

Born-regulated gravity in four dimensions

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Previous work involving Born-regulated gravity theories in two dimensions is extended to four dimensions. The action we consider has *two* dimensionful parameters. Black hole solutions are studied for typical values of these parameters. For masses above a critical value determined in terms of these parameters, the event horizon persists. For masses below this critical value, the event horizon disappears, leaving a “bare mass,” though of course no singularity. [S0556-2821(98)07324-X]

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I. INTRODUCTION

Recent developments in the theory of strings and branes have renewed interest in Born-Infeld Lagrangians [1] and their non-Abelian generalization [2–6]. In the 1930s, Born proposed a modified electromagnetic Lagrangian which removes the point-charge singularity that mars classical electrodynamics. If the Lagrangian is a nonpolynomial function of $F_{\mu\nu}F^{\mu\nu}$ with a branch point, then this branch point can impose an upper bound on field strengths, above which the Lagrangian will become imaginary. Specifically Born considered the theory [7]

$$\mathcal{L} = \Lambda^2 \left[\sqrt{1 - \frac{F_{\mu\nu}F^{\mu\nu}}{2\Lambda^2}} - 1 \right], \quad (1)$$

which requires $E^2 \leq \Lambda^2$.

Similar theories can be constructed for gravity, replacing the Maxwell field tensor with the Riemann curvature tensor. It is widely expected that quantum effects remove the singularities of classical general relativity, cutting off curvatures at the string scale. By integrating out all non-gravitational degrees of freedom in the full Lagrangian for the universe, one can obtain an effective Lagrangian for gravity which will be nonpolynomial in curvature components. This effective Lagrangian might be of the Born variety, and it is this possibility which we wish to explore in this paper.

Lagrangians of this type in two dimensions were considered by Feigenbaum, Freund, and Pigli in Ref. [8], where the four-dimensional case was briefly alluded to. Deser and Gibbons have considered the four-dimensional gravitational analogue of the Born-Infeld Lagrangian [9]. For reasons of simplicity, we will consider here black holes smoothed by an ordinary Born Lagrangian analogous to Eq. (1).

In Sec. II, we introduce the specific Born-regulated gravitational Lagrangian which we investigate here. Remarkably, on account of the two dimensionful parameters in this Lagrangian, we find two regimes. In one regime, an event horizon is present, as in the Einstein-Hilbert case, even though there is no singularity to “protect.” In the other regime, there is no singularity *and* no event horizon. One has a “bare mass,” the regularized version of a naked singularity. In Secs. III and IV we present an example of both kinds of

solutions. Then, in Sec. V, we will explore the regions in parameter space where each of these two types of black hole solutions occurs.

II. LAGRANGIAN IN FOUR DIMENSIONS

In Ref. [8], Born-regulated gravity theories in two dimensions were considered. The action

$$\mathcal{A} = \int d^2x \sqrt{-g} R [\ln R + \beta \ln(a - R)] \quad (2)$$

has Witten black hole solutions [10–12] in the limit as $\beta \rightarrow 0$ but imposes the bound $R < a$ on the scalar curvature for $\beta \neq 0$. As a result, for $\beta \neq 0$, instead of becoming singular the space-time goes asymptotically into a de Sitter space with $R = a$. Note that in two dimensions the scalar curvature is the sole independent curvature component.

In generalizing the notion of Born-regulated gravity theories to four dimensions, we must recognize that we now have 20 independent curvature components to play with and 3 scalar invariants which can be formed from the Riemann tensor and which can appear in a Lagrangian: i.e. $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $R_{\mu\nu}R^{\mu\nu}$, and R . We also have empirical data to contend with in four dimensions; so preferably a gravitational Lagrangian should reduce to the Einstein-Hilbert Lagrangian in the weak-field limit.

A candidate action to consider is

$$\mathcal{A} = \int d^4x \sqrt{-g} \times [R + \beta(\sqrt{1 - k_1 S - k_2 R_{\mu\nu}R^{\mu\nu} - k_3 R^2} - 1)], \quad (3)$$

where $S_{\beta}^{\alpha} \equiv R^{\alpha\mu\nu\rho}R_{\beta\mu\nu\rho}$ and S is the trace of this tensor. For the Schwarzschild black-hole solution, $R_{\mu\nu} = 0$. Assuming that $R_{\mu\nu} \sim 0$ for black hole solutions to the field equations obtained by varying this action, we simplify the action by setting $k_2 = k_3 = 0$ and $k_1 = k$ to obtain the action

$$\mathcal{A} = \int d^4x \sqrt{-g} [R + \beta(\sqrt{1 - kS} - 1)], \quad (4)$$

which imposes the bound $S \leq 1/k$ on the square of the Riemann tensor for $\beta \neq 0$.

The action (4) yields the field equations

$$R_{\beta}^{\alpha} - \frac{1}{2} \delta_{\beta}^{\alpha} [R + \beta(V-1)] - \frac{k\beta S_{\beta}^{\alpha}}{V} - 2k\beta \nabla^{\mu} \nabla_{\nu} \left(\frac{R_{\beta\mu}^{\alpha\nu}}{V} \right) = 0, \quad (5)$$

where

$$V = \sqrt{1 - kS}. \quad (6)$$

Two parameters β and k appear in the action (4). Here β has dimension $(\text{length})^{-2}$, and the dimension of k is $(\text{length})^4$. There will thus be *two* scales in the problem, not unlike string theory. This, as we shall see in Sec. IV, will have as an important consequence the existence of a critical mass below which the regulated analogue to the black hole solution sheds its event horizon.

III. BLACK HOLE SOLUTIONS FOR SMALL k AND β

We wish to consider solutions which behave as black holes at large distances and satisfy a spherically symmetric ansatz for the metric:

$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{h^2(r)} + r^2[d\theta^2 + \sin^2(\theta)d\varphi^2]. \quad (7)$$

Inserting this ansatz into Eq. (5), we obtain three nontrivial equations corresponding to the variation of the action with respect to g_{tt} , g_{rr} , and $g_{\theta\theta}$ (the equations corresponding to $g_{\theta\theta}$ and $g_{\varphi\varphi}$ being identical). However, since there are only two unknown functions in the ansatz, $f(r)$ and $h(r)$, these three equations are not independent.

In the Schwarzschild solution for a black hole of mass M ,

$$f_s(r) = h_s(r) = \sqrt{1 - \frac{2M}{r}}. \quad (8)$$

Consequently, in the limit of small k and β , for $r \rightarrow \infty$ we write

$$f(r) = \sqrt{1 - \frac{2M}{r}} [1 + \phi(r)] \quad (9)$$

and

$$h(r) = \sqrt{1 - \frac{2M}{r}} [1 + \eta(r)], \quad (10)$$

where $\eta(\infty) = \phi(\infty) = 0$. Let

$$\lambda = \frac{k^2\beta}{(2M)^6}. \quad (11)$$

This dimensionless parameter characterizes perturbations of the Schwarzschild solution. Solving Eqs. (5) for ϕ and η to lowest order in λ , we find

$$\phi(r) = \frac{-8k^2M^3\beta}{r^9} \left(\frac{8r-11M}{r-2M} \right) + O\left(\frac{k^3\beta^2M^3}{r^{11}} \right) \quad (12)$$

and

$$\eta(r) = \frac{-8k^2M^3\beta}{r^9} \left(\frac{36r-67M}{r-2M} \right) + O\left(\frac{k^3\beta^2M^3}{r^{11}} \right). \quad (13)$$

Clearly, in the limit of small k and β , deviations from the Schwarzschild solution are negligible for $r \gg 2M$.

We may note that the $k^2M^3\beta/r^9$ dependence of the prefactors in $\phi(r)$ and $\eta(r)$ is easily understood. For the unperturbed Schwarzschild solution, the lowest order nonvanishing terms in the field equations (5) derive from the $\beta k^2 S^2$ term in the expansion of the Born-regulating square root of the Lagrangian (4), and to lowest order in $1/r$ these terms go as $\beta k^2 M^4/r^{12}$. By contrast, the inclusion of $\phi(r)$ and $\eta(r)$ in the solution of Eqs. (9), (10) leads to nonvanishing Ricci tensor and scalar curvature terms in the field equations which go as $M\phi/r^3$ and $M\eta/r^3$ to lowest order in $1/r$. Consequently, in order for all these terms to cancel, we must have the prefactors seen in Eqs. (12), (13).

To analyze these solutions near and within the event horizon $r \approx 2M$, we must transform to Kruskal-like coordinates, exchanging r and t for the light-cone coordinates u and v . The spherically symmetric ansatz analogous to Eq. (7) for Kruskal-like coordinates is

$$ds^2 = -\exp[2\rho(w)]dudv + r^2(w)[d\theta^2 + \sin^2(\theta)d\varphi^2], \quad (14)$$

where $w = uv$ and the functions ρ and r are functions of w alone. Here $r(w)$ is precisely the coordinate r in the Schwarzschild-like ansatz of Eq. (7).

Inserting this ansatz into Eq. (5), we again obtain three separate but presumably not independent equations corresponding to the variation of Eq. (4) with respect to g_{uu} , g_{uv} , and $g_{\theta\theta}$. In order to integrate these differential equations, continuing from our solution of Eqs. (9), (10), we must know r and ρ and their first derivatives at some point.

The event horizon in u - v coordinates is the surface $w = 0$. In the region $w < 0$ which corresponds to the region outside the event horizon, we can make the coordinate transformation

$$u = -\sqrt{-w(r)} \exp\left(\frac{-t}{4M}\right) \quad (15)$$

and

$$v = \sqrt{-w(r)} \exp\left(\frac{t}{4M}\right). \quad (16)$$

Here $w(r)$ is the inverse of the function $r(w)$ in Eq. (14). The corresponding transformation of the metric then gives

$$f(r) = \frac{\sqrt{-w(r)} \exp[\rho(w(r))]}{4M} \quad (17)$$

and

$$h(r) = \frac{-2\sqrt{-w(r)} \exp[-\rho(w(r))]}{w'(r)}. \quad (18)$$

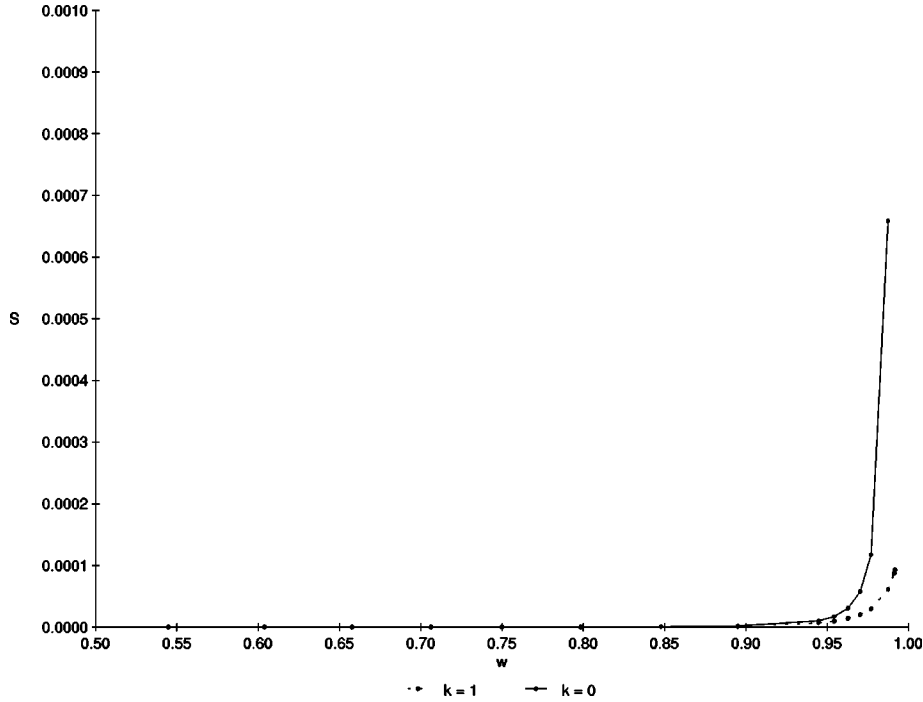


FIG. 1. The curvature invariant $S = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ as a function of the Kruskal-coordinate combination $w = uv$ for the ordinary Schwarzschild solution (solid curve) and for the Born regulated theory (dashed curve) to order w^{40} with $k = 1$ and $\beta = 10^9$, both for an object of mass 100.

We see from this identification that the event horizon in Schwarzschild-like coordinates is $r = r_s$, where

$$f(r_s) = h(r_s) = 0. \quad (19)$$

Solving Eq. (19) for r_s , we find to first order in λ that $r_s = 2M(1 + 5\lambda)$.

As always, at the event horizon, there is a coordinate singularity in r - t coordinates, but the Riemann tensor remains finite. Using the coordinate transformation of Eqs. (15) and (16), we can relate the components of the Riemann tensor in the two coordinate systems:

$$R_{uv}^{uv} = R_{tr}^{tr}, \quad (20)$$

$$R_{u\theta}^{u\theta} = R_{v\theta}^{v\theta} = \frac{1}{2}(R_{t\theta}^{t\theta} + R_{r\theta}^{r\theta}), \quad (21)$$

and

$$\frac{v}{u}R_{v\theta}^{u\theta} = \frac{u}{v}R_{u\theta}^{v\theta} = \frac{1}{2}(R_{r\theta}^{r\theta} - R_{t\theta}^{t\theta}). \quad (22)$$

We expand $r(w)$ and $\rho(w)$ around $w = 0$:

$$r(w) = r_s \left(1 + \sum_{n=1}^{\infty} c_n w^n \right) \quad (23)$$

and

$$\rho(w) = -\frac{1}{2} \ln(A) + \sum_{n=1}^{\infty} a_n w^n. \quad (24)$$

The scale of w is arbitrary; so we can set $c_1 = \exp(-1)$, which for the Schwarzschild solution would place the curvature singularity at $w = 1$. Using Eqs. (20)–(22), we compare at the event horizon the Riemann tensor components in the solution of Eqs. (9)–(13), written in Schwarzschild-like coordinates, with the series expansion of Eqs. (23), (24), written in Kruskal-like coordinates. In this manner, we obtain values for A , a_1 , and c_2 . This is the remaining information necessary to integrate Eq. (5).

As an example of a numerical solution, we choose $k = 1$, $M = 100$, and $\beta = 10^9$ to obtain a small value of the perturbation parameter $\lambda = .000015625$ from Eq. (11). In Fig. 1, S is plotted as a function of w for the Schwarzschild solution and the perturbed solution calculated to $O(w^{40})$. We see that as $w \rightarrow 1$, where the Schwarzschild solution is singular, S for the perturbed solution is less than S for the Schwarzschild solution. However, to 40th order in w , S remains much less than the upper bound of $S \leq 1$ at $w = 1$. In order to see the upper bound come into effect, we would need to calculate the solution to a very high order with such small values of k and β . In the next section, we will consider a solution with much larger values of these parameters, where the curvature bound will become evident.

IV. BLACK HOLE SOLUTIONS FOR LARGE β

For sufficiently large values of β , we can ignore the scalar curvature term of Eq. (4) and the field equations reduce to

$$\frac{1}{2} \delta_{\beta}^{\alpha} (V - 1) + \frac{k S_{\beta}^{\alpha}}{V} + 2k \nabla^{\mu} \nabla_{\nu} \left(\frac{R_{\beta\mu}^{\alpha\nu}}{V} \right) = 0. \quad (25)$$

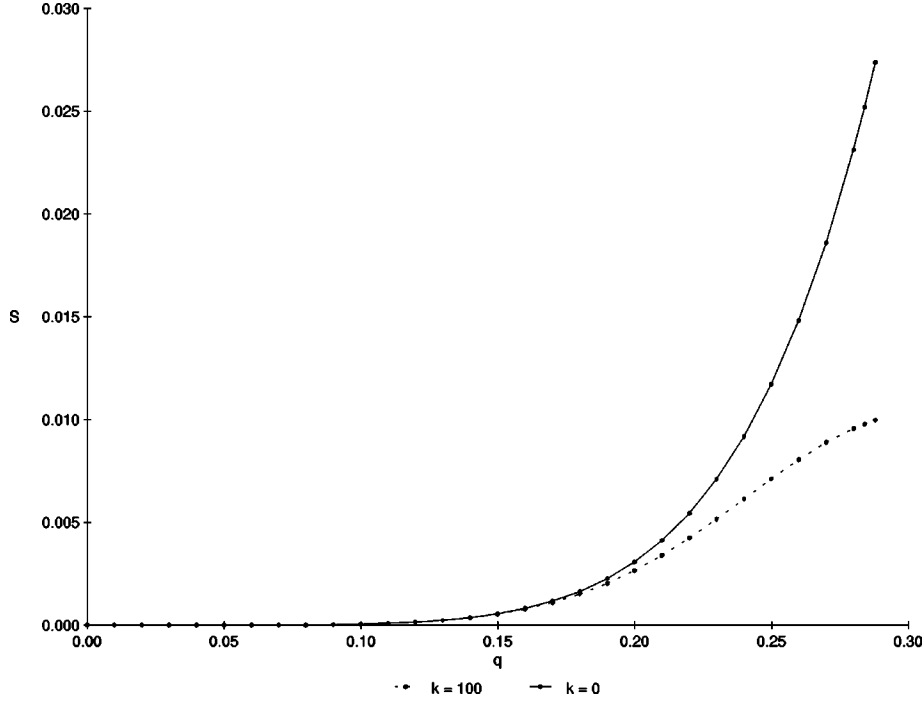


FIG. 2. The curvature invariant $S = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ as a function of $q = 1/r$ for the ordinary Schwarzschild solution (solid curve) and for the Born regulated theory (dashed curve) with $k = 100$, both for an object of unit mass.

Again, we wish to find solutions of these field equations that act like a black hole solution of mass M as $r \rightarrow \infty$. In order to expand around infinity, we introduce the variable $q = 1/r$. Replacing r with q in Eq. (7) we have the equivalent ansatz

$$ds^2 = -f^2(q)dt^2 + \frac{dq^2}{q^4 h^2(q)} + \frac{d^2\theta + \sin^2(\theta)d\varphi^2}{q^2}. \quad (26)$$

We expand $f(q)$ and $h(q)$ around $q = 0$:

$$f(q) = 1 + \sum_{n=1}^{\infty} b_n q^n \quad (27)$$

and

$$h(q) = 1 + \sum_{n=1}^{\infty} d_n q^n. \quad (28)$$

We require f and h to satisfy the same boundary conditions as the Schwarzschild solution; so we set $b_1 = d_1 = -M$. Then, if we solve recursively for the higher order coefficients, we obtain

$$f(q) = \sqrt{1 - 2Mq} + \frac{256kM^3 q^7}{336} + O(q^8) \quad (29)$$

and

$$h(q) = \sqrt{1 - 2Mq} + \frac{128kM^3 q^7}{48} + O(q^8). \quad (30)$$

Thus in the infinite β limit, this metric is indistinguishable from the Schwarzschild metric far from the black hole.

In Fig. 2, we plot S versus r for our solution here with $M = 1$ and $k = 100$ along with the Schwarzschild solution for $M = 1$. Here the curvature bound $S \leq .01$ is quite evident. As $q \sim .29$, S approaches .01. Numerical integration past $q = .29$ becomes exceedingly difficult but is fortunately unnecessary. Indeed, as $q \rightarrow \infty$, one can infer from the action principle that the solution goes asymptotically into a solution of constant S . With V as defined in Eq. (6), the field equations (25) can be rewritten as

$$\begin{aligned} \frac{1}{2}V^5(1-V)\delta_{\alpha}^{\beta} &= kV^4[S_{\alpha}^{\beta} + 2\nabla^{\mu}\nabla_{\nu}R_{\alpha\mu}^{\beta\nu}] \\ &+ k^2V^2[(\nabla^{\mu}R_{\alpha\mu}^{\beta\nu})(\nabla_{\nu}S) + (\nabla_{\nu}R_{\alpha\mu}^{\beta\nu})(\nabla^{\mu}S) \\ &+ R_{\alpha\mu}^{\beta\nu}(\nabla^{\mu}\nabla_{\nu}S)] + \frac{3}{2}k^3R_{\alpha\mu}^{\beta\nu}(\nabla^{\mu}S)(\nabla_{\nu}S). \end{aligned} \quad (31)$$

If S is constant at $1/k$, then $V = 0$, and clearly the field equation is satisfied. So if we have $S \rightarrow 1/k$ on the surface $q = q_0$, it follows that $S = 1/k$ for $q > q_0$.

Note that one glaring absence from the solution with $M = 1$ and $k = 100$ that we have described here is a coordinate singularity. In fact there can be no coordinate singularity for finite q . The curvature components in q - t coordinates are

$$R_{tq}^{tq} = -\frac{q^2 h}{f}(q^2 h f')', \quad (32)$$

$$R_{t\theta}^{t\theta} = q^3 h^2 (\ln(f))', \quad (33)$$

$$R_{q\theta}^{q\theta} = q^3 h h', \quad (34)$$

and

$$R_{\theta\phi}^{\theta\phi} = q^2(1 - h^2). \quad (35)$$

Since $S = 4(R_{tq}^{tq})^2 + 8(R_{t\theta}^{t\theta})^2 + 8(R_{q\theta}^{q\theta})^2 + 4(R_{\theta\phi}^{\theta\phi})^2$, we have the constraint

$$R_{\theta\phi}^{\theta\phi} \leq \frac{1}{2\sqrt{k}}. \quad (36)$$

For $q \geq .29$ and $k = 100$, it follows from Eq. (35) and the constraint (36) that $.6368 \leq h \leq 1.2627$. Since h and $R_{t\theta}^{t\theta}$ must be finite, it also follows, from Eq. (33), that $(d/dq)[\ln(f)]$ must be finite, and so $\ln(f)$ and f must remain finite for finite q .

Since f and h must remain finite for finite q , it follows that there can be no coordinate singularity and therefore no event horizon for finite q . The solution we have here describes what is not really a black hole but a ‘‘bare mass.’’ It is important to note that this is *not* a ‘‘naked singularity.’’ Although it is ‘‘bare’’ or ‘‘naked’’ in the sense that it is not hidden behind an event horizon, it is not a ‘‘naked singularity’’ because there is no singularity.

V. SHEDDING THE EVENT HORIZON

The absence of an event horizon is not a universal property of all solutions to Eq. (5) which behave as black holes for large r . If there is an event horizon, we must have $h = 0$ at this horizon. Then Eqs. (35) and (36) imply that the reciprocal Schwarzschild radius q_s satisfies $q_s^2 \leq 1/2\sqrt{k}$. For small β , $q_s \sim 1/2M$; so it follows that the dimensionless ratio k/M^4 will determine whether there can be an event horizon. For $k/M^4 \gg 1$, the event horizon must disappear. For $k/M^4 \ll 1$, there should still be an event horizon as in Sec. III. Since we used the original, unsimplified field equation (5) in Sec. III, it is important here to recognize that the argument at the end of Sec. IV depends on the metric ansatz (26) and the curvature bound $S \leq 1/k$ but not on the details of the Born Lagrangian. As such, this argument applies equally well for small β , the case considered here, as for large β , the case covered in Sec. IV. For a given value of the parameter k appearing in the Lagrangian, the event horizon will disappear as the mass falls below some critical mass of order $k^{1/4}$ (or $k^{1/4}/G$ if we include Newton’s gravitational constant explicitly).

For very small (yet nonzero) β , what makes this mechanism feasible is the presence of two dimensionless parameters in the problem, $M^2\beta$ and k/M^4 , corresponding to the two dimensionful parameters k and β in the action. We can choose βM^2 to be arbitrarily small. However, if we also choose $k/M^4 \gg 1$, then at the place where one would naively expect an event horizon to appear, $kS \rightarrow 1$. As a result, even though each individual term in the series expansion of $\beta\sqrt{1 - kS}$ in the action (4) may be very small, the terms do not diminish in magnitude. Consequently, their infinite sum will still dominate over the Einstein-Hilbert term, giving rise

to a very different solution from ordinary Schwarzschild. Besides being nonsingular, this solution lacks an event horizon.

The precise value of the critical mass will depend on the value of β . The surface in k - β - M space where the transition between having and not having an event horizon occurs should evidently take the form

$$\frac{k}{M^4} = \sigma(M^2\beta), \quad (37)$$

where σ is an unknown function. For general values of $M^2\beta$, the precise form of $\sigma(M^2\beta)$ will have to be determined numerically in the manner of the last section. In this way, we have found an upper bound on σ for large β of $\sigma(M^2\beta) < 1$.

One can infer the limiting value of $\sigma(M^2\beta)$ as $\beta \rightarrow 0$. For very small values of β , solutions of Eq. (5) with mass M should behave exactly like the Schwarzschild solution with mass M up until the point where $kS \rightarrow 1$, at which point S will flatten out and asymptotically approach the value $1/k$. For the Schwarzschild solution, $S = 3/4M^4$ at the event horizon. So if $3/4M^4 < 1/k$, there should still be an event horizon since the solution will not begin to deviate from Schwarzschild until we are inside the black hole. If $3/4M^4 > 1/k$, the solution will deviate from Schwarzschild before an event horizon can occur, and so there can be no event horizon. Thus we conclude that

$$\lim_{\beta \rightarrow 0} \sigma(M^2\beta) = \frac{4}{3}. \quad (38)$$

Evidently $\sigma(M^2\beta)$ decreases with β and so the critical mass, below which the event horizon disappears, increases with β , as one would intuitively expect since it would be very surprising if the effects of Born regulation should become less apparent as we increase β .

VI. CONCLUSION

In this paper, we have investigated a Born-regulated theory of gravity in four dimensions. We have found that solutions to this theory exist which behave asymptotically as black holes of mass M but become spaces of constant $S = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ at small radii. These spaces of constant S are analogous to the de Sitter spaces which we found in [8] and which Brandenberger found in [13,14].

For large values of k/M^4 in the Born-regulated Lagrangian, there is no event horizon in these solutions, and we have a ‘‘bare mass’’ instead of a black hole. If we assume that k is valued at the Planck scale, then the event horizon will be absent only for black hole masses below the Planck scale.

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