\mathrm{Re}\,\Phi^{[C]}_\mu = \Phi_\mu$. In the last line we used that $\{\Phi_\mu, P\} = \{\Phi^{[C]}_\mu, P\}_C$ due to Eqs. (34). Equation (58) means that one can replace χ by $\bar{\chi}$ in the gauge-fixing functions f^ν.

Consider the two columns in Eq. (56) corresponding to the Gauss law and Lorentz constraints. As a result of Eqs. (34) $\{\Phi^G, f(P)\} = i\{\Phi^L, f(P)\}$. Therefore, by multiplying the column with Φ^G by $-i$ and adding it to the column with Φ^G one obtains zeros everywhere, except for the lines corresponding to the gauge conditions $\chi^a - \overline{\chi}^a(P)$. As a result, one can represent the determinant (56) as a product of two determinants:

$$\Delta_3 \det\left(\frac{\partial g^\nu}{\partial n^\mu} \partial_t - \frac{\partial g^\nu}{\partial n^\rho} C^\rho_{\mu\sigma} n^\sigma + \{\Phi^{[C]}_\mu, f^\nu(\overline{\chi}(P),P)\}_C \right), \tag{59}$$

where

$$\begin{aligned}\Delta_3 &= \det\{\Phi^L_a - i\Phi^G_a, \chi^b\}\\ &= \det[(\delta_{ab} - \chi_a\chi_b) + i\epsilon^{abc}\chi_c]\\ &= \prod_{x,t} (1-\chi^2)^2.\end{aligned} \tag{60}$$

From expressions (49), (51), and (60) one can see that all Δ's cancel each other up to an overall minus sign which can be absorbed in the reversed orientation of the contour of the A integration. The path integral is now rewritten in terms of the Ashtekar variables:

$$\begin{aligned}Z[\overline{j},\overline{J}] = \int_R &\mathcal{D}A^a_i\, \mathcal{D}P^i_a\, \mathcal{D}\mathcal{N}^i_D\, \mathcal{D}\underset{\sim}{\mathcal{N}}\, \mathcal{D}A^a_0\, \mathcal{D}c^\mu\, \mathcal{D}\overline{c}_\mu\\ &\times \delta(f^\mu + g^\mu) \exp\left(i\int dt (L'_{eff} + \overline{j}^i_a A^a_i + \overline{J}^a_i P^i_a) \right),\end{aligned} \tag{61}$$

where

$$\begin{aligned}L'_{eff} = L_A - i\overline{c}_\nu &\left(\frac{\partial g^\nu}{\partial n^\mu} \partial_t - \frac{\partial g^\nu}{\partial n^\rho} C^\rho_{\mu\sigma} n^\sigma \right.\\ &\left. + \{\Phi^{[C]}_\mu, f^\nu(\overline{\chi}(P),P)\}_C \right) c^\mu.\end{aligned} \tag{62}$$

The subscript R means contour integration in complex spaces along lines defined by the reality conditions. Integration over $\mathcal{N}_L$ (which is essentially an imaginary part of A_0) has already been performed to produce a delta function of the Lorentz constraint. This delta function, in turn, has been used to integrate over ω. Thus in Eq. (61) we integrate over the real part of A_0. This integral gives $\delta(\Phi^G) = \delta(\mathcal{G})$. The equation $\mathcal{G}=0$ can be considered as a complex equation because $\mathrm{Im}\,\mathcal{G}=0$ is supplied by the reality conditions. The same is true for the gauge conditions $f^\mu + g^\mu = 0$. A fascinating property of these complex delta functions is possibility to integrate over complex variables without an explicit transition to real coordinates on a contour.

By comparing Eqs. (11) and (32), one can see that $C^\rho_{\mu\sigma}$ are just structure constants of Ashtekar gravity. (Note that this property does not hold in the variables used by Henneaux [9].) Therefore, the ghost term in Eq. (61) produces the ordinary Faddeev-Popov determinant for Ashtekar gravity. The path integral (61) coincides with what one would write naively, just ignoring any Jacobian factors which may arise from the reality conditions and fixing the Lorentz gauge freedom. Some remarks are in order. First of all, the result (61) is valid for a certain class of gauges only. We are not allowed to impose a gauge condition on A^a_0. This restriction is needed (i) to cancel contributions to the path integral of the second order structure functions (which are zero for Ashtekar gravity [6]) and (ii) to ensure delta functions of the complex Gauss law constraint. While (i) seems to depend on a particular choice of basic variables and constraints because rank of and algebra is not an invariant, the second point (ii) looks more fundamental. The complex Gauss law constraint is needed to prove the vanishing of the imaginary part of the Ashtekar action. We are not allowed to impose gauge conditions on the connection variables. The ultimate reason for this is that the last line of Eqs. (34) is not true if we replace P by A. This restriction will receive a natural explanation in the next section in the framework of the Faddeev path integral. In all other respects the gauge conditions $f^\alpha + g^\alpha$ are arbitrary. For a given set of admissible YM gauges one can first express χ^a from three of them and then denote the remaining gauge conditions by $f^\mu + g^\mu$. The path integral for Ashtekar gravity was previously considered by the present authors and Grigentch in the one-loop approximation over a de Sitter background [22] and for the Bianchi IX finite dimensional model [23]. In these simple cases the reality conditions do not lead to any Jacobian factors if one uses gauge conditions of the YM type. We observed also that one runs into trouble if gauge conditions are imposed on the connection variables.

Using this or that gauge condition is just a matter of convenience. In principle, it is enough to formulate the path integral in just one gauge. All physical results are to be gauge independent. However, extension of our results for arbitrary gauge conditions still poses an interesting problem from both technical and aesthetic points of view.

Note that we excluded sources for χ, ω, and Lagrange multipliers. Sources for χ and ω are not needed because in the present formulation these fields are absent. Moreover, χ and ω can be considered as composite fields. Sources for $\underset{\sim}{N}$ and $\mathcal{N}_D$ can be easily restored without any modification in our procedure. Therefore, we have enough sources to describe any Green functions of the four-metrics and three-dimensional connections. If, however, we introduce a source for A^a_0, it penetrates into the delta functions of the Gauss law and Lorentz constraints and destroys the reality of the Ashtekar action. Green functions of A_0 are not defined in our approach. At the last step we introduced sources $\overline{J}$ and $\overline{j}$ for P and A. This makes the exponential in Eq. (61) complex. Thus, strictly speaking, the path integral is not well defined, even though all finite order Green functions do exist. If one wishes to be on the safe side, one can easily return to the original sources J and j for $\tilde{E}$ and η.

VII. FADDEEV PATH INTEGRAL

In this section we give a more simple derivation of the Faddeev path integral [24] for Ashtekar gravity, which does not rely upon the heavy machinery of the BRST quantization. This also seems to be a proper place to discuss the triad form of the reality conditions. For a dynamical system with canonical variables q^s, p_s, first class constraints Φ_a, and weakly vanishing Hamiltonian, such as Hilbert-Palatini gravity, the Faddeev path integral reads

$$Z=\int \mathcal{D}q\,\mathcal{D}p\,\mathcal{D}n\,F\,\delta(f^\alpha)\exp\left(i\int dt(\dot{q}^s p_s+n^\alpha\Phi_\alpha)\right), \tag{63}$$

where f^α are gauge fixing-functions of the dynamical variables. F is the Faddeev-Popov determinant, $F=\det\{\Phi_\alpha,f^\beta\}$. We do not show the source terms explicitly. The expression (63) can be obtained from the path integral (40) by choosing $g^\alpha=0$ and integrating over the ghost fields c and $\bar{c}$. Of course, the starting point of the original derivation [24] of the Faddeev path integral was not the BRST approach.

To make the presentation as simple as possible, we fix Lorentz boosts by the condition

$$\chi=0. \tag{64}$$

Now we integrate over $\mathcal{N}_L^a$, χ, and ω. Again, integration over ω is equivalent to the following substitution:

$$\omega_a := \partial_j \tilde{E}_a^j. \tag{65}$$

If the remaining gauge-fixing conditions f^μ are functions of $\tilde{E}$ only, the Poisson brackets $\{f^\mu,\Phi_L^a\}$ vanish on the surface (64). Hence the Faddeev-Popov determinant takes the form

$$F=\det\{f^\mu(\tilde{E}),\Phi_\nu\}=\det\{f^\mu(-iP),\Phi_\nu^{[C]}\}_C. \tag{66}$$

The gauge (64) means that we are using reality conditions in the triad form

$$\mathrm{Re}\ P_a^i=0, \quad \mathrm{Re}(\partial_t P_a^i)=0, \tag{67}$$

instead of the metric reality conditions (8) and (9).

The change of variables $(\tilde{E},\eta)\to(P,A)$ gives a unit Jacobian factor. Our proof of the vanishing of the imaginary part of the Ashtekar action is still valid. Hence we arrive at the path integral for Ashtekar gravity in the Faddeev form:

$$Z=\int_R \mathcal{D}P\,\mathcal{D}A\,\mathcal{D}\underset{\sim}{N}\,\mathcal{D}\mathcal{N}_D\,\mathcal{D}A_0\,F\,\delta(f^\mu(-iP))\exp(iS_A), \tag{68}$$

where the subscript R means now that the contour of integration is defined by the reality conditions (67). Of course, most of the comments of the previous section apply here also.

VIII. DISCUSSION

The main result of the present paper is the path integral (61) for Ashtekar gravity, which is a kind of contour integral. As a by-product, we also constructed the BRST quantization of Hilbert-Palatini gravity. The main features of our approach were discussed in detail in Sec. VI. Here we speculate on perspectives of this approach.

The path integral (61) is obtained with certain restrictions on possible gauge conditions. In principle, one can transform Eq. (61) to any other gauge by means of the Faddeev-Popov trick [8]. However, this trick is not so easy to implement in the present context due to the reality conditions and quite unusual rules of the functional integration. Perhaps restrictions on the gauge conditions may be weakened or even lifted altogether. Anyhow, one should formulate the criteria of admissibility of gauge conditions for Ashtekar gravity in terms of the Ashtekar variables without referring to Hilbert-Palatini gravity. This definitely will not be easy to do. In general, a function of P is complex valued. Therefore, a condition $f=0$ implies two real gauge-fixing conditions $\mathrm{Re}\,f=0$ and $\mathrm{Im}\,f=0$ even if reality conditions are taken into account. Even the requirement that a given set of gauge conditions remove the correct number of degrees of freedom looks quite nontrivial. One may hope to overcome these difficulties by using the generalized Wick rotation [25].

In principle, one may include the Barbero connection [26] in our scheme. One may even keep the parameter β [26] (corresponding to γ of the later works) arbitrary. One must be very careful, however, with possible complex factors in the path integral measure. The criteria of admissibility of gauge conditions must be reconsidered after transformation to new variables. Given the growing interest [3] in the Barbero connection this can be an interesting topic to consider in future work.

We must admit that for the degenerate triad our analysis is incomplete. This reflects a well-known problem of Ashtekar gravity which exists already at the classical level.

An intriguing feature of Eq. (61) is that it is a contour integral. The contour of integration can be deformed as far as the reality conditions allow. (This corresponds to an arbitrariness of gauge fixing in the Hilbert-Palatini action.) One may hope that certain deformations are possible even beyond these limits. If this is really so, some interesting properties of quantum gravity can manifest themselves.

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APPENDIX A

Let us solve the second class constraint (27). The matrix $\mathcal{A}_{AB}$ is defined by the r^2 terms in the Hamiltonian constraint Φ^H. We have

$$\lambda^A\mathcal{A}_{AB}\lambda^B=r_b^c\mathcal{A}_{bb'}^{cc'}r_{b'}^{c'}, \tag{A1}$$

where $r_b^c=r_j^c\tilde{E}_b^j$. We can identify nondynamical components of the connection λ_A with the symmetric matrices r_a^b. The operator

$$\mathcal{A}_{bb'}^{cc'}=(1-\chi^2)\epsilon^{abb'}\epsilon^{dcc'}X_{ad}, \quad X_{ad}=\delta_{ad}-\chi_a\chi_d \tag{A2}$$

acts on the space of symmetric 3×3 matrices. One can represent it in the following form:

$$\mathcal{A}^{cc'}_{bb'}=(1-\chi^2)^2(X^{bc}X^{b'c'}-X^{bc'}X^{cb'}), \qquad \text{(A3)}$$

where X^{bc} is inverse of X_{bc}. The inverse of Eq. (A3) is easily found to be

$$(\mathcal{A}^{-1})^{cc'}_{bb'}=(1-\chi^2)^{-2}(\tfrac{1}{2}\ X_{bc}X_{b'c'}-X_{bc'}X_{cb'}). \quad \text{(A4)}$$

The linear part of the Hamiltonian constraint reads

$$\mathcal{B}_A\lambda^A=\epsilon^{abc}\tilde{E}^i_b[\tilde{E}^j_c r^a_d(\partial_i \underset{\sim}{E}^d_j)(1-\chi^2)+r^d_c\chi_d\partial_i\chi_a -(1-\chi^2)\chi_a r^d_c\eta^d_i+X_{ag}\eta^g_i r^d_c\chi_d]=\mathcal{B}^b_a r^a_b. \quad \text{(A5)}$$

Note that Eq. (A5) does not contain derivatives of r^b_a. Hence the second class constraint (27) can be solved for r^b_a:

$$r^c_d=\tfrac{1}{2}\ [(\mathcal{A}^{-1})^{ca}_{db}+(\mathcal{A}^{-1})^{cb}_{da}]\mathcal{B}^b_a. \qquad \text{(A6)}$$

Substitution of Eqs. (A4) and (A5) into Eq. (A6) gives expression (28). The Hamiltonian constraint takes the form $\Phi^H=\Phi^H_0+\mathcal{B}^b_a(\mathcal{A}^{-1})^{ac}_{bd}\mathcal{B}^d_c$, which is written explicitly in Eq. (29).

APPENDIX B

In this appendix we define structure functions $U^{(n)}$ of Hilbert-Palatini gravity. For $n=0$ and $n=1$ they are

$$U^{(0)}_\alpha=\Phi_\alpha, \quad U^{(1)\gamma}_{\alpha\beta}=-\tfrac{1}{2}\ C^\gamma_{\alpha\beta}, \qquad \text{(B1)}$$

with $C^\gamma_{\alpha\beta}$ defined by the algebra (32) through the relation $\{\Phi_\alpha,\Phi_\beta\}=C^\gamma_{\alpha\beta}\Phi_\gamma$. Higher order structure functions are defined through repeated Poisson brackets of the constraints

$$2U^{(2)\xi\eta}_{\alpha\beta\gamma}\Phi_\eta=D^{(1)\xi}_{\alpha\beta\gamma}=\tfrac{1}{2}\ (\{\Phi_\alpha,C^\xi_{\beta\gamma}\}-C^\delta_{\beta\gamma}C^\xi_{\alpha\delta})_{[\alpha\beta\gamma]}, \qquad \text{(B2)}$$

where $[\alpha_1\cdots\alpha_n]$ means antisymmetrization in $\alpha_1\cdots\alpha_n$ with the weight $1/n!$. In actual calculations it is convenient to replace antisymmetrization by multiplication by anticommuting ghosts. The indices $\alpha,\beta,\ldots$ denote constraints at different coordinate points. Therefore, antisymmetrization over coinciding indices does not necessarily give zero.

If fewer than two indices among α, β, and γ correspond to the Hamiltonian constraint, the structure functions C in Eq. (B2) become field-independent structure constants, and the second order structure functions $U^{(2)\xi\eta}_{\alpha\beta\gamma}$ vanish by virtue of ordinary Bianchi identities. Hence, one must calculate only the structure functions with a pair of indices, say, β and γ, corresponding to the Hamiltonian constraint. From now on, an index representing the Hamiltonian constraint will be denoted by 0, $\Phi^H\equiv\Phi_0$. We put $\gamma=\beta=0$.

It is convenient to introduce a connection field of the Lorentz group SO(3,1): $A^p_i=(\xi^a_i,\eta^x_i+\epsilon^{xgf}\xi^g_i\chi_f)$, $p=1,\ldots,6$. Here f^r_{pq} will denote structure constants of the corresponding Lie algebra.

From Eqs. (32) it is clear that canonical momenta enter the first order structure functions C through the vector $K^j[n,m]=(n\partial_i m-\partial_i n m)K^{ij}$, where K^{ij} is defined in Eqs. (33). Later n and m will be replaced by ghost fields. Thus an order is essential. n always precedes m. The tensor K has the following Poisson brackets with the constraints:

$$\{\Phi^G_a,K^{ij}\}=\{\Phi^L_a,K^{ij}\}=0, \quad \{c^0\Phi_0,K^j[c^0,c^0]\}=0,$$

$$\{c^k\Phi^D_k,K^{ij}\}=2(2K^{ij}\partial_k c^k+c^k\partial_k K^{ij}-\partial_k c^j K^{ik}-\partial_k c^i K^{kj}), \qquad \text{(B3)}$$

where contraction with anticommuting ghosts c is used for antisymmetrization in corresponding indices.

Let us calculate $c^\alpha D^{(1)\xi}_{\alpha00'}c^0(x)c^0(x')$. Consider various cases for α. If $\Phi_\alpha=\Phi_p=(\Phi^G,\Phi^L)$ and $\Phi_\xi=\Phi_0(=\Phi^H)$ or $\Phi_\xi=\Phi^D$, this quantity vanishes due to Eqs. (B3). For $\Phi_\xi=\Phi_q$ one obtains

$$c^pD^{(1)q}_{p00'}c^0(x)c^0(x')=\tfrac{2}{3}\ K^j[c_0(x),c_0(x')] \times(-\{c^p\Phi_p,A^q_j\}+f^q_{rp}A^r_j c^p+\partial_j c^q) \times\delta(x,x'). \qquad \text{(B4)}$$

As a part of our summation convention we assume integration over all continuous coordinates here and in the equations below. The expression (B4) is zero due to Eqs. (31). This implies that $U^{(2)\xi\eta}_{00p}=0$.

Let us put $\Phi_\alpha=\Phi^D_i$. We are to evaluate

$$c^iD^{(1)\xi}_{i0'0''}c^0(x')c^0(x'')=\tfrac{1}{6}\ c^i(\{\Phi_i,C^\xi_{0'0''}\}-C^\beta_{0'0''}C^\xi_{i\beta} -2C^\beta_{i0'}C^\xi_{0''\beta})c^0(x')c^0(x''). \qquad \text{(B5)}$$

First we observe that the only nonvanishing function C with zero upper index is C^0_{0i}. This immediately gives a vanishing of Eq. (B5) for $\xi=0$. Other components of Eq. (B5) vanish due to Eqs. (31) and (B3).

For $\alpha=0$ we have

$$\begin{aligned}c^0(x)D^{(1)0}_{00'0''}c^0(x')c^0(x'')&=-\tfrac{1}{2}\ c^0(x)C^i_{00'}C^0_{i0''}c^0(x')c^0(x'')\\&=8c^0\partial_i c^0\partial_j c^0K^{ij}\delta(x,x')\delta(x',x'')\\&=0,\end{aligned}$$

$$\begin{aligned}c^0(x)D^{(1)i}_{00'0''}c^0(x')c^0(x'')&=\tfrac{1}{2}\ c^0(x)\{\Phi_0,C^i_{00}\}c^0(x')c^0(x'')\\&=0,\end{aligned}$$

$$c^0(x)D^{(1)p}_{00'0''}c^0(x')c^0(x'') =-2\{c^0(x)\Phi^H(x),K^j[c^0(x'),c^0(x'')]A^p_0\}, \qquad \text{(B6)}$$

where the first line is zero due to the contraction of a symmetric tensor with an antisymmetric one. In the second line we used second equation of Eqs. (B3).

To calculate the remaining components of $D^{(1)}$ the following brackets are needed:

$$\{c^0\Phi_0,K^j[c^0,c^0]\xi_j^a\}=2c^0\partial_i c^0\partial_k c^0[(\tilde{E}_a^i\tilde{E}_b^k-\tilde{E}_a^k\tilde{E}_b^i)\Phi_b^G + (\tilde{E}_a^i\tilde{E}_g^k-\tilde{E}_a^k\tilde{E}_g^i)\epsilon^{gfb}\chi_f\Phi_b^L],$$

$$\{c^0\Phi_0,K^j[c^0,c^0](\eta_j^a+\epsilon^{abc}\xi_j^b\chi_c)\} = 2c^0\partial_i c^0\partial_k c^0[-\epsilon^{ade}\chi_d(\tilde{E}_e^i\tilde{E}_b^k-\tilde{E}_e^k\tilde{E}_b^i)\Phi_b^G - \epsilon^{ade}\chi_d(\tilde{E}_e^i\tilde{E}_g^k-\tilde{E}_e^k\tilde{E}_g^i)\epsilon^{gfb}\chi_f\Phi_b^L]. \tag{B7}$$

By introducing a 3×6 matrix field $\tilde{E}_p^i=(\tilde{E}_a^i,\epsilon_{abc}\tilde{E}_b^i\chi_c)$, $p=1,\ldots,6$, one can represent the nonvanishing second order structure functions in an elegant form

$$c^0(x)U^{(2)pq}_{00'0''}c^0(x')c^0(x'') = -8c^0\partial_i c^0\partial_k c^0\tilde{E}_p^i\tilde{E}_q^k\delta(x,x')\,\delta(x',x''). \tag{B8}$$

Third order structure functions are defined as

$$3U^{(3)\xi\eta\lambda}_{\alpha\beta\gamma\delta}\Phi_\lambda=(-\{U^{(2)\xi\eta}_{\alpha\beta\gamma},\Phi_\delta\}-\tfrac{1}{8}\{C^\xi_{\alpha\beta},C^\eta_{\gamma\delta}\} + \tfrac{3}{2}C^\lambda_{\alpha\beta}U^{(2)\xi\eta}_{\gamma\delta\lambda}+2U^{(2)\xi\lambda}_{\alpha\beta\gamma}C^\eta_{\delta\lambda})^{[\xi\eta]}_{[\alpha\beta\gamma\delta]}. \tag{B9}$$

As before, only the functions with $\alpha=\beta=\gamma=0$ could be nonzero. By straightforward calculations one can demonstrate that they vanish as well. There are no nonzero third or higher order structure functions in Hilbert-Palatini gravity.

APPENDIX C

In this appendix we prove that the imaginary part of the kinetic term of the Ashtekar action (7) vanishes for the fields (42) provided the real canonical variables satisfy the second class constraints (27) and the Gauss and Lorentz constraints.

Consider the kinetic term

$$\mathrm{Im}\,A_i^a\partial_t P_a^i=\{[\delta_{ab}(1-\chi^2)+\chi_a\chi_b]\xi_i^b-\epsilon^{abc}\chi_b\eta_i^c\}\partial_t\tilde{E}_a^i - (\epsilon^{abc}\eta_i^b\tilde{E}_c^i+\chi_a\xi_i^b\tilde{E}_b^i-\xi_i^a\tilde{E}_b^i\chi_b)\partial_t\chi_a, \tag{C1}$$

where expressions (42) were substituted. By making use of the constraints one can rewrite Eq. (C1) in the following form;

$$\begin{aligned}\mathrm{Im}\,A_i^a\partial_t P_a^i=&-\tfrac{1}{2}\partial_t\tilde{E}_a^i[\epsilon^{abc}\tilde{E}_b^k\tilde{E}_c^j(\delta_{gf}-\chi_g\chi_f)\underset{\sim}{E}_i^g\partial_k\underset{\sim}{E}_j^f-\underset{\sim}{E}_i^a\epsilon^{dbc}\tilde{E}_b^k\tilde{E}_c^j(\delta_{df}-\chi_d\chi_f)\partial_k\underset{\sim}{E}_j^f+\epsilon^{dbc}\tilde{E}_b^k\tilde{E}_c^j(\delta_{dg}-\chi_d\chi_g)\underset{\sim}{E}_i^g\partial_k\underset{\sim}{E}_j^a\\&-\epsilon^{abc}\partial_j\tilde{E}_c^j(\delta_{bg}-\chi_b\chi_g)\underset{\sim}{E}_i^g-\underset{\sim}{E}_i^a\epsilon^{dbc}\underset{\sim}{E}_b^j\partial_j\chi_d\chi_c+2\chi_g\underset{\sim}{E}_i^g\epsilon^{abc}\tilde{E}_c^j\partial_j\chi_b]\\&-\tfrac{1}{2}\partial_t\chi_a[\epsilon^{abc}\tilde{E}_b^k\tilde{E}_c^j\chi_g\partial_k\underset{\sim}{E}_j^g+\epsilon^{dbc}\tilde{E}_b^k\tilde{E}_c^j\chi_d\partial_k\underset{\sim}{E}_j^a+\epsilon^{abc}\partial_j\underset{\sim}{E}_b^j\chi_c+2\epsilon^{abc}\tilde{E}_b^j\partial_j\chi_c]\\=&\tfrac{1}{2}\partial_t[\epsilon^{dbc}\tilde{E}_b^k\tilde{E}_c^j(\delta_{dg}-\chi_d\chi_g)]\partial_k\underset{\sim}{E}_j^g-\tfrac{1}{2}\partial_t\underset{\sim}{E}_j^g\partial_k[\epsilon^{dbc}\tilde{E}_b^k\tilde{E}_c^j(\delta_{dg}-\chi_d\chi_g)]+\tfrac{1}{2}\partial_t(\tilde{E}_a^i\chi_b)\epsilon^{abc}\partial_i\chi_c\\&-\tfrac{1}{2}\epsilon^{abc}\partial_t\chi_c\partial_i(\tilde{E}_a^i\chi_b)=\tfrac{1}{2}\partial_t[\epsilon^{dbc}\tilde{E}_b^k\tilde{E}_c^j(\delta_{dg}-\chi_d\chi_g)\partial_k\underset{\sim}{E}_j^g]-\tfrac{1}{2}\partial_k[\epsilon^{dbc}\tilde{E}_b^k\tilde{E}_c^j(\delta_{dg}-\chi_d\chi_g)\partial_t\underset{\sim}{E}_j^g]\\&+\tfrac{1}{2}\partial_t(\epsilon^{abc}\tilde{E}_a^i\chi_b\partial_i\chi_c)-\tfrac{1}{2}\partial_i(\epsilon^{abc}\tilde{E}_a^i\chi_b\partial_t\chi_c).\end{aligned}$$

Thus the imaginary part of the kinetic term is a total derivative and can be neglected.

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